Research Article

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Nonlinear dynamics for different nonautonomous wave structure solutions

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Abstract: Based on the positive quadratic function method, the rich nonautonomous solutions of a generalized (2+1)-dimensional variable-coefficient breaking soliton equation with different wave structures are given. In this case, due to the influence of nonlinearity and dispersion, the characteristics, amplitude and velocity of nonautonomous wave will change with time. The breather wave and the interaction among lump wave, solitary wave and periodic wave solutions are studied. For different choices of arbitrary functions in these solutions, the corresponding dynamic properties are demonstrated.

Keywords: positive quadratic function method, lump wave, breather wave, breaking soliton equation

1 Introduction

In recent years, with the development of symbolic computation, people have begun to pay attention to the relevant theories of lump wave solutions [1–4]. In 2015, Ma [5] proposed a method of directly using the Hirota bilinear method to solve lump wave solutions and gave theoretical proofs and derivations, pushing the research of lump wave solutions to a new stage. At present, many researchers have successfully constructed lump wave solutions and interaction solutions of multiple high-dimensional nonlinear development equations using this method [6–9]. The research of these solutions has important significance and prospects for many high-dimensional nonlinear problems in mathematics, physics and other fields.

Variable coefficient integrability systems can more clearly describe real-world phenomena, such as in the context of ocean waves, the temporal variability of the variable coefficient may be due to the pressure dependence of the coefficient of thermal expansion of seawater, coupled with large-scale forward changes in ocean temperature–salinity relationship and other dynamic conditions [10]. In this article, under investigation is the generalized (2+1)-dimensional variable-coefficient breaking soliton equation [11]

$$\tau_1(t)u_xu_{xx} + \tau_4(t)(u_xu_{xy} + u_yu_{xx}) + \tau_3(t)u_{xxxy} + \tau_2(t)u_{xxxx} + u_{xt} = 0, \tag{1}$$

where $u = u(x, y, t)$ represents the interactions among two Riemann waves propagating along the y- or z-axis and a long wave propagating along the x-axis. $\tau_1(t)$, $\tau_2(t)$, $\tau_3(t)$ and $\tau_4(t)$ are arbitrary functions. Osman [12] presented the multi-soliton solutions of Eq. (1) by the generalized unified method. Li *et al.* [11] obtained the breather wave solutions and lump solutions of Eq. (1). However, the interaction among lump wave, solitary wave and periodic wave solutions has not been investigated, which will become our main work.

The article is organized as follows. Section 2 investigates the interaction between lump wave and solitary wave; Section 3 studies the interaction between lump wave and periodic wave; Section 4 obtains the breather wave solutions; Section 5 gives a conclusion.

2 Interaction between lump wave and solitary wave

Under the transformation

$$\begin{gathered}\tau_1(t) = 6\tau_2(t), \quad \tau_4(t) = 3\tau_3(t),\\ u = 2\,(\ln\kappa)_x, \quad \kappa = \kappa(x, y, t).\end{gathered} \tag{2}$$

Eq. (2) becomes

$$\begin{aligned}&[\tau_2(t)D_x^4 + \tau_3(t)D_x^3D_y + D_tD_x]\kappa\cdot\kappa\\ &\quad= 3\tau_2(t)\kappa_{xx}^2 + 3\tau_3(t)\kappa_{xy}\kappa_{xx} - 3\tau_3(t)\kappa_x\kappa_{xxy} - \tau_3(t)\kappa_y\kappa_{xxx}\\ &\qquad - 4\tau_2(t)\kappa_x\kappa_{xxx} + \kappa(\tau_3(t)\kappa_{xxxy} + \tau_2(t)\kappa_{xxxx} + \kappa_{xt})\\ &\qquad - \kappa_t\kappa_x = 0.\end{aligned} \tag{3}$$

