Research Article

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Breather wave and double-periodic soliton solutions for a (2+1)-dimensional generalized Hirota–Satsuma–Ito equation

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Abstract: In this work, a (2+1)-dimensional generalized Hirota–Satsuma–Ito equation realized to represent the propagation of unidirectional shallow water waves is investigated. We first study the breather wave solutions based on the three-wave method and the bilinear form. Second, the double-periodic soliton solutions are obtained *via* an undetermined coefficient method, which have not been seen in other literature. We present some illustrative figures to discuss the dynamic properties of the derived waves.

Keywords: double-periodic soliton, breather wave, three-wave method, Hirota–Satsuma–Ito equation

1 Introduction

The Hirota–Satsuma–Ito (HSI) equation emerges in the Jimbo–Miwa classification [1–3], which is useful in investigating the propagation of unidirectional shallow water waves [4]. Zhou and Manukure [5] presented the complexiton solutions of the HSI equation by the Hirota bilinear method and the linear superposition principle, and obtained the lump and interaction solutions to the HSI equation *via* the Hirota direct method [6]. Liu *et al.* [7] studied the N-soliton and localized wave interaction solutions of the HSI equation. Liu *et al.* [8] obtained the multi-wave, breather wave, and interaction solutions of the HSI equation based on the three-wave method and the homoclinic breather approach. Saima *et al.* [9] investigated the multiple rational rogue waves *via* symbolic computation approach. The aim of this work is to find the breather wave and the double-periodic soliton solutions *via* the undetermined coefficient method and the three-wave method.

In this article, under investigation is a (2+1)-dimensional generalized HSI equation [5]

$$u_x\left(3\int u_t\mathrm{d}x + \tau\right) + 3uu_t + u_{xxt} + \int u_{yt}\mathrm{d}x = 0, \tag{1}$$

where $u = u(x, y, t)$, τ is the arbitrary constant. Eq. (1) is the extension of the HS shallow water wave equation, which is proposed by Hirota and Satsuma *via* a Bäcklund transformation of the Boussinesq equation.

Making the following transformation

$$u = 2(\ln\xi)_{xx}, \quad \xi = \xi(x, y, t), \tag{2}$$

Eq. (1) becomes

$$\begin{aligned}(D_tD_x^3 + D_tD_y + \gamma D_x^2)\xi\cdot\xi \\ = -\gamma\xi_x^2 + \xi(\gamma\xi_{xx} + \xi_{yt} + \xi_{xxxt}) \\ -3\xi_{xxt}\xi_x - \xi_t\xi_y + 3\xi_{xt}\xi_{xx} - \xi_t\xi_{xxx} = 0.\end{aligned} \tag{3}$$

The solutions of Eq. (1) can be obtained by substituting the solutions of Eq. (3) into Eq. (2). Hence, the following work is mainly based on Eq. (3).

This article is organized as follows: Section 2 obtains the breather wave solutions by using the three-wave method; Section 3 investigates the double-periodic soliton solutions *via* an undetermined coefficient method; Section 4 gives a summary.

2 Breather wave solutions

Based on the three-wave method [10–12], we have

$$\begin{aligned}\xi = \mathrm{e}^{-t\omega_3-x\omega_1-y\omega_2} + \vartheta_1\mathrm{e}^{t\omega_3+x\omega_1+y\omega_2} + \vartheta_2\sin(t\omega_6 \\ + x\omega_4 + y\omega_5) + \vartheta_3\cos(t\omega_9 + x\omega_7 + y\omega_8),\end{aligned} \tag{4}$$

where ω_i $(i = 1, 2, \ldots, 9)$ and ϑ_i $(i = 1, 2, 3)$ are undetermined constants. Substituting Eq. (4) into Eq. (3), we obtain

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Case I:

$$
\begin{aligned}
&\vartheta_2 = 0,\ \ \omega_8 = \frac{\omega_1^2(\tau - 3\omega_7\omega_9) + \omega_7^2(\omega_7\omega_9 - \tau) + \omega_3\omega_1^3 - 3\omega_3\omega_7^2\omega_1 + \omega_2\omega_3}{\omega_9},\\
&\omega_5 = \frac{-\omega_1\omega_4(2\tau + 3\omega_1\omega_3) + \omega_3\omega_4^3 - (\omega_1^3 - 3\omega_4^2\omega_1 + \omega_2)\omega_6}{\omega_3},\\
&\omega_2 = \frac{\tau(-\omega_3\omega_1^2 - 2\omega_7\omega_9\omega_1 + \omega_3\omega_7^2)}{\omega_3^2 + \omega_9^2} - \omega_1^3 + 3\omega_7^2\omega_1,\\
&\tau = -\frac{3(\omega_1^2 + \omega_7^2)(\omega_3^2 + \omega_9^2)(\omega_7\omega_9\vartheta_3^2 + 4\omega_1\omega_3\vartheta_1)}{(\omega_3\omega_7 - \omega_1\omega_9)^2(4\vartheta_1 - \vartheta_3^2)}.
\end{aligned}
\tag{5}
$$

Case II:

$$
\omega_6 = -\frac{\omega_1\omega_3}{\omega_4},\ \omega_8 = \omega_7^3,\ \omega_5 = \omega_4^3,\ \tau = -3\omega_1\omega_3,\ \omega_2 = -\omega_1^3,\ \omega_9 = -\frac{\omega_1\omega_3}{\omega_7}.
\tag{6}
$$

