Research Article

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On theoretical analysis of nonlinear fractional order partial Benney equations under nonsingular kernel

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Abstract: In the present article, the first step is devoted to develop some results about existence and uniqueness of solution to a general problem of fractional order partial differential equations (FPDEs) *via* classical fixed point theory. In the second step, a novel technique is used to handle the semi-analytical approximate solution for the considered general problem. Then, we extend the said result to fractional order partial Benney equations (FOPBEs) of the second and third order, which are special cases of the general problem we considered. We study the proposed problem under the Caputo-Febrizo fractional derivative (CFFD). With the help of the proposed method, we derive a series type approximate (semi-analytical) solution. Some numerical interpretations and visualizations are also given.

Keywords: Benny equation, CFFD, theoretical results

1 Introduction

The subject of arbitrary order calculus has been gotten tremendous attraction of researchers in last many years. This branch was founded in the 17th century by Newton and Leibnitz. But due to some complexities in the nature of derivative of arbitrary order, the formal investigation was started in 1819 when for the first time the mentioned derivative was defined for a simple power function, and for detail, see ref. [1]. Later, the respective derivative was defined by various researchers in number of ways in which the definition of Riemann–Liouville, Hadamard, Grownwälad, Fourier, *etc*. were greatly adopted; see refs. [2–4]. In 1967, Caputo modified the Riemann–Liuville definition slightly and the new definition he called Caputo derivative [5]. The mentioned derivative was greatly used in studding of applied problems of biological and physical models, see refs. [6–11] and many others in fractional models and nonlinear dynamical system [12–18].

Fractional calculus are increasingly used by mathematicians for mathematical modeling of most of the real-world problems. Often it cause difficulties in treating fractional derivatives involving singular kernel. To omit this difficulties, recently, Caputo and Fabrizio have defined a new fractional order derivative with non-singular kernel [19,20,22]. Interesting observation is that their fractional integral is the fractional average of the Riemann–Liouville fractional integral of the given function and the function itself. In addition to the aforementioned benefits, the derivative was found very useful in thermal science, material sciences, *etc*. (see refs. [23–25]).

Inspired from the aforesaid work, we investigate nonlinear FOPDE under the aforesaid derivative. To the best of our knowledge, the FOPBE has not yet investigated under the said derivative. To study the considered problem, we utilize Laplace transform coupled with the Adomian decomposition method to form an hybrid method abbreviated as (LADM). In this regard, to investigate a physical problem first, we need to derive its existence. Therefore, we first developed some necessary conditions that guarantee the existence of solution of the considered problem by using some results from the nonlinear analysis. In this view, we first developed sufficient conditions for existence criteria of solution to the given general problem:

$$\begin{aligned} {}^{\mathrm{CF}}_{0}\mathbf{D}^{\omega}_{t} w(x,t) &= \psi(t, w(x,t)), \quad 0 < \omega \leq 1, \\ w(0) &= = g(x), \end{aligned} \tag{1}$$

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where ${}^{\mathrm{CF}}_{0}\mathbf{D}$ stands for CFFD and $\psi \in C([0,T] \times R, R)$, $g \in ([0,T], R)$. For the existence theory, we use the classical fixed theory of Krasnoselskii's and Banach. Then, we come across the solution for the FOPBE under the CFFD.

Classical PBE is provided [28] as follows:

$$\begin{aligned} &w_t(x,t) + (w^m(x,t))_x + w_{xx}(x,t) + \kappa w_{xxx}(x,t) \\ &\quad + w_{xxxx}(x,t) = 0, \\ &w(x,0) = g(x), \end{aligned}$$

where κ is the positive constant called characterizing dispersion and m is positive integer, $g : R \to R$. PBEs describes long waves progression in various problems in fluid dynamics.