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In order to investigate the interaction between lump wave and solitary wave of Eq. (1), the following assumptions are usually made

$$\kappa = v_7(t) + \left(\int v_3(t)\mathrm{d}t + v_1x + v_2y\right)^2 + \left(\int v_6(t)\mathrm{d}t + v_4x + v_5y\right)^2 + \phi_1(t)\mathrm{e}^{\int \varphi_3(t)\mathrm{d}t + \varphi_1 x + \varphi_2 y} + \phi_2(t)\mathrm{e}^{-\int \varphi_3(t)\mathrm{d}t - \varphi_1 x - \varphi_2 y}, \tag{4}$$

where v_i $(i = 1, 2, 4, 5)$ and φ_i $(i = 1, 2)$ are unknown constants. $v_i(t)$ $(i = 3, 6, 7)$, $\varphi_3(t)$ and $\phi_i(t)$ $(i = 1, 2)$ are unknown functions. Substituting Eq. (4) into Eq. (3), we obtain

$$\begin{aligned} &v_5 = \frac{v_2v_4}{v_1},\ v_6(t) = -\frac{v_1v_3(t)}{v_4},\ \varphi_2 = \frac{v_2\varphi_1}{v_1},\\ &\tau_2(t) = -\frac{v_2\tau_3(t)}{v_1},\ \phi_1(t) = \eta_1 e^{-\int \varphi_3(t)\mathrm{d}t},\\ &\phi_2(t) = \eta_2 e^{\int \varphi_3(t)\mathrm{d}t},\\ &v_7(t) = \int -\frac{2(v_1^2 + v_4^2)v_3(t)\left(\int v_3(t)\mathrm{d}t\right)}{v_4^2}\mathrm{d}t, \end{aligned} \tag{5}$$

where η_i $(i = 1, 2)$ is the integral constant. Substituting Eqs. (4) and (5) into Eq. (2), the interaction solution between lump wave and solitary wave of Eq. (1) is expressed as

$$\begin{aligned} u = &\left[2\left[2v_1\left(\int v_3(t)\mathrm{d}t + v_1x + v_2y\right) + 2v_4\left(v_4x - \frac{v_1\int v_3(t)\mathrm{d}t}{v_4} + \frac{v_2v_4y}{v_1}\right)\right.\right.\\ &\left.\left. + \eta_1\varphi_1\mathrm{e}^{\varphi_1x + \frac{v_2\varphi_1y}{v_1}} - \eta_2\varphi_1\mathrm{e}^{-\varphi_1x - \frac{v_2\varphi_1y}{v_1}}\right]\right] \Big/ \left[\left(\int v_3(t)\mathrm{d}t + v_1x + v_2y\right)^2\right.\\ & + \left[v_4x - \frac{v_1\int v_3(t)\mathrm{d}t}{v_4} + \frac{v_2v_4y}{v_1}\right]^2 + \eta_1\mathrm{e}^{\varphi_1x + \frac{v_2\varphi_1y}{v_1}} + \eta_2\mathrm{e}^{-\varphi_1x - \frac{v_2\varphi_1y}{v_1}}\\ &\left. - \int \frac{2(v_1^2 + v_4^2)v_3(t)\left(\int v_3(t)\mathrm{d}t\right)}{v_4^2}\mathrm{d}t\right]. \end{aligned} \tag{6}$$

When $\eta_1 = \eta_2 = 0$, Eq. (6) describes a lump wave, which has been studied in ref. [11]. When $\eta_1 = 0$, $\eta_2 \neq 0$, Eq. (6) represents the interaction solution between lump wave and solitary wave (Figure 1), which has not been discussed in other literature. When $\eta_1 \neq 0$ and $\eta_2 \neq 0$, Eq. (6) means the interaction solution between lump wave and two solitary waves (Figure 2), which has not been studied in other literature.

3 Interaction between lump wave and periodic wave

In order to study the interaction between lump wave and periodic wave of Eq. (1), the following assumptions are usually made

$$\begin{aligned} \kappa = &v_7(t) + \left(\int v_3(t)\mathrm{d}t + v_1x + v_2y\right)^2\\ &+ \left(\int v_6(t)\mathrm{d}t + v_4x + v_5y\right)^2\\ &+ \phi_1(t)\cos\left(\int \varphi_3(t)\mathrm{d}t + \varphi_1x + \varphi_2y\right). \end{aligned} \tag{7}$$