Case III:

$$
\begin{aligned}
&\vartheta_1 = \frac{\tau\omega_4^2\vartheta_2^2 + \tau\omega_7^2\vartheta_3^2 - 4\omega_6\omega_4^3\vartheta_2^2 + \omega_5\omega_6\vartheta_2^2 - 4\omega_7^3\omega_9\vartheta_3^2 + \omega_8\omega_9\vartheta_3^2}{4(\tau\omega_1^2 + 4\omega_3\omega_1^3 + \omega_2\omega_3)},\\
&\omega_8 = \frac{\tau\omega_1^2 - \tau\omega_7^2 + \omega_3\omega_1^3 - 3\omega_7\omega_9\omega_1^2 - 3\omega_3\omega_7^2\omega_1 + \omega_2\omega_3 + \omega_7^3\omega_9}{\omega_9},\\
&\omega_5 = \frac{-\omega_1\omega_4(2\tau + 3\omega_1\omega_3) + \omega_3\omega_4^3 - (\omega_1^3 - 3\omega_4^2\omega_1 + \omega_2)\omega_6}{\omega_3},\\
&\tau = -[3\omega_1(\omega_3^2 + \omega_6^2)(\omega_4^2 - \omega_7^2)(\omega_3^2 + \omega_9^2)]/[(-\omega_3\omega_1^2 - 2\omega_4\omega_6\omega_1 + \omega_3\omega_4^2)\omega_9^2\\
&\qquad + 2\omega_1(\omega_3^2 + \omega_6^2)\omega_7\omega_9 + \omega_3[(\omega_3\omega_4 - \omega_1\omega_6)^2 - (\omega_3^2 + \omega_6^2)\omega_7^2]],\\
&\omega_2 = \frac{-\tau\omega_3\omega_1^2 - 2\tau\omega_7\omega_9\omega_1 + \tau\omega_3\omega_7^2 - \omega_3^2\omega_1^3 - \omega_9^2\omega_1^3 + 3\omega_3^2\omega_7^2\omega_1 + 3\omega_7^2\omega_9^2\omega_1}{\omega_3^2 + \omega_9^2},\\
&\omega_9 = \pm\frac{\sqrt{(\omega_7^2 - \omega_4^2)\omega_3^2 + \omega_6^2(\omega_1^2 + \omega_7^2)}}{\sqrt{\omega_1^2 + \omega_4^2}}.
\end{aligned}
\tag{7}
$$

Case V:

$$
\begin{aligned}
&\vartheta_1 = \frac{\omega_1^2\vartheta_2^2 - \omega_1^2\vartheta_3^2 - 2\omega_7^2\vartheta_3^2}{4\omega_1^2},\ \omega_8 = \frac{\omega_1^2(\tau - 3\omega_7\omega_9) + \omega_7^2(\omega_7\omega_9 - \tau)}{\omega_9},\\
&\omega_5 = \frac{\omega_1^2(\tau - 3\omega_4\omega_6) + \omega_4^2(\omega_4\omega_6 - \tau)}{\omega_6},\ \omega_9 = \frac{2\tau\omega_6\omega_7}{2\tau\omega_4 - 3\omega_6\omega_4^2 + 3\omega_6\omega_7^2},\\
&\omega_6 = \pm\frac{2\tau\omega_1}{3\sqrt{\omega_7^4 + \omega_1^2\omega_7^2}},\ \omega_2 = \omega_1\left(\omega_4\left(3\omega_4 - \frac{2\tau}{\omega_6}\right) - \omega_1^2\right),\ \omega_3 = \omega_4 = 0.
\end{aligned}
\tag{8}
$$

Case VI:

$$
\begin{aligned}
&\vartheta_1 = \frac{\vartheta_2^2(\tau\omega_4^2 - 4\omega_6\omega_4^3 + \omega_5\omega_6) + \vartheta_3^2(\tau\omega_7^2 - 4\omega_9\omega_7^3 + \omega_8\omega_9)}{4\tau\omega_1^2 + 4(4\omega_1^3 + \omega_2)\omega_3},\\
&\omega_8 = \frac{\omega_1^2(\tau - 3\omega_7\omega_9) + \omega_7^2(\omega_7\omega_9 - \tau)}{\omega_9},\ \omega_5 = \frac{\omega_1^2(\tau - 3\omega_4\omega_6) + \omega_4^2(\omega_4\omega_6 - \tau)}{\omega_6},\\
&\omega_9 = \frac{2\tau\omega_6\omega_7}{2\tau\omega_4 - 3\omega_6\omega_4^2 + 3\omega_6\omega_7^2},\ \omega_2 = \omega_1\left(\omega_4\left(3\omega_4 - \frac{2\tau}{\omega_6}\right) - \omega_1^2\right),\ \omega_3 = 0,\\
&\omega_4 = \frac{2\tau\omega_6 \pm \frac{\sqrt{\omega_6^2\omega_7^2(\omega_1^2 + \omega_7^2)(4\tau^2 + 9\omega_6^2(\omega_1^2 + \omega_7^2))}}{\omega_1^2 + \omega_7^2}}{3\omega_6^2}.
\end{aligned}
\tag{9}
$$

By substituting Case I–Case VI into Eqs (2) and (4), respectively, the corresponding breather wave solutions of Eq. (1) can be obtained. In order to understand the dynamic properties of the breather wave solutions, we take the following solution corresponding to Case II as an example:

