Research Article

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Abundant optical soliton structures to the Fokas system arising in monomode optical fibers

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Abstract: Three effective methods, namely, the simplified extended tanh-function method (SETFM), variational method (VM) and He's frequency formulation method (HFFM) are employed to investigate the Fokas system that arises in the monomode optical fibers. Fifteen sets of the soliton solutions such as the bright soliton, dark soliton, bright-dark soliton, double-dark soliton, double-bright soliton, triple-bright soliton, kinky periodic soliton and perfect periodic soliton solutions are developed. The dynamic performances of the different soliton solutions are plotted *via* the 3-D contours and 2-D curves to interpret the physical behaviors by assigning reasonable parameters. From the results obtained from this study, it is found that three proposed methods are promising ways to seek various soliton solutions of the PDEs in optical physics.

Keywords: abundant soliton solutions, simplified extended tanh-function method, He's frequency formulation method, variational method, Fokas system

1 Introduction

The study of the nonlinear partial differential equations (NPDEs) is important since it can help us understand the complex phenomena occurring in optics [1–5], hydrodynamics [6–10], vibration [11–14], plasma physics [15,16] and so on [17–24]. And different effective methods, such as exp-function method [25–28], variational method (VM) [29–31], ancient Chinese algorithm [32], extended rational sine-cosine and sinh-cosh methods [33,34], are used to study the PDEs and many research results have been obtained. Under the current study, we consider the Fokas system which reads as [35]:

$$\begin{cases} i\varphi_t + \mu_1\varphi_{xx} + \mu_2\varphi\phi = 0, \\ \mu_3\phi_y - \mu_4(|\varphi|^2)_x = 0. \end{cases} \tag{1.1}$$

In the above equation, ϕ and φ are the complex functions which indicate the nonlinear pulse propagation in monomode optical fibers, and μ_1, μ_2, μ_3 and μ_4 are arbitrary constants. As for Eq. (1.1), different results are made by many outstanding scientific researchers. In ref. [36], the lump-soliton solutions were established by applying the Hirota's bilinear method. The exp-function based method was employed to seek the soliton solutions in ref. [37]. In ref. [38], the Jacobi elliptic function expansion method is employed to solve Eq. (1.1). In ref. [39], the extended rational sine-cosine and sinh-cosh methods were used to study Eq. (1.1). The simplified extended tanh-function method (SETFM) is a powerful tool to construct the soliton solutions, but it has not been used to solve Eq. (1.1). Recently, the VM and He's frequency formulation method (HFFM) have attracted wide attention and are used widely to develop the soliton solutions since they are simple and effective, which can construct the solutions *via* one or two step. Encouraged by the recent research results, the main purpose of this work is to make use of the SETFM and VM to develop abundant soliton solutions for Eq. (1.1). We arrange the structure of this article as follows: in Section 2, a brief introduction of the three methods is presented. In Section 3, we use the proposed methods to construct the different soliton solutions. In Section 4, we plot the behaviors of the results in the form of 3-D contours and 2-D curves, and discuss their physical explanations. In Section 5, a conclusion is made.

2 The three methods

The considered PDE is given as:

$$f(q, q_x, q_y, q_t, q_{xy}, q_{xt}, q_{yt}, \ldots) = 0. \tag{2.1}$$

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The traveling wave transformation is introduced as:

$$q(x, y, t) = \psi(\xi), \xi = \omega x + my + kt, \tag{2.2}$$

where ω, m and k are arbitrary nonzero constants.

Applying Eq. (2.2) in Eq. (2.1), we can get an ordinary differential equation (ODE) as:

$$f(\psi, \psi_\xi, \psi_{\xi\xi}, \ldots, \ldots) = 0. \tag{2.3}$$

2.1 The SETFM

The solution of Eq. (2.3) is supposed as [40–42]:

$$\psi(\xi) = \sum_{i=-m}^{m} \sigma_i \gamma^i, \tag{2.4}$$

where σ_i is an undetermined parameter. There is:

$$\gamma = \tanh(\xi). \tag{2.5}$$

With

$$\gamma' = 1 - \gamma^2. \tag{2.6}$$

Substituting Eq. (2.4) with Eq. (2.6) into Eq. (2.3) and using the balance principle, the value of m can be determined. With the obtained m, bringing Eq. (2.4) into Eq. (2.3) and merging the same power of γ, we have:

$$\sum \varepsilon_k \gamma^k = 0. \tag{2.7}$$

Setting $\varepsilon_k = 0$ leads to a system of equations. Solving the system, the coefficients σ_i can be obtained.

2.2 The VM

Adopting the semi-inverse method [43–45], the variational principle of Eq. (2.3) can be established as:

$$J(\psi) = \int \ell(\psi, \psi', \psi'', \ldots)\, d\xi. \tag{2.8}$$

Its solutions can be assumed as:

$$\text{Type-1: } \psi(\xi) = \Xi_1 \operatorname{sech}(\xi), \tag{2.9}$$

$$\text{Type-2: } \psi(\xi) = \frac{\Xi_2}{1 + \cosh(\xi)}, \tag{2.10}$$

$$\text{Type-3: } \psi(\xi) = \frac{\Xi_3 \sinh(\xi)}{[\cosh(\xi)]^{\frac{3}{2}}}. \tag{2.11}$$

Taking them into Eq. (2.8), respectively, and applying the stationary conditions [46,47]:

$$\text{condition 1: } \frac{dJ}{d\Xi_1} = 0, \tag{2.12}$$

$$\text{condition 2: } \frac{dJ}{d\Xi_2} = 0, \tag{2.13}$$

$$\text{condition 3: } \frac{dJ}{d\Xi_3} = 0. \tag{2.14}$$

By above conditions, we can easily determine Ξ_1, Ξ_2 and Ξ_3, respectively.

2.3 The HFFM

To study the Eq. (2.3):

$$\psi'' + f(\psi) = 0, \tag{2.15}$$

where $f(\psi)$ is a function about ψ and there is $\psi'' = \frac{d^2\psi}{d\xi^2}$. We assume Eq. (2.15) has the periodic solution as:

$$\psi(\xi) = \Theta \cos(\varpi \xi). \tag{2.16}$$

We can attain the amplitude-frequency relationship by one step *via* the HFFM [48–50] as:

$$\varpi = \sqrt{\frac{df}{d\psi}}\Bigg|_{\psi = \frac{\Theta}{2}}. \tag{2.17}$$

With Eq. (2.17), the period soliton solution of Eq. (1.1) can be easily gained.

3 Applications

Using the following transformation:

$$\begin{aligned} \varphi(x, y, t) &= p(\xi) e^{i(\varpi_1 x + \varpi_2 y + \varpi_3 t)}, \\ \phi(x, y, t) &= q(\xi),\ \xi = x + y - \chi t. \end{aligned} \tag{3.1}$$

where ω_1, ω_2, ω_3 and χ are non-zero arbitrary constants. Substituting Eq. (3.1) into Eq. (1.1), Eq. (1.1) can be decomposed into:

$$\begin{cases} [\mu_1 p'' - ip'(\chi - 2\omega_1\mu_1) - (\omega_1^2\mu_1 - \mu_2 q \\ \quad + \varpi_3)p] e^{i(\varpi_1 x + \varpi_2 y + \varpi_3 t)} = 0, \\ \mu_3 q' - 2\mu_4 pp' = 0. \end{cases} \tag{3.2}$$

Integrating the second equation of Eq. (3.2) with respect to ξ once and ignoring the integral constant yields:

$$q = \frac{\mu_4 p^2}{\mu_3}. \tag{3.3}$$

Taking Eq. (3.3) into the first equation of Eq. (3.2), the real and imaginary parts are obtained, respectively, as:

$$\mu_1 p'' - (\varpi_1^2\mu_1 + \varpi_3)p + \frac{\mu_2\mu_4}{\mu_3}p^3 = 0, \tag{3.4}$$

$$p'(\chi - 2\varpi_1\mu_1) = 0. \tag{3.5}$$

Setting that:

$$\chi = 2\varpi_1\mu_1. \tag{3.6}$$

Then, Eq. (3.5) vanishes, next we aim to consider the Fokas system as:

$$\mu_1 p'' - (\varpi_1^2\mu_1 + \varpi_3)p + \frac{\mu_2\mu_4}{\mu_3}p^3 = 0. \tag{3.7}$$