Substituting Eq. (7) into Eq. (3), we obtain

$$\begin{aligned} &v_5 = \frac{v_2v_4}{v_1},\quad v_6(t) = -\frac{v_1v_3(t)}{v_4},\quad \varphi_2 = \frac{v_2\varphi_1}{v_1},\\ &\tau_2(t) = -\frac{v_2\tau_3(t)}{v_1},\quad \phi_1(t) = \eta_3,\quad \varphi_3(t) = 0,\\ &v_7(t) = \int -\frac{2(v_1^2 + v_4^2)v_3(t)\left(\int v_3(t)\mathrm{d}t\right)}{v_4^2}\mathrm{d}t, \end{aligned} \tag{8}$$

where η_3 is the integral constant. Substituting Eqs. (7) and (8) into Eq. (2), the interaction solution between lump wave and periodic wave of Eq. (1) is derived as

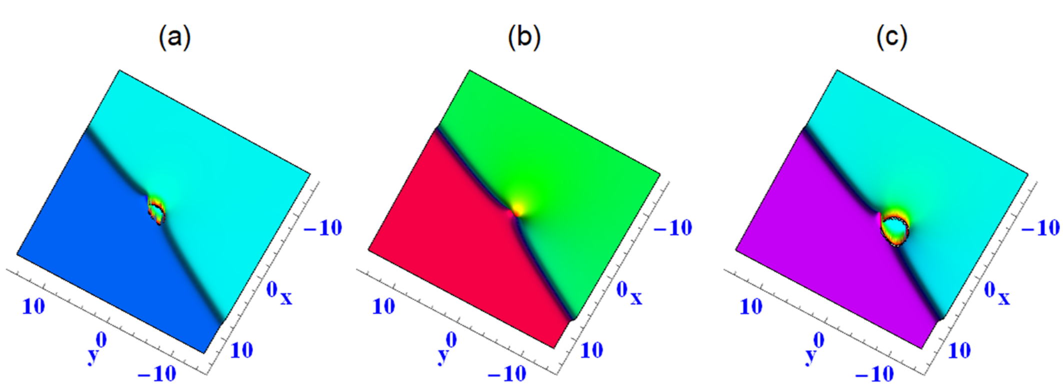

Figure 1: Solution (6) with $v_1 = 2$, $v_2 = v_4 = v_3(t) = \varphi_3(t) = 1$, $\varphi_1 = \eta_1 = 3$, $\eta_2 = 0$; (a) $t = -2$, (b) $t = 0$, and (c) $t = 2$.

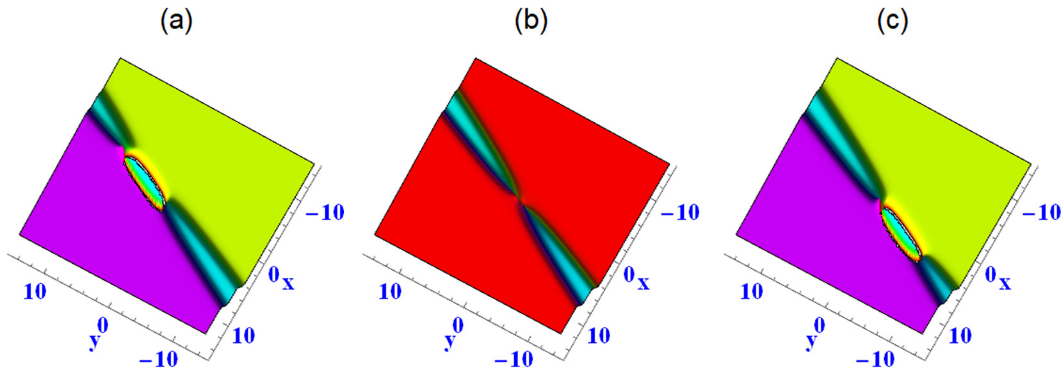

Figure 2: Solution (6) with $v_2 = v_4 = v_3(t) = \varphi_3(t) = 1$, $\varphi_1 = \eta_1 = \eta_2 = 3$, $v_1 = 2$; (a) $t = -5$, (b) $t = 0$, and (c) $t = 5$.