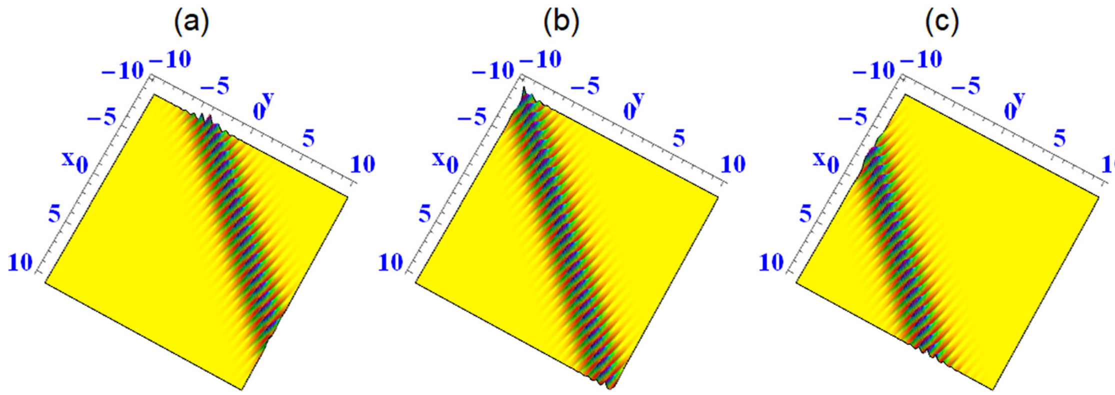

Figure 1: Solution (10) with $\omega_1 = \vartheta_3 = -1$, $\vartheta_1 = \omega_7 = 2$, $\omega_4 = 1$, $\omega_3 = 3$, $\vartheta_2 = 0$. (a) $t = -2$, (b) $t = 0$, and (c) $t = 2$.

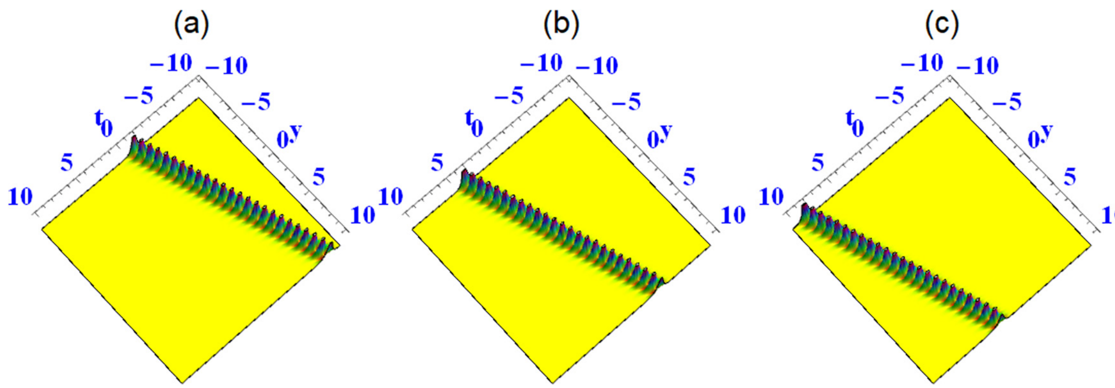

Figure 2: Solution (10) with $\omega_1 = \vartheta_3 = -1$, $\vartheta_1 = \omega_7 = 2$, $\omega_4 = 1$, $\omega_3 = 3$, $\vartheta_2 = 0$. (a) $x = -15$, (b) $x = 0$, and (c) $x = 15$.

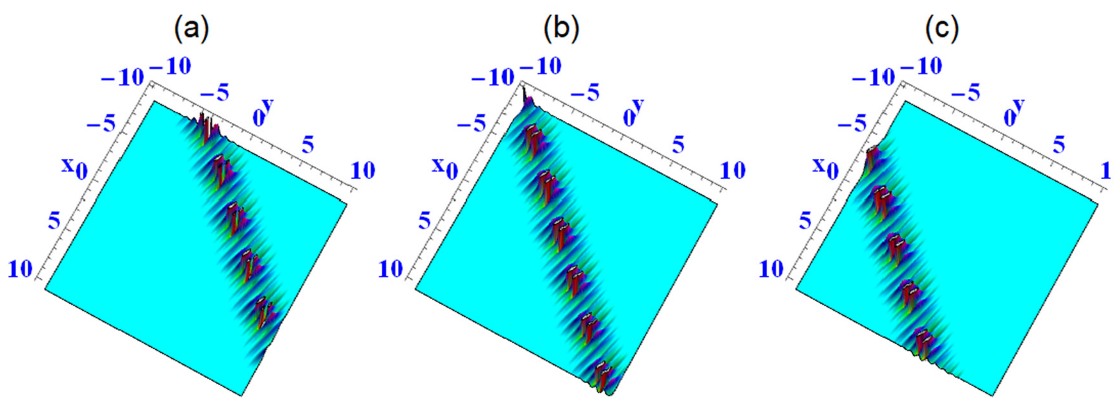

Figure 3: Solution (10) with $\omega_1 = \vartheta_3 = -1$, $\vartheta_1 = \omega_7 = \vartheta_2 = 2$, $\omega_4 = 1$, $\omega_3 = 3$. (a) $t = -2$, (b) $t = 0$, and (c) $t = 2$.