3.1 Application of the SETFM

By the SETFM, the solution of Eq. (3.7) can be written as:

$$p(\xi) = \sum_{i=-m}^{m} \sigma_i \gamma^i. \tag{3.8}$$

Taking Eq. (3.8) into Eq. (3.7) and applying the balance theory, we have:

$$m + 2 = 3m. \tag{3.9}$$

It gives:

$$m = 1. \tag{3.10}$$

Then, Eq. (3.8) reduces to:

$$p(\xi) = \sigma_1\gamma + \sigma_0 + \frac{\sigma_{-1}}{\gamma}. \tag{3.11}$$

Substituting above equation into Eq. (3.7), merging the same power of γ and setting the coefficients as zero, it yields:

$$\gamma^{-2}:\ 3\sigma_{-1}^2\sigma_0\mu_2\mu_4 = 0,$$

$$\gamma^{-1}:\ -2\sigma_{-1}\mu_1\mu_3 + 3\sigma_{-1}\sigma_0^2\mu_2\mu_4 + 3\sigma_{-1}^2\sigma_1\mu_2\mu_4 - \sigma_{-1}\mu_1\mu_3\varpi_1^2 - \sigma_{-1}\mu_3\varpi_3 = 0,$$

$$\gamma^{0}:\ \sigma_0^3\mu_2\mu_4 + 6\sigma_{-1}\sigma_0\sigma_1\mu_2\mu_4 - \sigma_0\mu_1\mu_3\varpi_1^2 - \sigma_0\mu_3\varpi_3 = 0,$$

$$\gamma^{1}:\ -2\sigma_1\mu_1\mu_3 + 3\sigma_0^2\sigma_1\mu_2\mu_4 + 3\sigma_{-1}\sigma_1^2\mu_2\mu_4 - \sigma_1\mu_1\mu_3\varpi_1^2 - \sigma_1\mu_3\varpi_3 = 0,$$

$$\gamma^{2}:\ 3\sigma_1^2\sigma_0\mu_2\mu_4 = 0,$$

$$\gamma^{3}:\ 2\sigma_1\mu_1\mu_3 + \sigma_1^3\mu_2\mu_4 = 0.$$

Solving them, we have:

Family 1:

$$\left\{\sigma_1 \to 0, \sigma_0 \to 0, \sigma_{-1} \to \pm i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}},\ \varpi_3 \to -2\mu_1 - \mu_1\varpi_1^2\right\}.$$

So the solution of Eq. (3.7) can be obtained as:

$$p(\xi) = \pm i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}}\coth(\xi). \tag{3.12}$$

In the view of Eqs. (3.1) and (3.3), the soliton solutions of Eq. (1.1) are attained as:

Set 1:

$$\begin{cases} \varphi_1(x,y,t) = i\sqrt{\dfrac{2\mu_1\mu_3}{\mu_2\mu_4}}\coth(x+y \\ \qquad - 2\varpi_1\mu_1 t)e^{i[\varpi_1 x + \varpi_2 y - \mu_1(2+\varpi_1^2)t]}, \\ \phi_1(x,y,t) = -\dfrac{2\mu_1}{\mu_2}\coth^2(x+y-2\varpi_1\mu_1 t). \end{cases} \tag{3.13}$$

Set 2:

$$\begin{cases} \varphi_2(x,y,t) = -i\sqrt{\dfrac{2\mu_1\mu_3}{\mu_2\mu_4}}\coth(x+y \\ \qquad - 2\varpi_1\mu_1 t)e^{i[\varpi_1 x + \varpi_2 y - \mu_1(2+\varpi_1^2)t]}, \\ \phi_2(x,y,t) = -\dfrac{2\mu_1}{\mu_2}\coth^2(x+y-2\varpi_1\mu_1 t). \end{cases} \tag{3.14}$$

Family 2:

$$\left\{\sigma_1 \to -i\sqrt{\frac{2\mu_3}{\mu_2\mu_4}}, \sigma_0 \to 0, \sigma_{-1} \to -i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}}, \varpi_3 \to -8\mu_1 - \mu_1\varpi_1^2\right\}.$$

The solution of Eq. (3.7) is:

$$p(\xi) = -i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}}[\tanh(\xi) + \coth(\xi)]. \tag{3.15}$$

The soliton solutions of Eq. (1.1) can be expressed as:

Set 3:

$$\begin{cases} \varphi_3(x,y,t) = -i\sqrt{\dfrac{2\mu_1\mu_3}{\mu_2\mu_4}}[\tanh(x+y-2\varpi_1\mu_1 t) \\ \qquad + \coth(x+y-2\varpi_1\mu_1 t)]e^{i[\varpi_1 x+\varpi_2 y-\mu_1(8+\varpi_1^2)t]}, \\ \phi_3(x,y,t) = -\dfrac{2\mu_1}{\mu_2}[\tanh(x+y-2\varpi_1\mu_1 t) + \coth(x+y-2\varpi_1\mu_1 t)]^2. \end{cases} \tag{3.16}$$

Family 3:

$$\left\{\sigma_1 \to i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}},\ \sigma_0 \to 0,\ \sigma_{-1} \to i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}},\ \varpi_3 \to -8\mu_1 - \mu_1\varpi_1^2\right\}.$$

The solution of Eq. (3.7) is:

$$p(\xi) = i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}}[\tanh(\xi) + \coth(\xi)]. \quad (3.17)$$

The soliton solutions of Eq. (1.1) can be obtained as:

Set 4:

$$\begin{cases} \varphi_4(x,y,t) = i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}}[\tanh(x+y-2\varpi_1\mu_1 t) \\ \qquad + \coth(x+y-2\varpi_1\mu_1 t)]e^{i[\varpi_1 x+\varpi_2 y-\mu_1(8+\varpi_1^2)t]}, \\ \phi_4(x,y,t) \\ \quad = -\frac{2\mu_1}{\mu_2}[\tanh(x+y-2\varpi_1\mu_1 t) + \coth(x+y-2\varpi_1\mu_1 t)]^2. \end{cases} \quad (3.18)$$

Family 4:

$$\left\{\sigma_1 \to i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}},\ \sigma_0 \to 0,\ \sigma_{-1} \to -i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}},\ \varpi_3 \to -4\mu_1 - \mu_1\varpi_1^2\right\}.$$

We get the solution of Eq. (3.7) as:

$$p(\xi) = i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}}[\tanh(\xi) - \coth(\xi)]. \quad (3.19)$$

Then the soliton solutions of Eq. (1.1) can be attained as:

Set 5:

$$\begin{cases} \varphi_5(x,y,t) = i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}}[\tanh(x+y-2\varpi_1\mu_1 t) \\ \qquad - \coth(x+y-2\varpi_1\mu_1 t)]e^{i[\varpi_1 x+\varpi_2 y-\mu_1(4+\varpi_1^2)t]}, \\ \phi_5(x,y,t) = -\frac{2\mu_1}{\mu_2}[\tanh(x+y-2\varpi_1\mu_1 t) - \coth(x+y-2\varpi_1\mu_1 t)]^2. \end{cases} \quad (3.20)$$