Corresponding to $m = 2$, the PBE has been investigated by extended homotopy perturbation method [30] given by

$$\begin{aligned} &w_t(x,t) + w(x,t)w_x(x,t) + w_{xx}(x,t) + \kappa w_{xxx}(x,t) \\ &\quad + w_{xxxx}(x,t) = 0, \\ &w(x,0) = g(x). \end{aligned} \tag{2}$$

Here, we remark that Eq. (2) is utilized in plasma for the description of the long waves on a viscous fluid flowing down an inclined plane and the unstable drift waves. If $\kappa = 0$, the said PDE reduces to chemical reaction equations, which is increasingly used in chemical mathematics. Further, if $m = 3$, one obtains

$$\begin{aligned} &w_t(x,t) + 3w^2(x,t)w(x,t)_x + w_{xx}(x,t) + \kappa w_{xxx}(x,t) \\ &\quad + w_{xxxx}(x,t) = 0, \\ &w(x,0) = g(x). \end{aligned} \tag{3}$$

Eq. (3) has many applications in solitons theory, computational fluid mechanics, dynamics, and physics; see refs. [26–29]. The mentioned equations have been investigated by Wavelets for ordinary fractional derivative in ref. [28]. Therefore, in this article, we study the following FOPBEs in nonsingular derivative for order $m = 2$ as well as $m = 3$ given by

$$\begin{aligned} &{}^{\mathrm{CF}}_{0}\mathbf{D}^{\omega}_{t} w(x,t) + w(x,t)w_x(x,t) + w_{xx}(x,t) + \kappa w_{xxx}(x,t) \\ &\quad + w_{xxxx}(x,t) = 0, \\ &w(x,0) = g(x) \end{aligned} \tag{4}$$

and

$$\begin{aligned} &{}^{\mathrm{CF}}_{0}\mathbf{D}^{\omega}_{t} w(x,t) + 3w^2(x,t)w(x,t)_x + w_{xx}(x,t) \\ &\quad + \kappa w_{xxx}(x,t) + w_{xxxx}(x,t), \\ &w(x,0) = g(x), \end{aligned} \tag{5}$$

where $\omega \in (0,1)$. With the help of the proposed technique, we derive the approximate analytical solution to Eqs. (4) and (5). Also some existence results are provided *via* using the nonlinear analysis. Graphical representations are also provided.

2 Preliminaries

Definition 2.1. [20] Let $\psi \in \mathbf{H}^1(a,b)$, $b > a$, $\omega \in (0,1)$, and then, the CF derivative is recalled as follows:

$${}^{\mathrm{CF}}\mathbf{D}^{\omega}_{t}(\psi(x,t)) = \frac{\mathcal{H}(\omega)}{1-\omega}\int_a^t \psi'(\theta,t)\exp\left[-\omega\frac{t-\theta}{1-\omega}\right]\mathrm{d}\theta,$$

where $\mathcal{H}(\omega)$ is the normalization function with $\mathcal{H}(1) = \mathcal{H}(0) = 1$.

Definition 2.2. [20] Let $\omega \in (0,1)$, then CF integral of Ψ is recalled as follows:

$${}^{\mathrm{CF}}_{a}\mathbf{I}^{\omega}_{t}[\psi(x,t)] = \frac{(1-\omega)}{\mathcal{H}(\omega)}\psi(x,t) + \frac{\omega}{\mathcal{H}(\omega)}\int_0^t \psi(x,\theta)\mathrm{d}\theta, \quad t \geq 0.$$

Remark 2.3. For some functions, we have [21]

(1) If $\psi(x,t)$ is constant then, ${}^{\mathrm{CF}}_{0}\mathbf{D}^{\omega}_{t}(\Psi(x,t)) = 0$.

(2) If $\psi(x,t) = t + x$, then ${}^{\mathrm{CF}}_{0}\mathbf{D}^{\omega}_{t}[t+x] = \frac{1}{\omega}\left[1 - \exp\left(\frac{\omega}{1-\omega}t\right)\right]$.

(3) If $\psi(x,t) = \sin t$, then

$${}^{\mathrm{CF}}_{0}\mathbf{D}^{\omega}_{t}[\sin t] = \frac{-\omega\exp\left(\frac{\omega}{1-\omega}t\right) + \omega\cos t + (1-\omega)\sin t}{1 - 2\omega + \omega^2}.$$

(4) If $\psi(x,t) = t\exp(t)$, then

$$\begin{aligned} &{}^{\mathrm{CF}}_{0}\mathbf{D}^{\omega}_{t}[t\exp(t)] \\ &\quad = \exp\left(\frac{\omega}{1-\omega}t\right)\left[-\omega + (\omega + t)\exp\left(\frac{1}{1-\omega}t\right)\right]. \end{aligned}$$