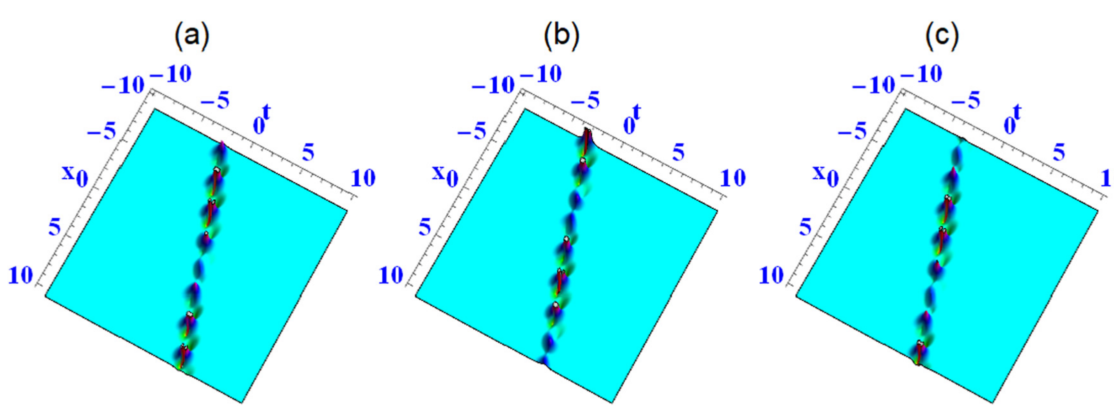

Figure 4: Solution (10) with $\omega_1 = \vartheta_3 = -1$, $\vartheta_1 = \omega_7 = \vartheta_2 = 2$, $\omega_4 = 1$, $\omega_3 = 3$. (a) $y = -2$, (b) $y = 0$, and (c) $y = 2$.

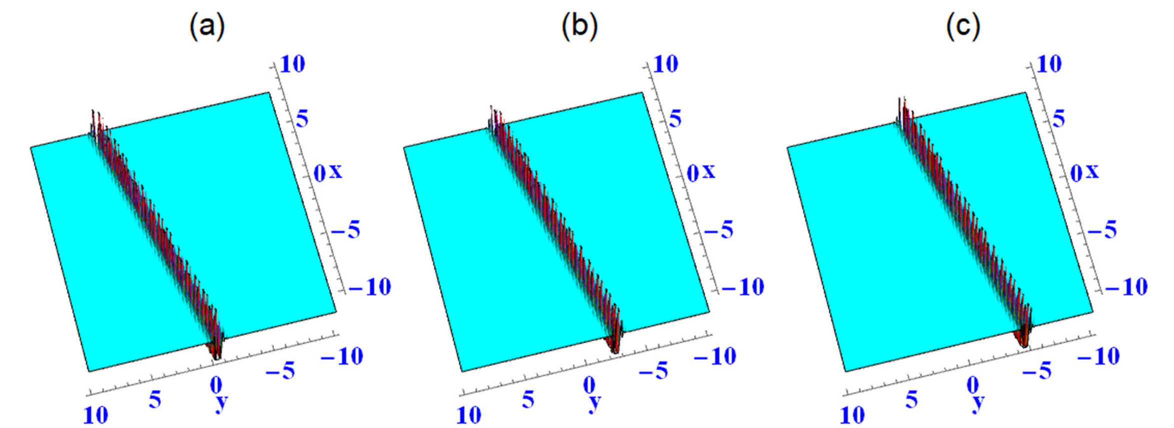

Figure 5: Solution (13) with $\beta_1 = k_1 = -1$, $\tau = \alpha_4 = \delta_4 = \delta_1 = k_2 = 1$, $\alpha_2 = \alpha_3 = 3$, $\alpha_1 = 2$, $\gamma_1 = -2$, $\gamma_3 = 0$. (a) , (b) $t = 0$, and (c) $t = 100$.

$$\begin{aligned}
u = \Bigg[2\Bigg[\Bigg[\Big[& e^{-t\omega_3 - x\omega_1 + y\omega_1^3} + \vartheta_1 e^{t\omega_3 + x\omega_1 - y\omega_1^3} \\
& - \vartheta_2 \sin\left[\frac{t\omega_1\omega_3}{\omega_4} - \omega_4(x + y\omega_4^2)\right] \\
& + \vartheta_3 \cos\left[\frac{t\omega_1\omega_3}{\omega_7} - \omega_7(x + y\omega_7^2)\right]\Big]\Bigg]\Big[\omega_1^2(e^{-t\omega_3 - x\omega_1 + y\omega_1^3} + \vartheta_1 e^{t\omega_3 + x\omega_1 - y\omega_1^3}) \\
& + \omega_4^2\vartheta_2 \sin\left[\frac{t\omega_1\omega_3}{\omega_4} - \omega_4(x + y\omega_4^2)\right] \\
& - \omega_7^2\vartheta_3 \cos\left[\frac{t\omega_1\omega_3}{\omega_7} - \omega_7(x + y\omega_7^2)\right]\Big]\Bigg] \qquad (10) \\
& - \Big[\omega_1(\vartheta_1 e^{t\omega_3 + x\omega_1 - y\omega_1^3} - e^{-t\omega_3 - x\omega_1 + y\omega_1^3}) \\
& + \omega_7\vartheta_3 \sin\left[\frac{t\omega_1\omega_3}{\omega_7} - \omega_7(x + y\omega_7^2)\right] \\
& + \omega_4\vartheta_2 \cos\left[\frac{t\omega_1\omega_3}{\omega_4} - \omega_4(x + y\omega_4^2)\right]\Big]^2\Bigg]\Bigg] \\
& /\Big[\Big[e^{-t\omega_3 - x\omega_1 + y\omega_1^3} + \vartheta_1 e^{t\omega_3 + x\omega_1 - y\omega_1^3} \\
& - \vartheta_2 \sin\left[\frac{t\omega_1\omega_3}{\omega_4} - \omega_4(x + y\omega_4^2)\right] \quad . \\
& + \vartheta_3 \cos\left[\frac{t\omega_1\omega_3}{\omega_7} - \omega_7(x + y\omega_7^2)\right]\Big]\Big]^2
\end{aligned}$$

When $\vartheta_3 = 0$, the breather wave solution (10) has been studied in ref. [8]. When $\vartheta_3 \neq 0$, Eq. (10) has not been seen in other literature. The corresponding dynamic properties are shown in Figures 1–4 by selecting different values for parameters in Eq. (10).