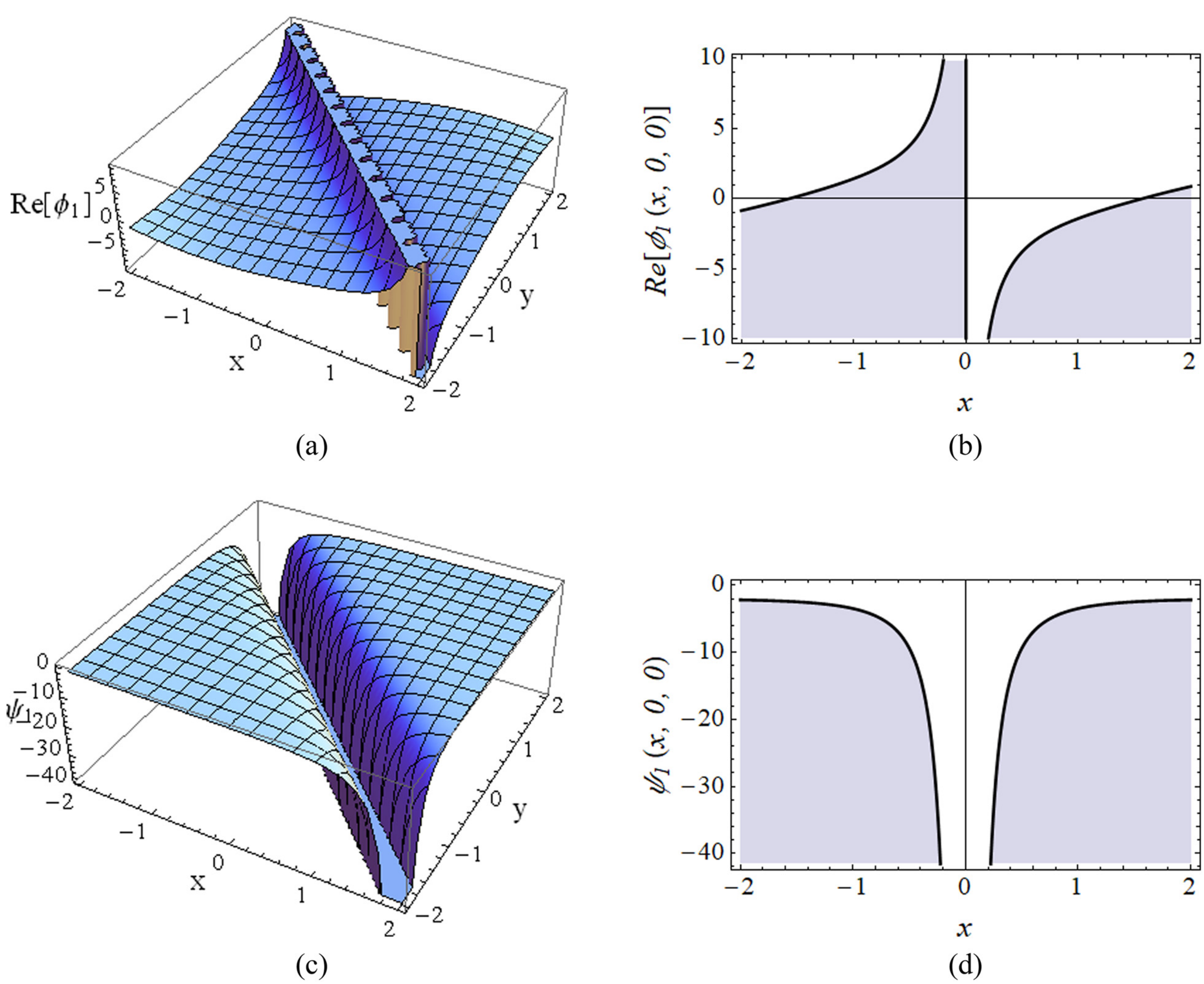

Figure 1: The behavior of Eq. (3.13) for $\mu_1 = 2$, $\mu_2 = 1$, $\mu_3 = -1$, $\mu_4 = 1$, $\varpi_1 = 1$ and $\varpi_2 = 1$ (a) and (c) for $t = 0$. (b) and (d) for $y = 0$ and $t = 0$.

Family 5:

$$\left\{\sigma_1 \to -i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}},\, \sigma_0 \to 0,\, \sigma_{-1} \to i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}},\, \varpi_3 \to -4\mu_1 - \mu_1\varpi_1^2\right\}.$$

Here the solution of Eq. (3.7) can be expressed as:

$$p(\xi) = i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}}[-\tanh(\xi) + \coth(\xi)]. \tag{3.21}$$

We can develop the soliton solutions of Eq. (1.1) as:

Set 6:

$$\begin{cases} \varphi_6(x,y,t) = i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}}[-\tanh(x+y-2\varpi_1\mu_1 t) \\ \qquad + \coth(x+y-2\varpi_1\mu_1 t)]e^{i[\varpi_1 x+\varpi_2 y-\mu_1(4+\varpi_1^2)t]} \\ \phi_6(x,y,t) = -\frac{2\mu_1}{\mu_2}[-\tanh(x+y-2\varpi_1\mu_1 t) + \coth(x+y-2\varpi_1\mu_1 t)]^2. \end{cases} \tag{3.22}$$

Family 6:

$$\left\{\sigma_1 \to \pm i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}},\, \sigma_0 \to 0,\, \sigma_{-1} \to 0,\, \varpi_3 \to -2\mu_1 - \mu_1\varpi_1^2\right\}.$$

We can get the expression of Eq. (3.7) as:

$$p(\xi) = \pm i\sqrt{\frac{2\mu_1\mu_3}{\mu_2\mu_4}}\tanh(\xi). \tag{3.23}$$

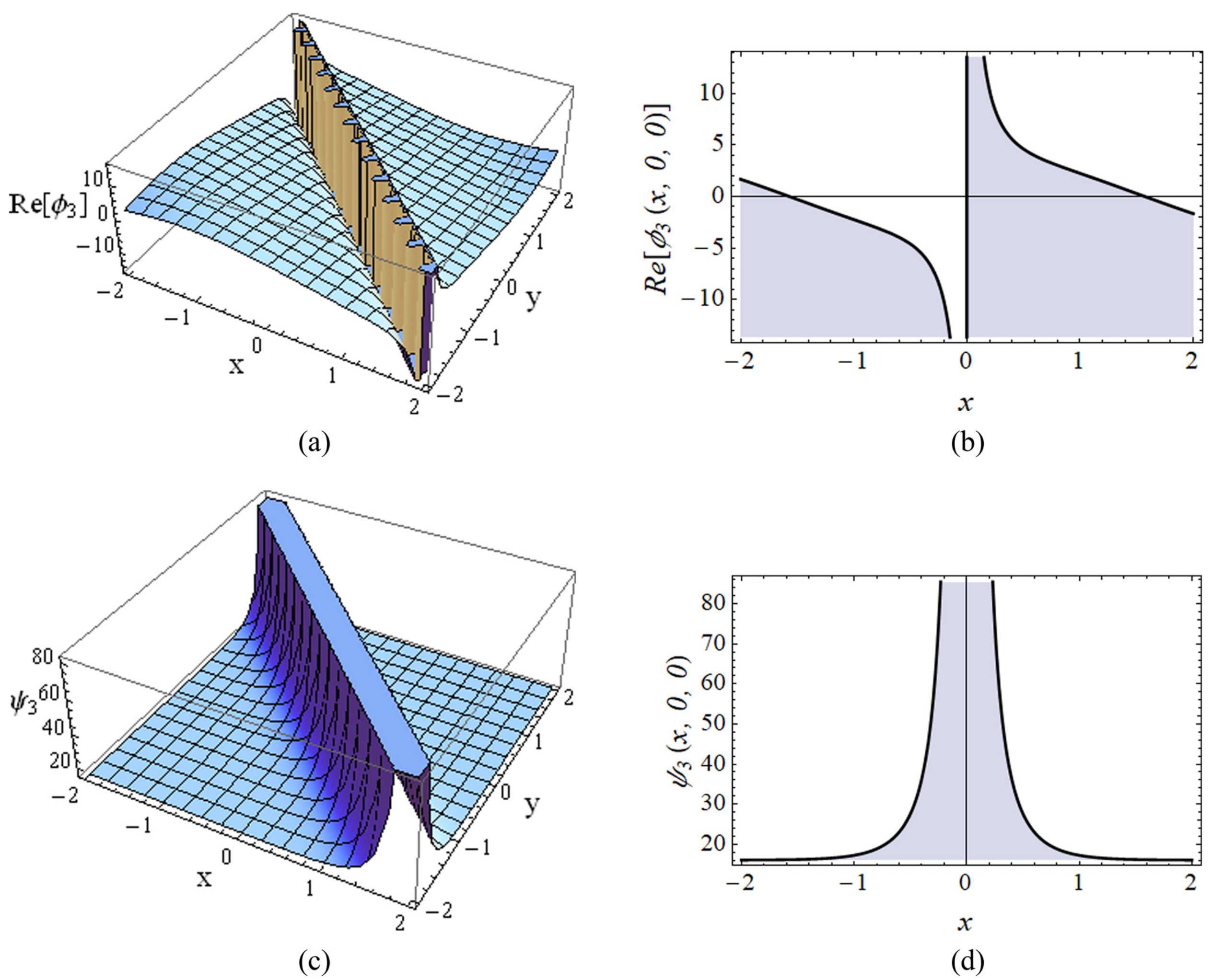

Figure 2: The behavior of Eq. (3.16) for $\mu_1 = 2$, $\mu_2 = -1$, $\mu_3 = 1$, $\mu_4 = 1$, $\varpi_1 = 1$ and $\varpi_2 = 1$. (a) and (c) for $t = 0$. (b) and (d) for $y = 0$ and $t = 0$.