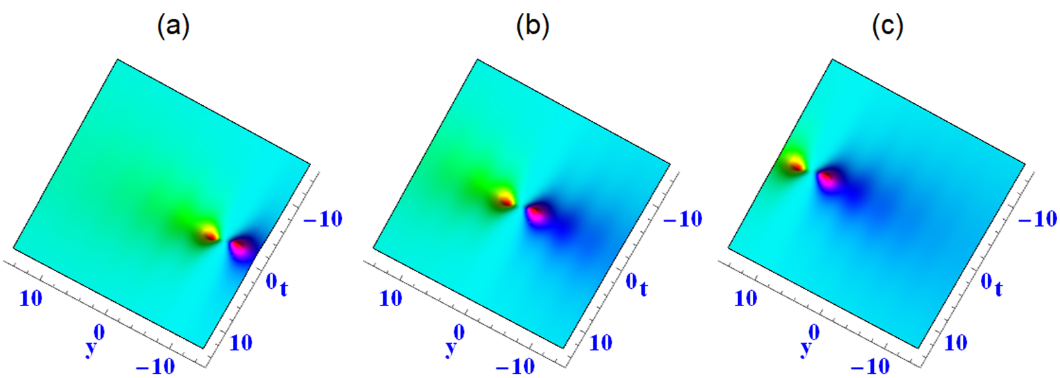

Figure 3: Solution (9) with $v_1 = 2$, $v_2 = v_4 = v_3(t) = \eta_3 = 1$, $\varphi_1 = 3$, (a) $x = -5$, (b) $x = 0$, (c) $x = 5$.

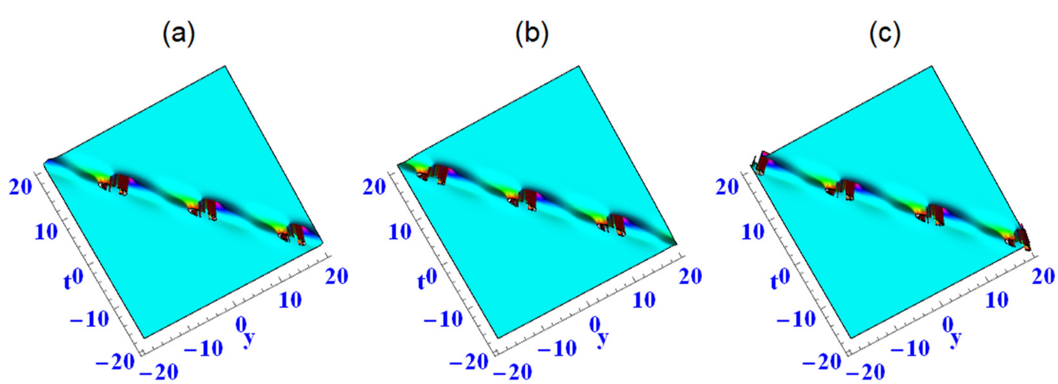

Figure 4: Solution (12) with $C_1 = -2$, $v_2 = \beta_1 = k_1(t) = \beta_2 = \vartheta_6(t) = 1$, $\beta_3 = 0$; (a) $x = -15$, (b) $x = 0$, and (c) $x = 15$.

$$u = \left[2\left[2\nu_1\left(\int \nu_3(t)\mathrm{d}t + \nu_1 x + \nu_2 y \right) + 2\nu_4\left[\nu_4 x - \frac{\nu_1 \int \nu_3(t)\mathrm{d}t}{\nu_4} + \frac{\nu_2\nu_4 y}{\nu_1} \right] - \eta_3\varphi_1 \sin\left(\varphi_1 x + \frac{\nu_2\varphi_1 y}{\nu_1} \right) \right] \right] \Big/ \left[\left(\int \nu_3(t)\mathrm{d}t + \nu_1 x + \nu_2 y \right)^2 + \left[\nu_4 x - \frac{\nu_1 \int \nu_3(t)\mathrm{d}t}{\nu_4} + \frac{\nu_2\nu_4 y}{\nu_1} \right]^2 + \eta_3 \cos\left(\varphi_1 x + \frac{\nu_2\varphi_1 y}{\nu_1} \right) - \int \frac{2(\nu_1^2 + \nu_4^2)\nu_3(t)\left(\int \nu_3(t)\mathrm{d}t \right)}{\nu_4^2}\mathrm{d}t \right]. \tag{9}$$

The interaction between lump wave and periodic wave is shown in Figure 3. It is easy to see from Figure 3 that with the continuous change of x, a lump wave and a periodic wave propagate forward together, and the amplitude of the lump wave remains unchanged during propagation, which is an elastic collision.