3 Double-periodic soliton solutions

In ref. [13], a new ansätz function was proposed to construct double-periodic soliton structures. Subsequently, some important conclusions of nonlinear evolution equation were obtained by the new ansätz function [14–16]. According to the idea of refs [12–14], the solutions of Eq. (3) can be obtained as follows:

$$\begin{aligned}
\xi = {} & e^{\theta_1}[\gamma_1 \cos(\theta_2) + \gamma_2 \sin(\theta_2)] + k_1 e^{2\theta_1} \\
& + e^{\theta_3}[\gamma_3 \cos(\theta_4) + \gamma_4 \sin(\theta_4)] + k_2 e^{\theta_4},
\end{aligned} \qquad (11)$$

where $\theta_i = \alpha_i\, x + \beta_i\, y + \delta_i\, t$, $i = 1, 2, 3, 4$ and α_i, β_i, and δ_i are unknown constants. Substituting Eq. (11) into Eq. (2) yields the double periodic soliton solutions of Eq. (1). Substituting Eq. (11) into Eq. (3), we obtain

Case (1)

$$\begin{aligned}
& \tau = \gamma_3 = \delta_2 = \gamma_4 = \beta_4 = 0,\ \delta_1 = \delta_4, \\
& \gamma_2 = \frac{6\alpha_2\alpha_1\gamma_1}{-3\alpha_1^2 + 7\alpha_4^2 + 3(\alpha_2^2 + \alpha_3^2)}, \\
& \beta_2 = \alpha_2(\alpha_2^2 - 3\alpha_1^2), \quad \beta_1 = -\alpha_1^3 + 3\alpha_2^2\alpha_1.
\end{aligned} \qquad (12)$$

Case (2)

$$\begin{aligned}
& \alpha_4 = \beta_4 = \delta_4 = \gamma_4 = 0,\ \delta_2 = -\frac{\gamma_2\tau(\alpha_2\gamma_1 - \alpha_1\gamma_2)^2}{3\alpha_1(\alpha_1^2 + \alpha_2^2)\gamma_1(\gamma_1^2 + \gamma_2^2)}, \\
& \delta_3 = -\frac{\alpha_3\tau}{3\alpha_1(\alpha_3 - \alpha_1)}, \quad \gamma_2 = -\frac{\alpha_1\gamma_1}{\alpha_2}, \\
& \delta_1 = -\frac{\alpha_1^2\tau}{4\alpha_1^3 + \beta_1 - \beta_4}, \quad \beta_2 = \alpha_2^3, \\
& \beta_3 = -\alpha_3(3\alpha_1^2 - 3\alpha_3\alpha_1 + \alpha_3^2), \quad \beta_1 = -\alpha_1^3.
\end{aligned} \qquad (13)$$

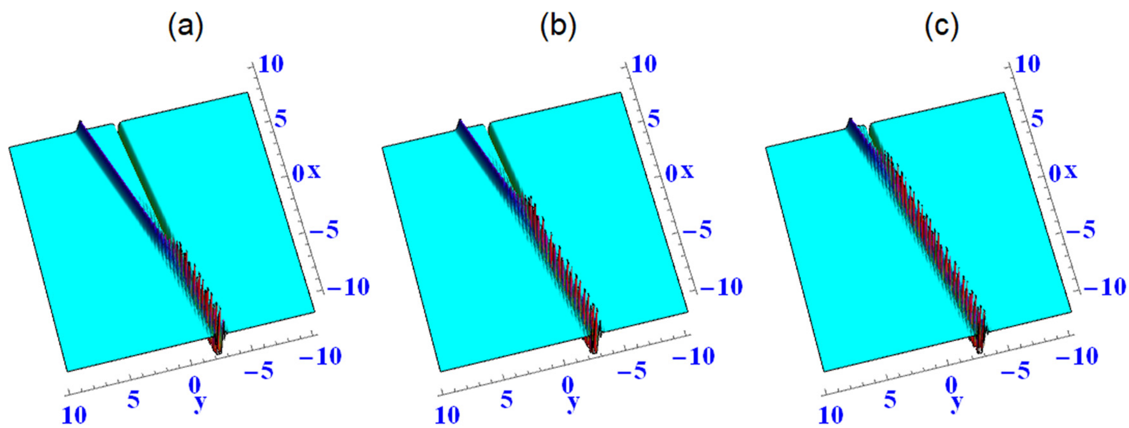

Figure 6: Solution (13) with $\beta_1 = k_1 = -1$, $\tau = \alpha_4 = \delta_4 = \delta_1 = k_2 = 1$, $\alpha_2 = \alpha_3 = 3$, $\alpha_1 = 2$, $\gamma_1 = -2$, $\gamma_3 = 3$. (a) $t = -10$, (b) $t = 0$, and (c) $t = 10$.