Thus, we can obtain the soliton solutions of Eq. (1.1) as:

Set 7:

$$\begin{cases}\varphi_7(x,y,t)=i\sqrt{\dfrac{2\mu_1\mu_3}{\mu_2\mu_4}}\tanh(x+y\\ \qquad -2\varpi_1\mu_1 t)e^{i[\varpi_1 x+\varpi_2 y-\mu_1(2+\varpi_1^2)t]},\\ \phi_7(x,y,t)=-\dfrac{2\mu_1}{\mu_2}\tanh^2(x+y-2\varpi_1\mu_1 t).\end{cases} \tag{3.24}$$

Set 8:

$$\begin{cases}\varphi_8(x,y,t)=-i\sqrt{\dfrac{2\mu_1\mu_3}{\mu_2\mu_4}}\tanh(x+y\\ \qquad -2\varpi_1\mu_1 t)e^{i[\varpi_1 x+\varpi_2 y-\mu_1(2+\varpi_1^2)t]},\\ \phi_8(x,y,t)=-\dfrac{2\mu_1}{\mu_2}\tanh^2(x+y-2\varpi_1\mu_1 t).\end{cases} \tag{3.25}$$

3.2 Application of the VM

We can establish the variational principle of Eq. (3.7) as:

$$J(p)=\int_0^\infty\left\{-\frac{1}{2}\mu_1(p')^2-\frac{\varpi_1^2\mu_1+\varpi_3}{2}p^2+\frac{\mu_2\mu_4}{4\mu_3}p^4\right\}d\xi. \tag{3.26}$$

Now we seek the solutions of Eq. (3.7) in the following forms:

Form 1. The solution of Eq. (3.7) is supposed as:

$$p(\xi)=\Xi_1\operatorname{sech}(\xi). \tag{3.27}$$

Taking it into Eq. (3.26) yields:

$$J(\Xi_1)=\frac{\Xi_1^2[\Xi_1^2\mu_2\mu_4-\mu_1\mu_2(1+3\varpi_1^2)-3\mu_3\varpi_3]}{6\mu_3}. \tag{3.28}$$

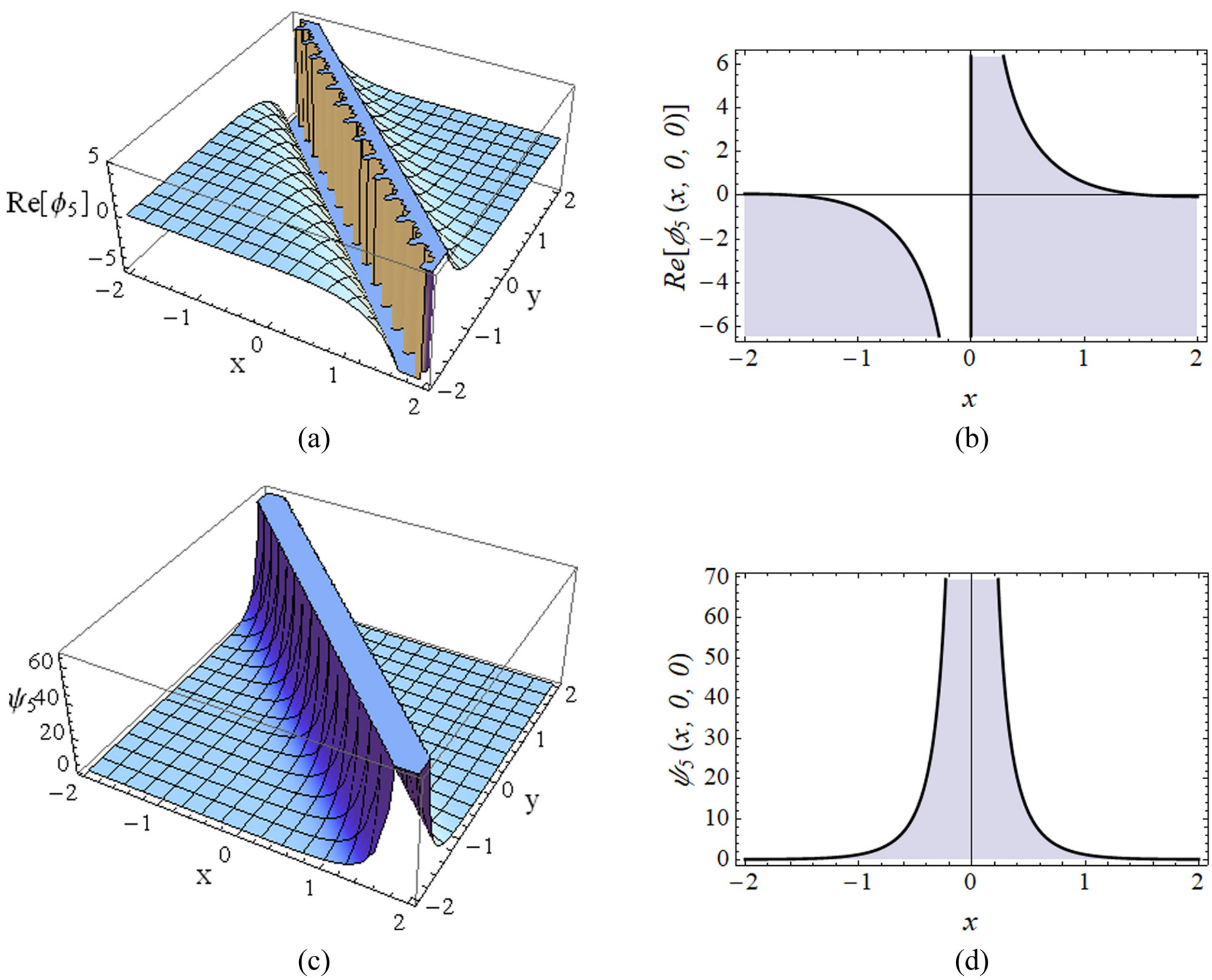

Figure 3: The behavior of Eq. (3.20) for $\mu_1=2$, $\mu_2=-1$, $\mu_3=1$, $\mu_4=1$, $\varpi_1=1$ and $\varpi_2=1$. (a) and (c) for $t=0$. (b) and (d) for $y=0$ and $t=0$.

Its stationary condition is:

$$\frac{dJ(\Xi_1)}{d\Xi_1} = 0. \tag{3.29}$$

which gives:

$$\frac{1}{3}\Xi_1\left[\frac{2\Xi_1^2\mu_2\mu_4}{\mu_3} - \mu_1(1 + 3\varpi_1^2) - 3\varpi_3\right] = 0. \tag{3.30}$$

Solving Eq. (3.30), we have:

$$\Xi_1 = \pm\sqrt{\frac{\mu_3(\mu_1 + 3\mu_1\varpi_1^2 + 3\varpi_3)}{2\mu_2\mu_4}}. \tag{3.31}$$

Equation (3.27) can be re-written as:

$$\Xi_1 = \pm\sqrt{\frac{\mu_3(\mu_1 + 3\mu_1\varpi_1^2 + 3\varpi_3)}{2\mu_2\mu_4}}. \tag{3.31}$$

Taking Eqs. (3.1), (3.3), and (3.32), we can develop the soliton solutions of Eq. (1.1) as:

Set 9:

$$\begin{cases} \varphi_9(x, y, t) = \sqrt{\dfrac{\mu_3(\mu_1 + 3\mu_1\varpi_1^2 + 3\varpi_3)}{2\mu_2\mu_4}}\operatorname{sech}(x + y \\ \quad - 2\varpi_1\mu_1 t)e^{i[\varpi_1 x + \varpi_2 y + \varpi_3 t]}, \\ \phi_9(x, y, t) = \dfrac{(\mu_1 + 3\mu_1\varpi_1^2 + 3\varpi_3)}{2\mu_2}\operatorname{sech}^2(x + y \\ \quad - 2\varpi_1\mu_1 t). \end{cases} \tag{3.33}$$

Set 10:

$$\begin{cases} \varphi_{10}(x, y, t) = -\sqrt{\dfrac{\mu_3(\mu_1 + 3\mu_1\varpi_1^2 + 3\varpi_3)}{2\mu_2\mu_4}}\operatorname{sech}(x \\ \quad + y - 2\varpi_1\mu_1 t)e^{i[\varpi_1 x + \varpi_2 y + \varpi_3 t]}, \\ \phi_{10}(x, y, t) = \dfrac{(\mu_1 + 3\mu_1\varpi_1^2 + 3\varpi_3)}{2\mu_2}\operatorname{sech}^2(x + y \\ \quad - 2\varpi_1\mu_1 t). \end{cases} \tag{3.34}$$