4 Breather wave

In order to discuss the breather wave solution of Eq. (1), the following assumptions are usually made

$$\kappa = k_3(t)\sin(\varsigma_3(t) + \nu_3 x + \mu_3 y) + k_2(t)\cos(\varsigma_2(t) + \nu_2 x + \mu_2 y) + k_1(t)e^{\varsigma_1(t) + \nu_1 x + \mu_1 y} + e^{-\varsigma_1(t) - \nu_1 x - \mu_1 y}, \tag{10}$$

where ν_i and μ_i $(i = 1, 2, 3)$ are unknown constants. $k_i(t)$ and $\varsigma_i(t)$ $(i = 1, 2, 3)$ are unknown functions. Eq. (10) is different from the breather wave solutions in ref. [11]. Substituting Eq. (10) into Eq. (3), we obtain

$$\begin{aligned} &\nu_3 = -\nu_2,\ \nu_1 = 0, \\ &\tau_2(t) = \frac{(\mu_3 - \mu_2)\tau_3(t)}{2\nu_2}, \\ &k_2(t) = k_3(t) = C_1\sqrt{k_1(t)}, \\ &\varsigma_1(t) = \frac{1}{2}\int\left(2\nu_2^2\mu_1\tau_3(t) - \frac{k_1'(t)}{k_1(t)} \right)\mathrm{d}t, \\ &\varsigma_2(t) = \frac{1}{2}\nu_2^2(\mu_2 + \mu_3)\int \tau_3(t)\mathrm{d}t, \\ &\varsigma_3(t) = \frac{1}{2}\nu_2^2(\mu_2 + \mu_3)\int \tau_3(t)\mathrm{d}t, \end{aligned} \tag{11}$$

where C_1 is the integral constant. Substituting Eqs. (10) and (11) into Eq. (2), the breather wave solution of Eq. (1) is presented as

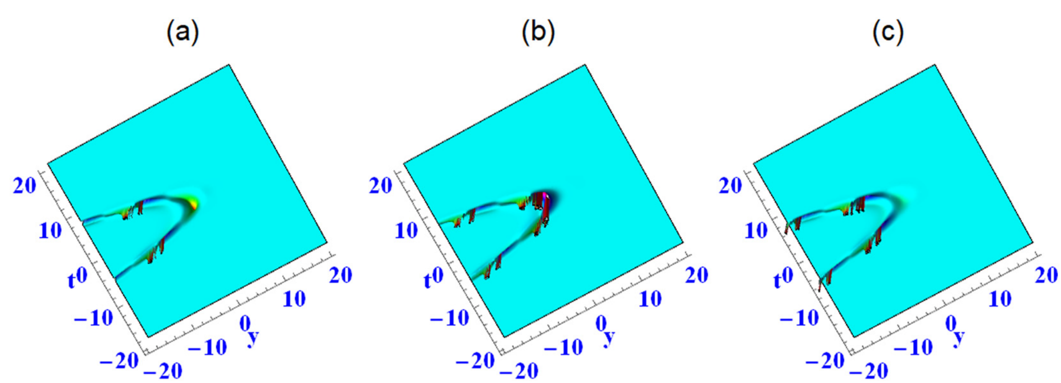

Figure 5: Solution (12) with $C_1 = -2$, $\nu_2 = \mu_1 = k_1(t) = \mu_2 = 1$, $\tau_3(t) = t$, $\mu_3 = 0$; (a) $x = -15$, (b) $x = 0$, and (c) $x = 15$.