Case (3)

$$
\begin{aligned}
&k_1 = k_2 = \gamma_1 = \gamma_3 = 0,\ \beta_4 = \frac{4\alpha_4^3\delta_4 - \alpha_4^2\tau}{\delta_4},\\
&\beta_2 = \frac{4\alpha_2^3\delta_2 - \alpha_2^2\tau}{\delta_2},\quad \delta_2 = \frac{\alpha_2\delta_4}{\alpha_4},\\
&\delta_4 = \frac{4(\alpha_1-\alpha_3)^2\alpha_4\tau}{3(\alpha_4^4 + 2((\alpha_1-\alpha_3)^2 - \alpha_2^2)\alpha_4^2 + (\alpha_2^2 + (\alpha_1-\alpha_3)^2)^2)},\\
&\delta_3 = \frac{4(\alpha_1-\alpha_3)^3\tau}{3(\alpha_4^4 + 2((\alpha_1-\alpha_3)^2 - \alpha_2^2)\alpha_4^2 + (\alpha_2^2 + (\alpha_1-\alpha_3)^2)^2)} + \delta_1,\\
&\beta_3 = [4\alpha_1[\alpha_3(3\alpha_2^2 - \alpha_3^2 + 3\alpha_4^2) + \beta_1]\\
&\quad - 4\alpha_3\beta_1 + \alpha_1^4 - 4\alpha_3\alpha_1^3 - 6(\alpha_2^2 - \alpha_3^2 + \alpha_4^2)\alpha_1^2\\
&\quad + 9\alpha_2^4 + (\alpha_3^2 - 3\alpha_4^2)^2 - 6\alpha_2^2(\alpha_3^2 + 3\alpha_4^2)]/[4(\alpha_1-\alpha_3)].
\end{aligned}
\tag{14}
$$

Since the formula is too long, see Appendix A for other solutions. By substituting Case (1)–Case (9) into equations (2) and (11), respectively, the corresponding double periodic soliton solutions of Eq. (1) can be derived. In order to understand the dynamic properties of the double periodic soliton solutions, we take the solution corresponding to Case (2) as an example. By selecting different values for parameters in the solution corresponding to Case (2), the dynamic properties are described in Figures 5 and 6.

4 Conclusion

In this article, the (2+1)-dimensional generalized HSI equation is studied, which is used for describing the propagation of unidirectional shallow water waves. By the three-wave method, the breather wave solutions for Eq. (1) are presented. By an undetermined coefficient method, we obtain abundant double-periodic soliton solutions, which have not been seen in other literature. The dynamic properties for these derived results are demonstrated in Figures 1–6. In Figures 1 and 2, we can observe a periodic breather wave with the change of t and x values. In Figures 3 and 4, the interaction of breather waves formed by two different periodic functions is described. In Figures 5 and 6, the double-periodic soliton structures can be found. Obviously, it is easy to obtain double periodic soliton solutions of nonlinear integrable equations using this method of undetermined coefficients. Generally speaking, if a nonlinear integrable equation has a Hirota bilinear form [17–19], the undetermined coefficient method can be used to obtain the double periodic soliton solutions of the equation *via* symbolic computation [20–25].

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Conflict of interest: The authors state no conflict of interest.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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Appendix

Case (4)

$$\begin{aligned}
&k_1 = k_2 = \gamma_2 = \gamma_3 = 0, \beta_4 = \frac{4\alpha_4^3\delta_4 - \alpha_4^2\tau}{\delta_4},\\
&\beta_2 = \frac{4\alpha_2^3\delta_2 - \alpha_2^2\tau}{\delta_2},\\
&\delta_3 = [-\alpha_4^2\delta_2^2\tau + \alpha_4\delta_4\delta_2[2\alpha_2[3(\alpha_1 - \alpha_3)\delta_1 + \tau]\\
&\quad + 3[-\alpha_2^2 + (\alpha_1 - \alpha_3)^2 + \alpha_4^2]\delta_2]\\
&\quad + \alpha_2\delta_4^2[3[\alpha_2^2 + (\alpha_1 - \alpha_3)^2 - \alpha_4^2]\delta_2 - \alpha_2\tau]]/[6\alpha_2(\alpha_1\\
&\quad - \alpha_3)\alpha_4\delta_2\delta_4],\\
&\beta_3 = [\alpha_4^2\delta_2^2\tau[(\alpha_3 - \alpha_1)[(\alpha_1 - \alpha_3)^2 - 3(\alpha_2^2 + \alpha_4^2)] - \beta_1]\\
&\quad + \alpha_4\delta_4\delta_2[3\delta_2[[-\alpha_2^2 + (\alpha_1 - \alpha_3)^2 + \alpha_4^2]\beta_1\\
&\quad + (\alpha_1 - \alpha_3)[\alpha_1^4 - 4\alpha_3\alpha_1^3 + 2(\alpha_2^2 + 3\alpha_3^2 - \alpha_4^2)\alpha_1^2\\
&\quad - 4\alpha_3(\alpha_2^2 + \alpha_3^2 - \alpha_4^2)\alpha_1 + 9\alpha_2^4 + \alpha_3^4 - 3\alpha_4^4\\
&\quad + 2\alpha_2^2\alpha_3^2 - 2(3\alpha_2^2 + \alpha_3^2)\alpha_4^2]]\\
&\quad + 2\alpha_2\tau[(\alpha_3 - \alpha_1)[2(\alpha_1 - \alpha_3)^2 + 3(\alpha_2^2 + \alpha_4^2)] + \beta_1]]\\
&\quad + \alpha_2\delta_4^2[3\delta_2[[\alpha_2^2 + (\alpha_1 - \alpha_3)^2 - \alpha_4^2]\beta_1\\
&\quad + (\alpha_1 - \alpha_3)[\alpha_1^4 - 4\alpha_3\alpha_1^3 + 2(-\alpha_2^2 + 3\alpha_3^2 + \alpha_4^2)\alpha_1^2\\
&\quad - 4\alpha_3(-\alpha_2^2 + \alpha_3^2 + \alpha_4^2)\alpha_1 - 3\alpha_2^4 + \alpha_3^4 + 9\alpha_4^4\\
&\quad + 2\alpha_3^2\alpha_4^2 - 2\alpha_2^2(\alpha_3^2 + 3\alpha_4^2)]] + \alpha_2\tau\\
&\quad \times [(\alpha_3 - \alpha_1)[(\alpha_1 - \alpha_3)^2 - 3(\alpha_2^2 + \alpha_4^2)] - \beta_1]]]/[-\alpha_4^2\delta_2^2\tau\\
&\quad + \alpha_4\delta_4\delta_2[3[-\alpha_2^2 + (\alpha_1 - \alpha_3)^2 + \alpha_4^2]\delta_2 + 2\alpha_2\tau]\\
&\quad + \alpha_2\delta_4^2[3[\alpha_2^2 + (\alpha_1 - \alpha_3)^2 - \alpha_4^2]\delta_2 - \alpha_2\tau]],\\
&\delta_2 = (\alpha_2^2\delta_4^2\tau^2)/[\pm 2\sqrt{3}\sqrt{\alpha_2^2(\alpha_1 - \alpha_3)^2\alpha_4\delta_4^3\tau^2(\tau - 3\alpha_4\delta_4)}\\
&\quad + \alpha_2\delta_4\tau[3[\alpha_2^2 + (\alpha_1 - \alpha_3)^2 - \alpha_4^2]\delta_4 + \alpha_4\tau]].
\end{aligned}\tag{A1}$$