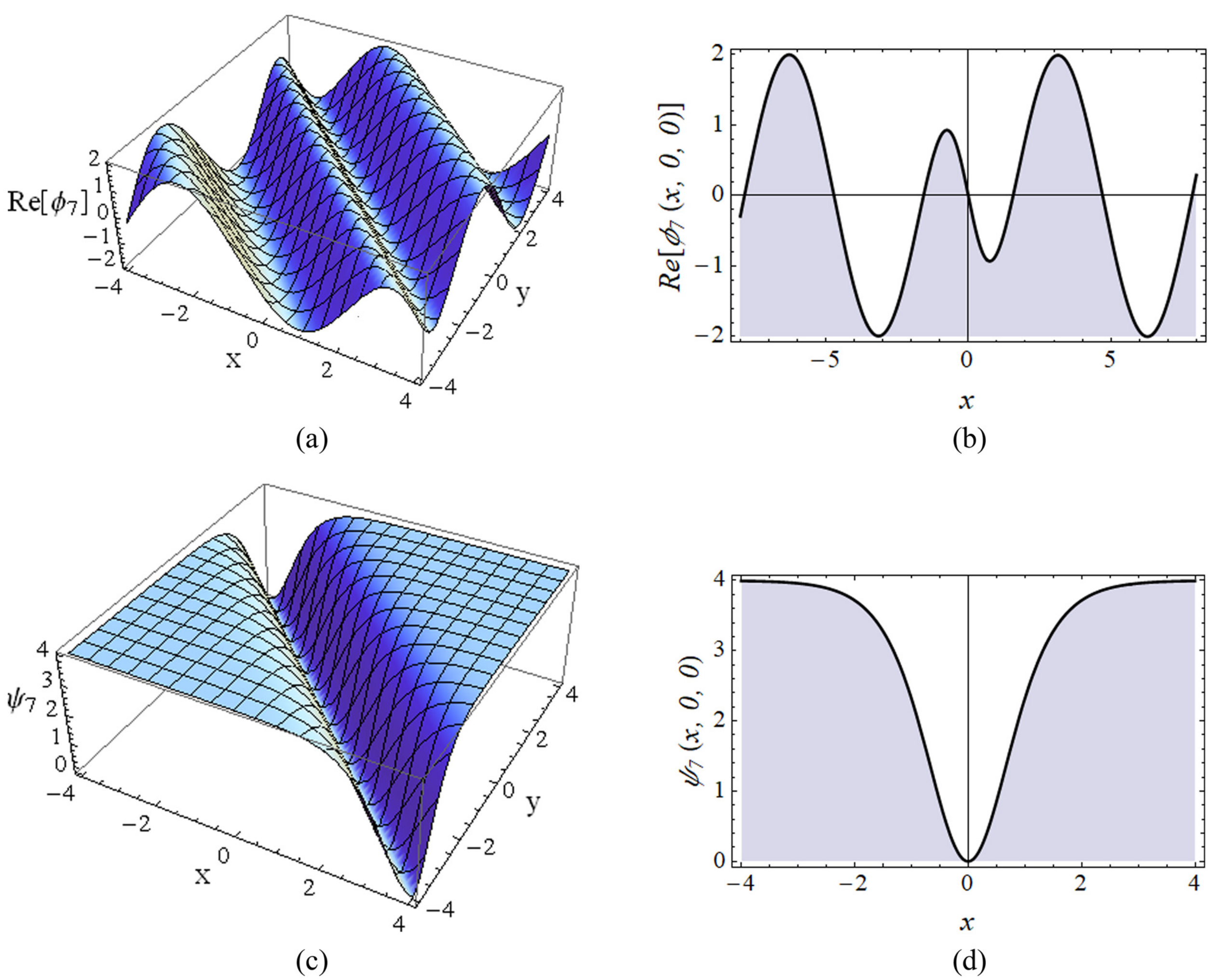

Figure 4: The behavior of Eq. (3.24) for $\mu_1 = 2$, $\mu_2 = -1$, $\mu_3 = 1$, $\mu_4 = 1$, $\varpi_1 = 1$ and $\varpi_2 = 1$. (a) and (c) for $t = 0$. (b) and (d) for $y = 0$ and $t = 0$.

Form 2. The solution of Eq. (3.7) is supposed as:

$$p(\xi) = \frac{\Xi_2}{1 + \cosh(\xi)}. \tag{3.35}$$

By the same way, taking Eq. (3.35) into Eq. (3.7) and applying the stationary condition, it yields:

$$\frac{\Xi_2[6\Xi_2^2\mu_2\mu_4 - 7\mu_1\mu_3(1 + 5\varpi_1^2) - 35\mu_3\varpi_3]}{105\mu_3} = 0. \tag{3.36}$$

We can determine Ξ_2 by solving Eq. (3.36):

$$\Xi_2 = \pm\sqrt{\frac{7\mu_3(\mu_1 + 5\mu_1\varpi_1^2 + 5\varpi_3)}{6\mu_2\mu_4}}. \tag{3.37}$$

So the solution of Eq. (3.7) is obtained, which is:

$$p(\xi) = \pm\sqrt{\frac{7\mu_3(\mu_1 + 5\mu_1\varpi_1^2 + 5\varpi_3)}{6\mu_2\mu_4}}\frac{1}{1 + \cosh(\xi)}. \tag{3.38}$$

Thus, the soliton solutions of Eq. (1.1) are got as:

Set 11:

$$\begin{cases} \varphi_{11}(x,y,t) = \sqrt{\frac{7\mu_3(\mu_1 + 5\mu_1\varpi_1^2 + 5\varpi_3)}{6\mu_2\mu_4}}\frac{1}{1 + \cosh(x + y - 2\varpi_1\mu_1 t)} \\ \qquad e^{i[\varpi_1 x + \varpi_2 y + \varpi_3 t]}, \\ \phi_{11}(x,y,t) = \frac{7(\mu_1 + 5\mu_1\varpi_1^2 + 5\varpi_3)}{6\mu_2}\left[\frac{1}{1 + \cosh(x + y - 2\varpi_1\mu_1 t)}\right]^2. \end{cases} \tag{3.39}$$

Set 12:

$$\begin{cases} \varphi_{12}(x,y,t) = \sqrt{\frac{7\mu_3(\mu_1 + 5\mu_1\varpi_1^2 + 5\varpi_3)}{6\mu_2\mu_4}}\frac{1}{1 + \cosh(x + y - 2\varpi_1\mu_1 t)} \\ \qquad e^{i[\varpi_1 x + \varpi_2 y + \varpi_3 t]}, \\ \phi_{12}(x,y,t) = \frac{7(\mu_1 + 5\mu_1\varpi_1^2 + 5\varpi_3)}{6\mu_2}\left[\frac{1}{1 + \cosh(x + y - 2\varpi_1\mu_1 t)}\right]^2. \end{cases} \tag{3.40}$$

Form 3. We also assume the solution of Eq. (3.7) with the following form:

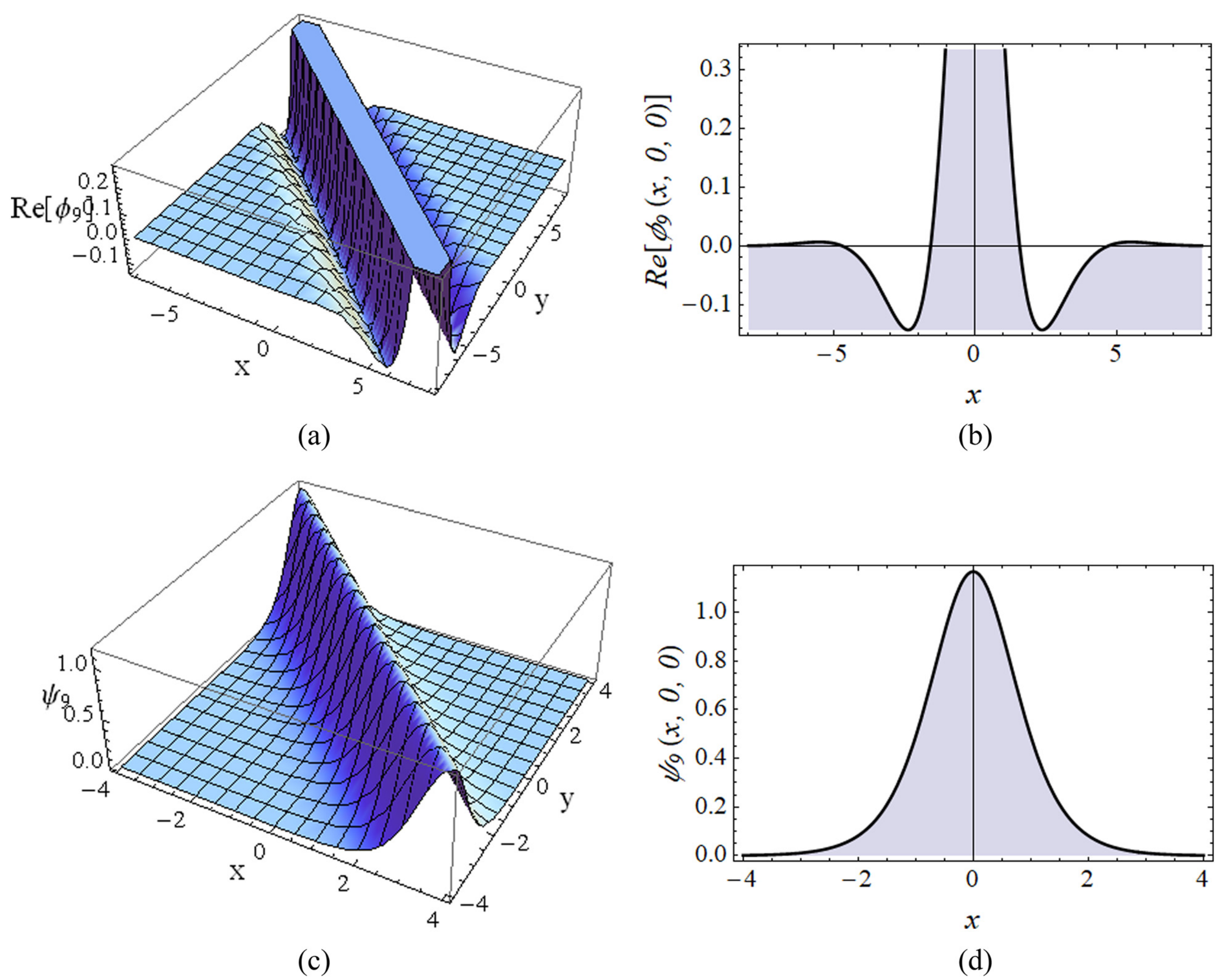

Figure 5: The behavior of Eq. (3.33) for $\mu_1 = 1$, $\mu_2 = 3$, $\mu_3 = 1$, $\mu_4 = 1$, $\varpi_1 = 1$ and $\varpi_2 = 1$, $\varpi_3 = 1$. (a) and (c) for $t = 0$. (b) and (d) for $y = 0$ and $t = 0$.