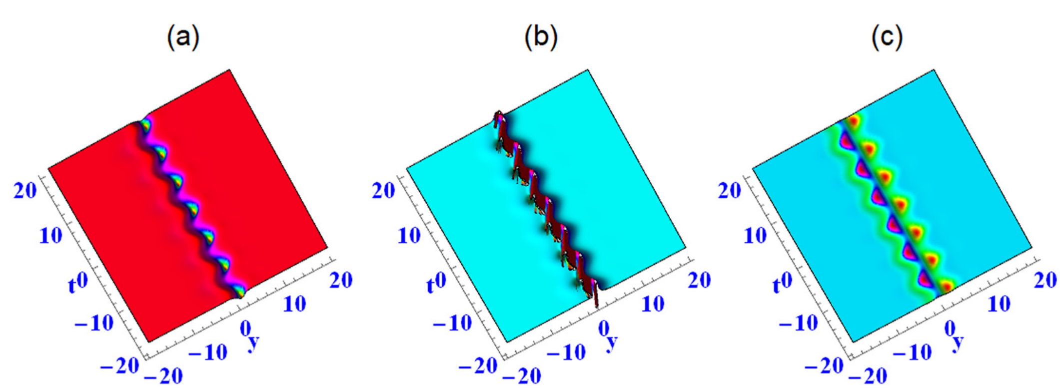

Figure 6: Solution (12) with $C_1 = -2$, $\nu_2 = \mu_1 = k_1(t) = \mu_2 = 1$, $\tau_3(t) = \sin t$, $\mu_3 = 0$; (a) $x = -15$, (b) $x = 0$, and (c) $x = 15$.

$$
\begin{aligned}
u = -\Bigg[& 2v_2C_1\sqrt{k_1(t)}\exp\left[\int\left(v_2^2\mu_1\tau_3(t) - \frac{k_1'(t)}{2k_1(t)}\right)\mathrm{d}t + \mu_1 y\right] \\
& \times \left[v_2x + \mu_2 y + \sin\left[\frac{1}{2}v_2^2(\mu_2 + \mu_3)\int\tau_3(t)\mathrm{d}t\right]\right. \\
& \left.+ \cos\left[-\frac{1}{2}v_2^2(\mu_2 + \mu_3)\int\tau_3(t)\mathrm{d}t + v_2x - \mu_3 y\right]\right]\Bigg] \Bigg/ \\
& \times \left[C_1\sqrt{k_1(t)}\exp\left[\int\left(v_2^2\mu_1\tau_3(t) - \frac{k_1'(t)}{2k_1(t)}\right)\mathrm{d}t + \mu_1 y\right]\right. \\
& \times \left[\cos\left[\frac{1}{2}v_2^2(\mu_2 + \mu_3)\int\tau_3(t)\mathrm{d}t + v_2x + \mu_2 y\right]\right. \\
& \left.- \sin\left[v_2x - \mu_3 y - \frac{1}{2}v_2^2(\mu_2 + \mu_3)\int\tau_3(t)\mathrm{d}t\right]\right] \\
& \left.+ k_1(t)\exp\left[2\left[\int\left(v_2^2\mu_1\tau_3(t) - \frac{k_1'(t)}{2k_1(t)}\right)\mathrm{d}t + \mu_1 y\right]\right] + 1\right].
\end{aligned}
\tag{12}
$$

Figure 4 demonstrates the dynamic properties of breather wave solution when variable coefficient $\tau_3(t)$ takes a constant. The dynamic properties of breather wave solution are shown in Figures 5 and 6 when $\tau_3(t)$ takes different functions.

5 Conclusion

In this work, we investigate a generalized (2+1)-dimensional variable-coefficient breaking soliton equation. Rich nonautonomous solutions with different wave structures are obtained by using the positive quadratic function method and symbolic computation [13–27]. The breather wave and the interaction among lump wave, solitary wave and periodic wave solutions are studied. The dynamic properties are shown in Figures 1–6. The breather wave and the interaction among lump wave, solitary wave and periodic wave solutions in this work can also be constructed by using the bilinear neural network method [28–30]. The assumptions used in this article contain more arbitrary parameters, which can be used to describe more complex physical background and obtain more different forms of solutions. The method is simple, direct and effective. If the bilinear form of a nonlinear partial differential equation can be obtained, the method of this article can be used to solve the equation.

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