Case (5)

$$\begin{aligned}
&k_1 = k_2 = \gamma_2 = \gamma_4 = 0, \quad \beta_4 = \frac{4\alpha_4^3\delta_4 - \alpha_4^2\tau}{\delta_4}, \quad \beta_2 = \frac{4\alpha_2^3\delta_2 - \alpha_2^2\tau}{\delta_2},\\
&\delta_3 = \frac{4(\alpha_1 - \alpha_3)^3\tau}{3(\alpha_4^4 + 2((\alpha_1 - \alpha_3)^2 - \alpha_2^2)\alpha_4^2 + (\alpha_2^2 + (\alpha_1 - \alpha_3)^2)^2)} + \delta_1,\\
&\beta_3 = [4\alpha_1[\alpha_3(3\alpha_2^2 - \alpha_3^2 + 3\alpha_4^2) + \beta_1] - 4\alpha_3\beta_1 + \alpha_1^4 - 4\alpha_3\alpha_1^3\\
&\quad - 6(\alpha_2^2 - \alpha_3^2 + \alpha_4^2)\alpha_1^2 + 9\alpha_2^4 + (\alpha_3^2 - 3\alpha_4^2)^2\\
&\quad - 6\alpha_2^2(\alpha_3^2 + 3\alpha_4^2)]/[4(\alpha_1 - \alpha_3)],\\
&\delta_2 = \frac{\alpha_2\delta_4}{\alpha_4}, \; \delta_4 = [4(\alpha_1 - \alpha_3)^2\alpha_4\tau]/[3(\alpha_1^2 - 2\alpha_3\alpha_1\\
&\quad + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 - 2\alpha_2\alpha_4)(\alpha_1^2 - 2\alpha_3\alpha_1 + \alpha_2^2\\
&\quad + \alpha_3^2 + \alpha_4^2 + 2\alpha_2\alpha_4)].
\end{aligned}\tag{A2}$$

Case (6)

$$\begin{aligned}
&k_1 = k_2 = \gamma_1 = \gamma_2 = 0, \; \beta_4 = \frac{4\alpha_4^3\delta_4 - \alpha_4^2\tau}{\delta_4},\\
&\beta_2 = \frac{4\alpha_2^3\delta_2 - \alpha_2^2\tau}{\delta_2}.
\end{aligned}\tag{A3}$$

Case (7)