Definition 2.4. [22] The Laplace transform of any function ${}^{\mathrm{CF}}_{0}\mathbf{D}^{p+\omega}_{t}[\psi(x,t)]$ with $0 < \omega \leq 1$ is given by

$$\mathscr{L}[{}^{\mathrm{CF}}_{0}\mathbf{D}^{p+\omega}_{t}\psi(x,t)] = \frac{\mathcal{H}(\omega)}{(s + \omega(1-s))}[s\mathscr{L}[\psi(x,t) - \psi(x,0)]].$$

Definition 2.5. Further, if we take $\mathcal{H}(\omega) = 1$, then the Laplace transform of any function $\psi(x,t)$ is given as follows:

$$\begin{aligned} &\mathscr{L}[{}^{\mathrm{CF}}_{0}\mathbf{D}^{p+\omega}_{t}\psi(x,t)] \\ &\quad = \frac{1}{\omega - 1}\mathscr{L}[\psi^{(p+\omega)}(x,t)]\mathscr{L}\left[\exp\left(\frac{-\omega t}{1-\omega}\right)\right] \\ &\quad = \frac{s^{p+1}}{s + \omega(1-s)}\mathscr{L}[\psi^{(p+\omega)}(x,t)] - \sum_{k=0}^{p}\frac{s^{p-k}\psi^{(k)}(x,0)}{s + \omega(1-s)}. \end{aligned}$$

Let define space as $\mathscr{Y} = C[0,T]$ with the norm defined by $\|w\| = \max_{t\in[0,T]}|w(t)|$. For existence of at least one result as a solution of the problem (1), we recall the given theorem.

Theorem 2.6. [31] *If $D \subset \mathscr{Y}$ be closed nonempty and convex subset, $\exists$ two operators $\top_1$, $\top_2$ with the conditions*
(1) $\top_1 w_1 + \top_2 w_2 \in D, \quad \forall w_i \ (i = 1, 2) \in D$;
(2) $\top_1$ *is condensing map*;
(3) $\top_2$ *is compact and continues*;

then $\exists$ at least one result $w \in D$ satisfies

$$\top_1(w) + \top_2(w) = w.$$

3 Construction of some existence results

In this section, we construct a general procedure regarding existence, uniqueness, and then for iterative solution of the considered problem.

Lemma 3.1. *The solution of the considered problem* (1) *by using Definitions* 2.1 *and* 2.2 *is provided as follows*:

$$\begin{aligned} w(x,t) &= g(x) + \frac{(1-\omega)}{\mathcal{H}(\omega)}[\psi(t, w(x,t)) - \psi(0, w(x,0))] \\ &+ \frac{\omega}{\mathcal{H}(\omega)}\int_0^t \psi(\theta, w(x,\theta))\mathrm{d}\theta. \end{aligned} \tag{6}$$

We provide some prior assumptions needed onward as follows:

($A1$) The nonlinear function $\psi(t, w)$ satisfies the growth conditions as follows:

$$|\psi(t,w)| \le a_\psi + C|w|^q, \quad q \in (0,1), \quad C \ge 0.$$

($A2$) $\exists$ constant $L_\psi > 0$ such that $\forall w_1, w_2 \in R$ one has

$$|\psi(t,w_1) - \psi(t,w_2)| \le L_\psi |w_1 - w_2|, \quad \forall t \in [0,T].$$

Moreover, for ψ, the result $\psi(t, 0) = 0$ holds.

Now we define the operators $\top_1, \top_2 : \mathscr{Y} \to \mathscr{Y}$ as follows:

$$\begin{cases} \top_1 w(x,t) = g(x) + \dfrac{(1-\omega)}{\mathcal{H}(\omega)}[\psi(t, w(x,t)) \\ \quad - \psi(0, w(x,0))], \\ \top_2 w(x,t) = \dfrac{\omega}{\mathcal{H}(\omega)}\displaystyle\int_0^t \psi(\theta, w(x,\theta))\mathrm{d}\theta. \end{cases} \tag{7}$$

Theorem 3.2. *Inview of hypothesis* (A_i), $i = 1, 2$, *if* $1 > \frac{(1-\omega)}{\mathcal{H}(\omega)}L_\psi$, *then problem* (1) *has at least one solution.*