$$p(\xi) = \frac{\Xi_3 \sinh(\xi)}{[\cosh(\xi)]^{\frac{3}{2}}}. \tag{3.41}$$

We can get the following expression by taking Eq. (3.41) into Eq. (3.26) *via* the stationary condition:

$$\frac{1}{320}\Xi_3\left[\frac{64\Xi_3^2\mu_2\mu_4}{\mu_3} - 5\pi\mu_1(11 + 16\varpi_1^2) - 80\pi\varpi_3\right] = 0. \tag{3.42}$$

Solving Eq. (3.42) gives:

$$\Xi_3 = \pm\sqrt{\frac{5\pi\mu_3(11\mu_1 + 16\mu_1\varpi_1^2 + 16\varpi_3)}{8\mu_2\mu_4}}. \tag{3.43}$$

So Eq. (3.41) can be expressed as:

$$p(\xi) = \pm\sqrt{\frac{5\pi\mu_3(11\mu_1 + 16\mu_1\varpi_1^2 + 16\varpi_3)}{8\mu_2\mu_4}}\,\frac{\sinh(\xi)}{[\cosh(\xi)]^{\frac{3}{2}}}. \tag{3.44}$$

Then, we get the soliton solutions of Eq. (1.1) as:

Set 13:

$$\begin{cases} \varphi_{13}(x,y,t) = \sqrt{\dfrac{5\pi\mu_3(11\mu_1 + 16\mu_1\varpi_1^2 + 16\varpi_3)}{8\mu_2\mu_4}}\,\dfrac{\sinh(x + y - 2\varpi_1\mu_1 t)}{[\cosh(x + y - 2\varpi_1\mu_1 t)]^{\frac{3}{2}}}\, e^{i[\varpi_1 x + \varpi_2 y + \varpi_3 t]}, \\ \phi_{13}(x,y,t) = \dfrac{5\pi(11\mu_1 + 16\mu_1\varpi_1^2 + 16\varpi_3)}{8\mu_2}\,\dfrac{\sinh^2(x + y - 2\varpi_1\mu_1 t)}{[\cosh(x + y - 2\varpi_1\mu_1 t)]^3}. \end{cases} \tag{3.45}$$

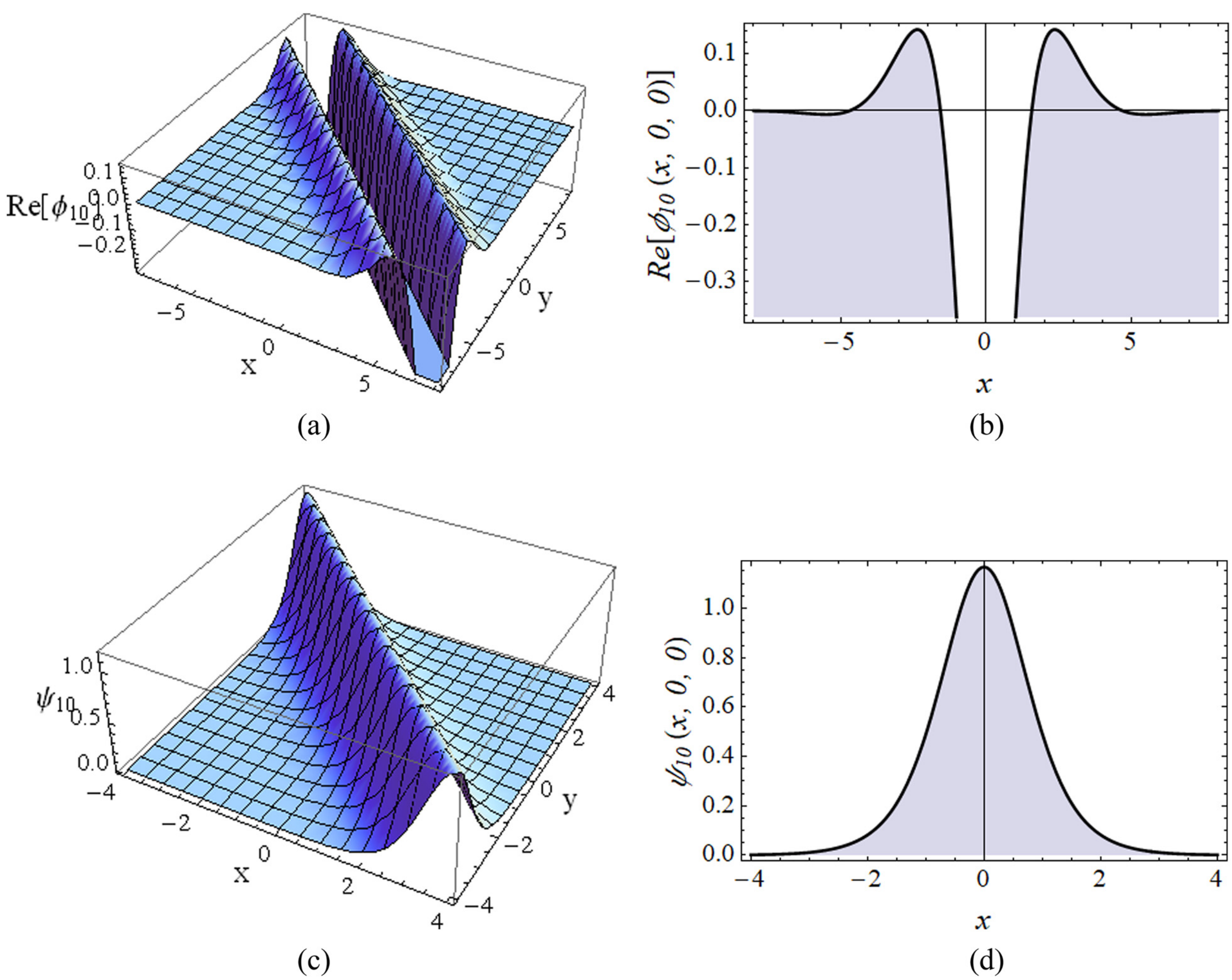

Figure 6: The behavior of Eq. (3.34) for $\mu_1 = 1$, $\mu_2 = 3$, $\mu_3 = 1$, $\mu_4 = 1$, $\varpi_1 = 1$, $\varpi_2 = 1$ and $\varpi_3 = 1$. (a) and (c) for $t = 0$. (b) and (d) for $y = 0$ and $t = 0$.

Set 14:

$$\begin{cases} \varphi_{14}(x,y,t) \\ \quad = -\sqrt{\dfrac{5\pi\mu_3(11\mu_1+16\mu_1\varpi_1^2+16\varpi_3)}{8\mu_2\mu_4}} \\ \quad \dfrac{\sinh(x+y-2\varpi_1\mu_1 t)}{[\cosh(x+y-2\varpi_1\mu_1 t)]^{\frac{3}{2}}} e^{i[\varpi_1 x+\varpi_2 y+\varpi_3 t]}, \\ \phi_{14}(x,y,t) = \dfrac{5\pi(11\mu_1+16\mu_1\varpi_1^2+16\varpi_3)}{8\mu_2} \\ \quad \dfrac{\sinh^2(x+y-2\varpi_1\mu_1 t)}{[\cosh(x+y-2\varpi_1\mu_1 t)]^3}. \end{cases} \tag{3.46}$$

3.3 Application of the HFFM

Eq. (3.7) can be expressed as:

$$p'' - \frac{\varpi_1^2\mu_1+\varpi_3}{\mu_1}p + \frac{\mu_2\mu_4}{\mu_1\mu_3}p^3 = 0. \tag{3.47}$$

That is:

$$p'' + f(p) = 0. \tag{3.48}$$

where:

$$f(p) = -\frac{\varpi_1^2\mu_1+\varpi_3}{\mu_1}p + \frac{\mu_2\mu_4}{\mu_1\mu_3}p^3. \tag{3.49}$$