$$\begin{aligned}
&k_1 = \gamma_3 = \gamma_3 = 0, \; \beta_4 = \frac{4\alpha_4^3\delta_4 - \alpha_4^2\tau}{\delta_4}, \beta_2 = \frac{4\alpha_2^3\delta_2 - \alpha_2^2\tau}{\delta_2},\\
&\tau = \frac{3\alpha_2(\alpha_2^2 + (\alpha_1 - 2\alpha_4)^2)\delta_2(\delta_2^2 + (\delta_1 - 2\delta_4)^2)}{((\alpha_1 - 2\alpha_4)\delta_2 - \alpha_2(\delta_1 - 2\delta_4))^2},\\
&\beta_1 = [-2\alpha_4^2\delta_2\tau(\gamma_1\delta_1 + \gamma_2\delta_2) + 2\delta_4^2[\delta_2[[\alpha_1^3 - 6\alpha_4\alpha_1^2\\
&\quad - 3(\alpha_2^2 - 4\alpha_4^2)\alpha_1 - 16\alpha_4^3 + 6\alpha_2^2\alpha_4]\gamma_1\\
&\quad + 3\alpha_2[\alpha_2^2 + (\alpha_1 - 2\alpha_4)^2]\gamma_2] - \alpha_2^2\gamma_2\tau] + \delta_4[\alpha_2^2[\gamma_2[3(\alpha_1 - 2\alpha_4)\delta_2^2 + \delta_1\tau]\\
&\quad + 3(\alpha_1 - 2\alpha_4)\gamma_1\delta_1\delta_2] + (\alpha_1 - 2\alpha_4)\alpha_2\delta_2[\gamma_2[-3(\alpha_1 - 2\alpha_4)\delta_1 - 2\tau]\\
&\quad + 3(\alpha_1 - 2\alpha_4)\gamma_1\delta_2] + (\alpha_1 - 4\alpha_4)\delta_2[\gamma_1(\alpha_1(-\tau) - (\alpha_1^2 - 2\alpha_4\alpha_1 + 4\alpha_4^2)\delta_1)\\
&\quad - (\alpha_1^2 - 2\alpha_4\alpha_1 + 4\alpha_4^2)\gamma_2\delta_2] + 3\alpha_2^3\delta_2(\gamma_1\delta_2 - \gamma_2\delta_1)]]/\\
&\quad [\delta_2\delta_4[\gamma_2\delta_2 + \gamma_1(\delta_1 - 2\delta_4)]].
\end{aligned}\tag{A4}$$

Case (8)

$$\begin{aligned}
&k_1 = k_2 = \gamma_3 = \gamma_4 = 0, \; \beta_4 = \frac{4\alpha_4^3\delta_4 - \alpha_4^2\tau}{\delta_4},\\
&\beta_2 = \frac{4\alpha_2^3\delta_2 - \alpha_2^2\tau}{\delta_2}.
\end{aligned}\tag{A5}$$

Case (9)

$$\begin{aligned}
&k_1 = k_2 = \gamma_1 = \gamma_4 = 0, \; \beta_4 = \frac{4\alpha_4^3\delta_4 - \alpha_4^2\tau}{\delta_4},\\
&\quad \beta_2 = \frac{4\alpha_2^3\delta_2 - \alpha_2^2\tau}{\delta_2},\\
&\delta_3 = [\alpha_4^2\delta_2^2(-\tau) + \alpha_4\delta_4\delta_2[2\alpha_2[3(\alpha_1 - \alpha_3)\delta_1 + \tau]\\
&\quad + 3[-\alpha_2^2 + (\alpha_1 - \alpha_3)^2\\
&\quad + \alpha_4^2]\delta_2] + \alpha_2\delta_4^2[3[\alpha_2^2 + (\alpha_1 - \alpha_3)^2 - \alpha_4^2]\delta_2\\
&\quad - \alpha_2\tau]]/[6\alpha_2(\alpha_1 - \alpha_3)\alpha_4\delta_2\delta_4],\\
&\beta_3 = [\alpha_2(\alpha_1 - \alpha_3)\delta_4^2\tau[(\alpha_1 - \alpha_3)[(\alpha_1 - \alpha_3)^2\\
&\quad - 3(\alpha_2^2 + \alpha_4^2)] + \beta_1]\\
&\quad + \alpha_2(\alpha_1 - \alpha_3)^2\alpha_4\delta_4\tau^2\\
&\quad - 2\sqrt{3}\,\alpha_3\alpha_1\sqrt{\alpha_2^2(\alpha_1 - \alpha_3)^2\alpha_4\delta_4^3\tau^2(\tau - 3\alpha_4\delta_4)}\\
&\quad + \sqrt{3}\,\alpha_1^2\sqrt{\alpha_2^2(\alpha_1 - \alpha_3)^2\alpha_4\delta_4^3\tau^2(\tau - 3\alpha_4\delta_4)}\\
&\quad - \sqrt{3}\,\alpha_2^2\sqrt{\alpha_2^2(\alpha_1 - \alpha_3)^2\alpha_4\delta_4^3\tau^2(\tau - 3\alpha_4\delta_4)}\\
&\quad + \sqrt{3}\,\alpha_4^2\sqrt{\alpha_2^2(\alpha_1 - \alpha_3)^2\alpha_4\delta_4^3\tau^2(\tau - 3\alpha_4\delta_4)}\\
&\quad + \sqrt{3}\,\alpha_3^2\sqrt{\alpha_2^2(\alpha_3 - \alpha_1)^2\alpha_4\delta_4^3\tau^2(\tau - 3\alpha_4\delta_4)}\,]/[\alpha_2(\alpha_1\\
&\quad - \alpha_3)\delta_4^2\tau],\\
&\delta_2 = [\alpha_2^2\delta_4^2\tau^2]/[2\sqrt{3}\sqrt{\alpha_2^2(\alpha_1 - \alpha_3)^2\alpha_4\delta_4^3\tau^2(\tau - 3\alpha_4\delta_4)}\\
&\quad + \alpha_2\delta_4\tau[3[\alpha_2^2 + (\alpha_1 - \alpha_3)^2 - \alpha_4^2]\delta_4 + \alpha_4\tau]],\\
&\delta_4 = \frac{4(\alpha_1 - \alpha_3)^2\alpha_4\tau}{3(\alpha_4^4 + 2((\alpha_1 - \alpha_3)^2 - \alpha_2^2)\alpha_4^2 + (\alpha_2^2 + (\alpha_1 - \alpha_3)^2)^2)}.
\end{aligned}\tag{A6}$$