Proof. Let us utilize Theorem 2.6, so we define bounded set $D = \{w \in \mathscr{Y} : \|w\|_{\mathscr{Y}} \le \mathcal{R}\}$. Thank to the continuity of $\psi(t, w)$, operators $\top_1$, $\top_2$ are continues. To prove that $\top_1$ is condensing map, we take $w_1, w_2 \in D$ in view of A_1 and obtain

$$\begin{aligned} &\|\top_1(w_1) - \top_1(w_2)\|_{\mathscr{Y}} \\ &= \max_{t\in[0,T]} \left| \frac{(1-\omega)}{\mathcal{H}(\omega)}\psi(t, w_1(x,t)) \right. \\ &\quad \left. - \frac{(1-\omega)}{\mathcal{H}(\omega)}\psi(t, w_2(x,t)) \right| \\ &\le \frac{(1-\omega)}{\mathcal{H}(\omega)} L_\psi \|w_2 - w_1\|_{\mathscr{Y}}. \end{aligned} \tag{8}$$

Hence, $\top_1$ is a condensing map. Further to derive the results about compactness and continuity of $\top_2$, $\forall w \in D$, we take

$$\begin{aligned} \|\top_2(w)\|_{\mathscr{Y}} &= \max_{t\in[0,T]} \left| \frac{\omega}{\mathcal{H}(\omega)} \int_0^t \psi(\theta, w(x,\theta))\mathrm{d}\theta \right| \\ &\le \frac{\omega}{\mathcal{H}(\omega)} \max_{t\in[0,T]} \int_0^t |\psi(\theta, w(x,\theta))\mathrm{d}\theta \\ &\le \frac{\omega}{\mathcal{H}(\omega)} \int_0^t [a_\psi + C|w|^q]\mathrm{d}\theta \\ &\le \frac{\omega}{\mathcal{H}(\omega)}[a_\psi + C\mathcal{R}^q]T. \end{aligned}$$

Hence, $\top_2$ is bounded on D. Further, for equicontinuty, we take $t_1 > t_2$, such that

$$\begin{aligned} &|\top_2 w_1 - \top_2 w_2| \\ &= \left| \frac{\omega}{\mathcal{H}(\omega)} \left[\int_0^{t_1} \psi(\theta, w(x,\theta))\mathrm{d}\theta \right.\right. \\ &\quad \left.\left. - \int_0^{t_2} \psi(\theta, w(x,\theta))\mathrm{d}\theta \right] \right| \\ &\le \frac{\omega}{\mathcal{H}(\omega)}[a_\psi + C\mathcal{R}^q](t_1 - t_2). \end{aligned} \tag{9}$$

From Eq. (9), we observe that $\|\top_2 w_1 - \top_2(w_2)\| \to 0$, as $t_1 \to t_2$. In this way, we come to the conclusion that $\top_2$ is compact as well as equi-continuous due to *Arzelá-Ascoli's* theorem. Hence, by using Theorem 2.6, the considered problem (1) has at least one result, which lies in D. □

Theorem 3.3. *Under the hypothesis A_2 and if $\frac{1+\omega(T-1)}{\mathcal{H}}L_\psi < 1$, then the problem* (1) *under investigation has unique result.*

Proof. From Lemma 6, we define the operator $\top$ as follows:

$$\begin{aligned}\top w(x,t) &= g(x)+\frac{(1-\omega)}{\mathcal{H}(\omega)}[\psi(t,w(x,t))-\psi(0,w(x,0))]\\ &\quad+\frac{\omega}{\mathcal{H}(\omega)}\int_0^t\psi(\theta,w(x,\theta))\mathrm{d}\theta.\end{aligned}\tag{10}$$

Now consider $w_1, w_2 \in \mathscr{Y}$, we have

$$\begin{aligned}\|\top w_1-\top w_2\|_{\mathscr{Y}} &\le \max_{t\in[0,T]}\left|\frac{(1-\omega)}{\mathcal{H}(\omega)}[\psi(t,w_1(x,t))-\psi(t,w_2(x,t))]\right|\\ &\quad+\max_{t\in[0,T]}\left|\frac{\omega}{\mathcal{H}(\omega)}\int_0^t[\psi(\theta,w_1(x,\theta))-\psi(\theta,w_2(x,\theta))]\mathrm{d}\theta\right|\\ &\le \frac{(1-\omega)}{\mathcal{H}(\omega)}L_\psi\|w_1-w_2\|_{\mathscr{Y}}+\frac{\omega}{\mathcal{H}(\omega)}TL_\psi\|w_1-w_2\|_{\mathscr{Y}}\\ &= \left(\frac{1+\omega(T-1)}{\mathcal{H}(\omega)}L_\psi\right)L_\psi\|w_1-w_2\|_{\mathscr{Y}}.\end{aligned}$$