Suppose the solution of Eq. (3.47) is:

$$p(\xi) = \Theta\cos(\varpi\xi), \varpi > 0, \tag{3.50}$$

In the light of Eq. (2.7), there is:

$$\varpi = \sqrt{\frac{\mathrm{d}f}{\mathrm{d}p}}\Bigg|_{p=\frac{\Theta}{2}} = \sqrt{-\frac{\varpi_1^2\mu_1+\varpi_3}{\mu_1} + \frac{3\mu_2\mu_4}{4\mu_1\mu_3}\Theta^2}, \tag{3.51}$$

So the period soliton solution of Eq. (1.1) is:

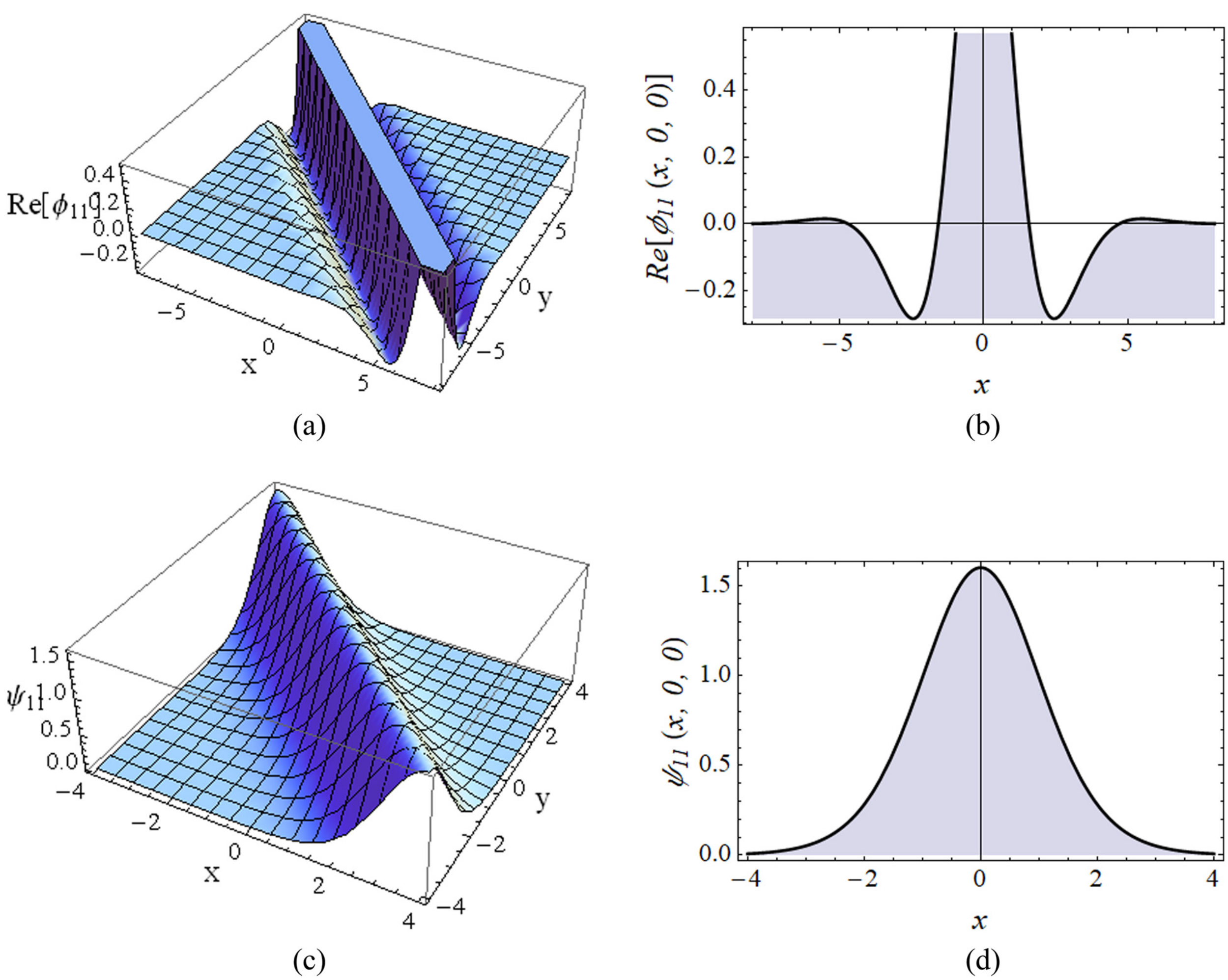

Figure 7: The behavior of Eq. (3.39) for $\mu_1 = 1$, $\mu_2 = 3$, $\mu_3 = 1$, $\mu_4 = 1$, $\varpi_1 = 1$, $\varpi_2 = 1$ and $\varpi_3 = 1$. (a) and (c) for $t = 0$. (b) and (d) for $y = 0$ and $t = 0$.

Set 15:

$$\begin{cases} \varphi_{15}(x,y,t) = \Theta\cos\left(\sqrt{-\dfrac{\varpi_1^2\mu_1+\varpi_3}{\mu_1}+\dfrac{3\mu_2\mu_4}{4\mu_1\mu_3}\Theta^2}(x+y - 2\varpi_1\mu_1 t)\right)e^{i(\varpi_1 x+\varpi_2 y+\varpi_3 t)}, \\ \phi_{15}(x,y,t) = \dfrac{\mu_4\Theta^2}{\mu_3}\cos^2\left(\sqrt{-\dfrac{\varpi_1^2\mu_1+\varpi_3}{\mu_1}+\dfrac{3\mu_2\mu_4}{4\mu_1\mu_3}\Theta^2}(x+y - 2\varpi_1\mu_1 t)\right). \end{cases} \quad (3.52)$$

A comparison between our solutions and those given by ref. [39] shows that Eqs. (3.13) and (3.14) from our work are the same as Eq. (3.37) and Eq. (3.38) given by ref. [39], Eqs. (3.24) and (3.25) in our study are the same as Eqs. (3.27) and (3.28) provided by ref. [39] *via* using the special parameters. However, the other solutions obtained in our study have not been reported in the literature [39].

4 Physical explanation

In this section, the numerical simulation of some solutions are presented *via* the 3-D plot and 2-D curve to interpret the physical behaviors by assigning the proper parameters. It should be noted that in the following content, the labels (a) and (b) represent the 3-D plot and 2-D curve of the real part of $\varphi(x,y,t)$, the labels (c) and (d) indicate the 3-D plot and 2-D curve of $\phi(x,y,t)$, respectively.

We plot the solution Eq. (3.13) in Figure 1. It can be seen that the wave of the real part (RE) of $\varphi_1(x,y,t)$ is bright-dark soliton, and performance of the $\phi_1(x,y,t)$ is the dark soliton.

The behavior of Eq. (3.16) is drawn in Figure 2, where we can observe the contour of the RE of $\varphi_3(x,y,t)$ is bright-dark soliton, and the contour of the $\phi_3(x,y,t)$ is the bright soliton.

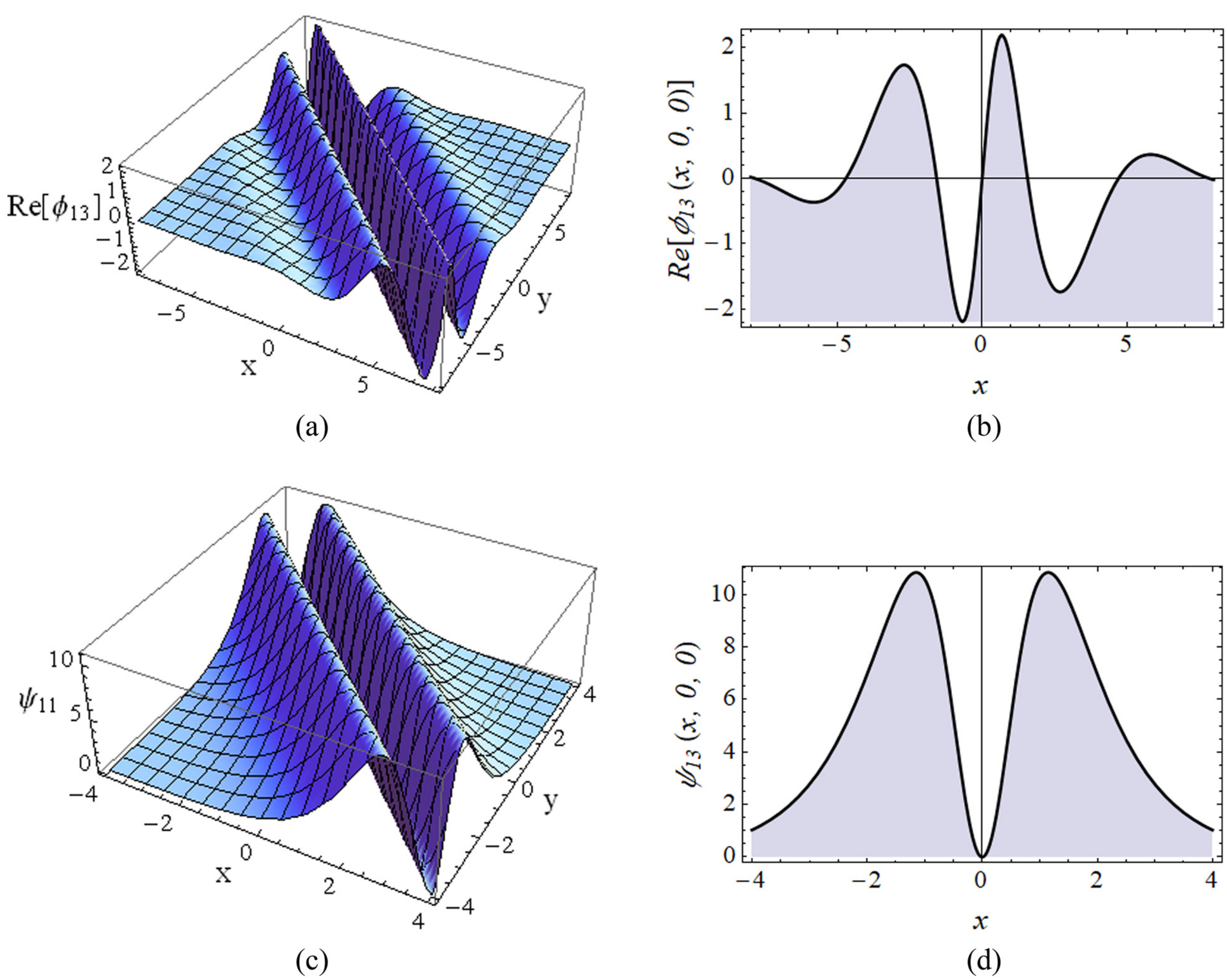

Figure 8: The behavior of Eq. (3.45) for $\mu_1=1$, $\mu_2=3$, $\mu_3=1$, $\mu_4=1$, $\varpi_1=1$, $\varpi_2=1$ and $\varpi_3=1$. (a) and (c) for $t=0$. (b) and (d) for $y=0$ and $t=0$.

Figure 3 plots the performance of Eq. (3.20), the contours are the same as Figure 2.

From the Eq. (3.24) plotted in Figure 4, it is found the wave contour of the RE of the $\varphi_7(x, y, t)$ is a kinky periodic soliton, and the contour of the $\phi_7(x, y, t)$ is dark soliton.

The performance of Eq. (3.33) are presented in Figure 5. Obviously, the contour of the RE of $\varphi_9(x, y, t)$ is the double-dark soliton, and the contour of $\phi_9(x, y, t)$ is the bright soliton.

From the description of the solution Eq. (3.34) in Figure 6, we can find the contour of the RE of $\varphi_9(x, y, t)$ is the double-bright soliton, on the other hand, the contour of $\phi_9(x, y, t)$ is the bright soliton.

From Figure 7, it can be found that the contour of the RE of $\varphi_{11}(x, y, t)$ is the double-dark soliton and the contour of $\phi_{11}(x, y, t)$ is the bright soliton.

As shown in Figure 8, it can be seen that the contour of the RE of $\varphi_{13}(x, y, t)$ is the triple-bright soliton and the contour of $\phi_{13}(x, y, t)$ is the double-bright soliton.

Figure 9 illustrates the solutions of $\phi_{15}(x, y, t)$ and $\varphi_{15}(x, y, t)$. It is noticeable that the wave form of the RE of the $\phi_{15}(x, y, t)$ is the kinky periodic soliton and that of the $\varphi_{15}(x, y, t)$ is a perfect periodic soliton.

5 Conclusion

The Fokas system has been studied in this work by three powerful approaches, the SETFM, VM and the HFFM. Fifteen sets of the soliton solutions in the term of bright soliton, dark soliton, bright-dark soliton, double-dark soliton, double-bright soliton, triple-bright soliton, kinky periodic soliton and periodic soliton solutions are obtained. By comparing our results with those provided by ref. [39], it is found that some solutions obtained by our study are new, which provide a good supplement to the existing literature. Finally, the performance of the solutions are presented t-

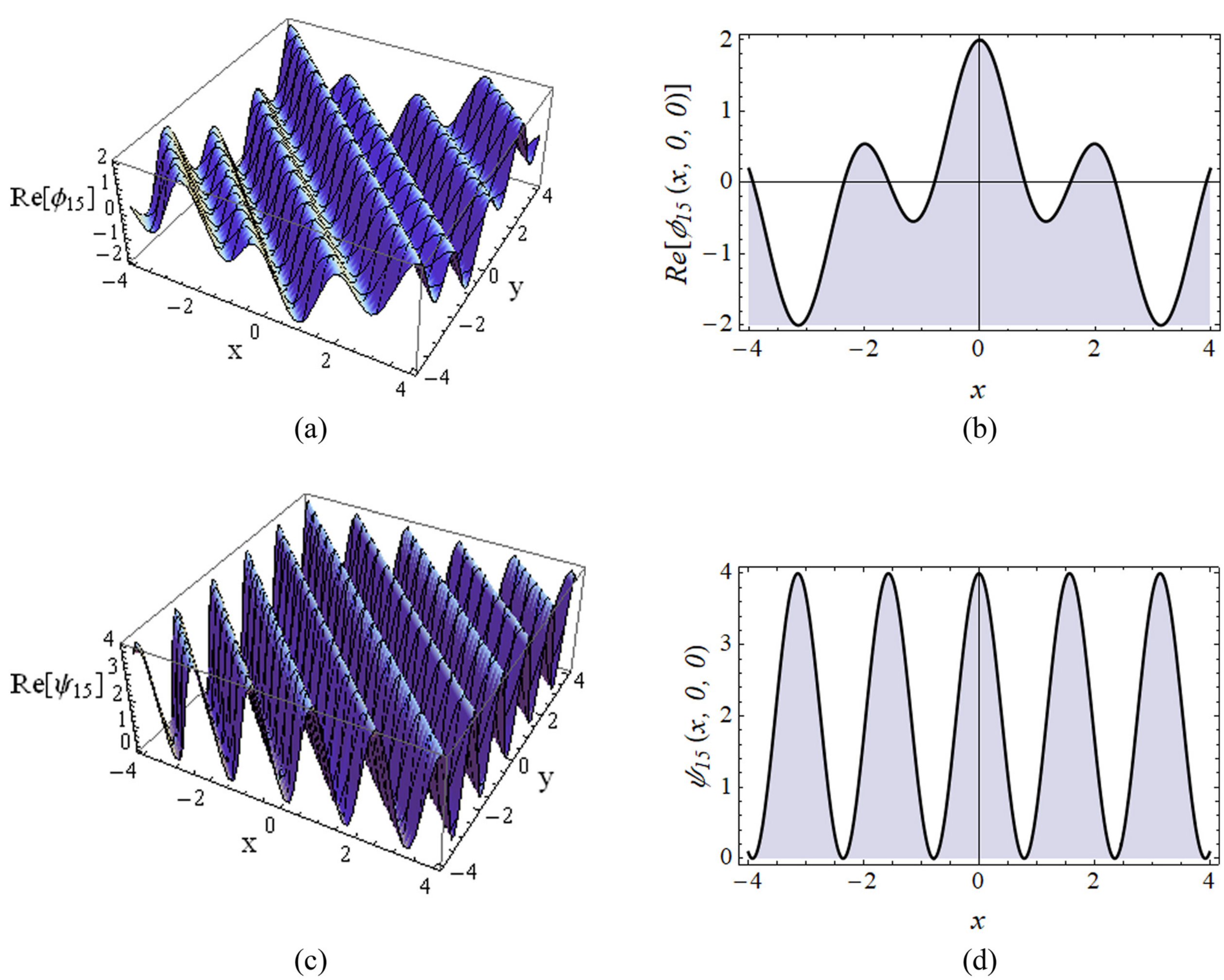

Figure 9: The behavior of Eq. (3.52) for $\mu_1 = 1$, $\mu_2 = 3$, $\mu_3 = 1$, $\mu_4 = 1$, $\varpi_1 = 1$, $\varpi_2 = 1$, $\varpi_3 = -2$ and $\Theta = 2$. (a) and (c) for $t = 0$. (b) and (d) for $y = 0$ and $t = 0$.

hrough the 3-D contours and 2-D curves to interpret the physical behaviors by using proper parameters. The results reveal that the presented methods are powerful and effective, which can be used to develop the abundant solition solutions of the PDEs arising in physics.

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