Hence, the operator $\top$ is condensing map, which leads us to the uniqueness result of the problem. □

4 General procedure of solution

Here, we consider a general problems of CFFDEs as follows:

$$\begin{cases}{}^{\mathrm{CF}}_{0}\mathbf{D}_t^{p+\omega}w(x,t)=\mathcal{N}w(x,t)+Rw(x,t)+f(x,t),\\ l-1<\omega\le l,\\ w^{(k)}(x,t)|_{t=0}=g_k(x),\ \ g_k\in([0,T],R),\\ \forall k=0,1,\ldots,l-1,\end{cases}\tag{11}$$

where $l=[\omega]+1$, $\mathcal{N}w(x,t)$ is nonlinear terms, while $Mw(x,t)$ represents linear terms and f is the source term of the considered problem (11). Employing Laplace transform on (11) and using initial condition, we obtain

$$\begin{aligned}&\frac{\mathcal{H}(\omega)}{(s+\omega(1-s))}[s^{p+1}\mathscr{L}[w(x,t)]-s^pw(x,0)-s^{p-1}w^{(1)}(x,0)\\ &-\cdots-w(x,0)]\\ &=\mathscr{L}[\mathcal{N}w(x,t)+Rw(x,t)+f(x,t)],\end{aligned}$$

$$\begin{aligned}\mathscr{L}[w(x,t)]&=\frac{1}{s^{p+1}}[s^pg_0(x)+s^{p+1}g_1(x)+\cdots+g_p(x)]\\ &\quad+\frac{1}{s^{p+1}}\left[\frac{(s+\omega(1-s))}{\mathcal{H}(\omega)}f(x,t)\right]\\ &\quad+\frac{(s+\omega(1-s))}{\mathcal{H}(\omega)}\frac{1}{s^{p+1}}\mathscr{L}[\mathcal{N}w(x,t)\\ &\quad+Mw(x,t)].\end{aligned}\tag{12}$$

Now let us assume that we want to compute the solution $w(x,t)$ in the form infinite series as $w(x,t)=\sum_{n=0}^{\infty}w_n(x,t)$ and expressing nonlinear terms in terms of Adomian polynomials as follows:

$$\begin{aligned}&\mathcal{N}w(x,t)=\sum_{p=0}^{\infty}\mathcal{P}_p(x,t),\\ &\text{where }\mathcal{P}_p=\frac{1}{\Gamma(p+1)}\frac{d^k}{\mathrm{d}\rho^k}\left[\mathcal{N}\left(\sum_{k=0}^{p}\rho^kw_k(x,t)\right)\right]\Bigg|_{\rho=0}.\end{aligned}\tag{13}$$

For instance, if $\mathcal{N}w(x,t)=w^2(x,t)$, then we have the following polynomials for $p=0,1,2,3,\ldots$:

$$\begin{aligned}\mathcal{P}_0(x,t)&=w_0^2(x,t),\\ \mathcal{P}_1(x,t)&=2w_0(x,t)w_1(x,t),\\ \mathcal{P}_2(x,t)&=2w_0(x,t)w_2(x,t)+w_1^2(x,t),\\ \mathcal{P}_3(x,t)&=2w_1(x,t)w_2(x,t)+2w_0(x,t)w_3(x,t),\end{aligned}$$

and so on. Hence, using the aforementioned representation in (12) and one has on comparing terms:

$$\begin{cases}\mathscr{L}[w_0(x,t)]=\frac{1}{s^{p+1}}[s^pg_0(x)+s^{p+1}g_1(x)+\cdots+g_p(x)]\\ \qquad+\frac{1}{s^{p+1}}\left[\frac{(s+\omega(1-s))}{\mathcal{H}(\omega)}f(x,t)\right],\\ \mathscr{L}[w_1(x,t)]=\frac{(s+\omega(1-s))}{\mathcal{H}(\omega)}\frac{1}{s^{p+1}}\mathscr{L}[\mathcal{P}_0(x,t)\\ \qquad+Mw_0(x,t)],\\ \mathscr{L}[w_2(x,t)]=\frac{(s+\omega(1-s))}{\mathcal{H}(\omega)}\frac{1}{s^{p+1}}\mathscr{L}[\mathcal{P}_1(x,t)\\ \qquad+Mw_1(x,t)],\\ \vdots\\ \mathscr{L}[w_{p+1}(x,t)]=\frac{(s+\omega(1-s))}{\mathcal{H}(\omega)}\frac{1}{s^{p+1}}\mathscr{L}[\mathcal{P}_p(x,t)\\ \qquad+Mw_p(x,t)],\quad p\ge 0.\end{cases}\tag{14}$$

Evaluating inverse Laplace transform in each step of (14), we obtain following series solution:

$$w(x,t)=w_0(x,t)+w_2(x,t)+w_3(x,t)+\cdots.\tag{15}$$

The obtained series is in the form of infinite series. Such series are mostly convergent (see ref. [32]) for ordinary Caputo derivative. In the same manner, we can also prove the convergence for the said derivative.

5 Applications of our method to proposed problem

Series type solutions to FOPBEs (4) and (5) under different initial conditions are computed as follows:

Example 5.1. Consider FOPBE in the second order $m=2$ under initial and boundary conditions as follows:

$$\begin{aligned}&{}^{CF}_{0}\mathbf{D}_t^{\omega}w(x,t)+w(x,t)w_x(x,t)+w_{xx}(x,t)\\&\quad+\kappa w_{xxx}(x,t)+w_{xxxx}(x,t)=0,\\&w(x,0)=\frac{1}{x},\quad w(0,t)=0.\end{aligned}\tag{16}$$

Inview of the proposed scheme, few terms of solution are as follows:

$$\begin{aligned}w_0(x,t)&=\frac{1}{x},\\w_1(x,t)&=\frac{-1}{\mathcal{H}(\omega)}\left[\frac{1}{x^2}+\frac{2}{x^3}+\frac{6\kappa}{x^4}+\frac{24}{x^5}\right](1+\omega(t-1)),\\w_2(x,t)&=\frac{-2}{(\mathcal{H}(\omega))^2}\left[\frac{2}{x^3}+\frac{7}{x^4}+\frac{(2+24\kappa)}{x^5}+\frac{(120+6\kappa)}{x^6}\right.\\&\quad\left.+\frac{24}{x^7}\right](1+\omega^2(t-1)+2\omega(t-1))\\&\quad+\frac{2}{(\mathcal{H}(\omega))^2}\left[\frac{6}{x^4}+\frac{24}{x^5}+\frac{120\kappa}{x^6}+\frac{720}{x^7}\right]\\&\quad\times(1+\omega^2(t-1)+2\omega(t-1))\\&\quad+\frac{2\kappa}{(\mathcal{H}(\omega))^2}\left[\frac{24}{x^5}+\frac{120}{x^6}+\frac{720\kappa}{x^7}+\frac{5{,}040}{x^8}\right]\\&\quad\times(1+\omega^2(t-1)+2\omega(t-1))\\&\quad+\frac{2}{(\mathcal{H}(\omega))^2}\left[\frac{120}{x^6}+\frac{720}{x^7}+\frac{5{,}040\kappa}{x^8}+\frac{40{,}320}{x^9}\right]\\&\quad\times(1+\omega^2(t-1)+2\omega(t-1)),\\&\;\vdots,\end{aligned}$$

and so on. In this manner, we compute the remaining terms. Therefore, the series solution becomes

$$w(x,t)=w_0(x,t)+w_1(x,t)+w_2(x,t)+\cdots.\tag{17}$$

We present the 3D plot in Figure 1, the approximate solutions up to first three terms for different values of ω and taking $\kappa=0.5$. We observe as the fractional order ω approaches to its integer value, the approximate solution tending to the classical order result.

In Figure 2, we provide plots at fixed time $t=1$ and different fractional order ω.

Example 5.2. Consider the FOPBE (5) of third order as follows:

$$\begin{aligned}&{}^{CF}_{0}\mathbf{D}_t^{\omega}w(x,t)+3w^2(x,t)w_x(x,t)+w_{xx}(x,t)\\&\quad+\kappa w_{xxx}(x,t)+w_{xxxx}(x,t)=0,\\&w(x,0)=x^3.\end{aligned}\tag{18}$$

Here, we see that $g_0(x)=x^3$, $f(x,t)=0$ and $\omega\in(0,1]$. Also

$$\begin{aligned}\mathcal{N}w(x,t)&=3w^2(x,t)w_x(x,t),\\Mw(x,t)&=w_{xx}(x,t)+\kappa w_{xxx}(x,t)+w_{xxxx}(x,t).\end{aligned}$$

Hence, inview of Eq. (14), we have

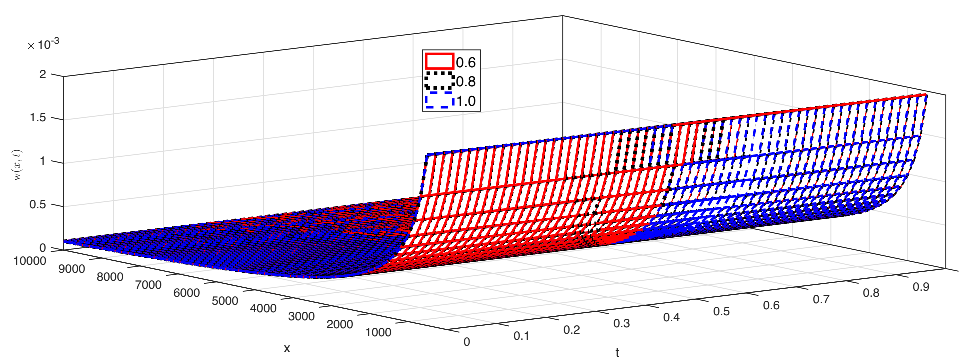

Figure 1: Plot of first three terms approximate solution at various values of ω for Example 5.1 with $\kappa=0.5$.

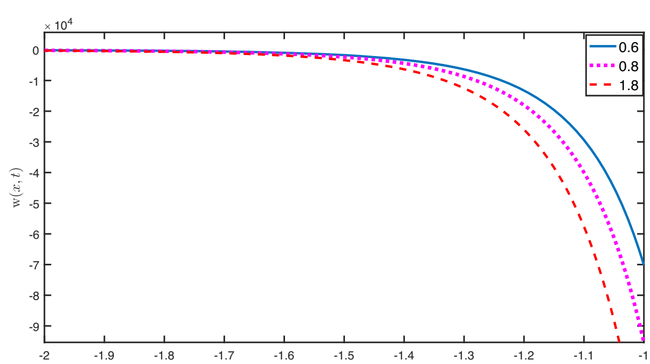

Figure 2: Plot of first three terms approximate solution at various values of ω for Example 5.1 with $\kappa=0.5$ and $t=1$.

$$
\begin{aligned}
w_0(x,t) &= x^3,\\
w_1(x,t) &= \frac{1}{\mathcal{H}(\omega)}[9x^8 + 6(x+\kappa)](1+\omega(t-1)),\\
w_2(x,t) &= \frac{2}{(\mathcal{H}(\omega))^2}[162x^{13} + 216x^{12} + 630x^6\\
&\quad + 3{,}132\kappa x^5 + 15{,}120x^4]\\
&\quad \times \left(1 + \omega^2\left(\frac{t^2}{2!} + 1 - 2t\right) + 2\omega\left(\frac{t^2}{2!} - t\right)\right),\\
w_3(x,t) &= \frac{16}{(\mathcal{H}(\omega))^3}(162x^{16} + 216x^{15} + 630x^9 + 3{,}132\kappa x^8\\
&\quad + 15{,}120x^7)\left[1 + 2\omega\left(\frac{t^2}{2!} - t\right)\right.\\
&\quad + \omega^2\left(\frac{2t^3}{3!} - \frac{3t^2}{2!} + 2t - 1\right)\\
&\quad \left. + \omega^3\left(\frac{t^3}{3!} - \frac{3t^2}{2!} + 3t - 1\right)\right]\\
&\quad + \frac{216(3x^8 + 2(x+\kappa))^2}{(\mathcal{H}(\omega))^3}[1 + 3\omega(t-1)\\
&\quad \left. + \omega^2(2t^2 - 6t + 3) + \omega^3\left(\frac{t^3}{3} - 2t^2 + 3t - 1\right)\right]\\
&\quad + (25{,}272x^{11} + (28{,}512 + 277{,}992\kappa)x^{10}\\
&\quad + (285{,}120\kappa + 2{,}779{,}920)x^9 + 2{,}566{,}080x^8\\
&\quad + 18{,}900x^4 + (62{,}640 + 75{,}600\kappa)x^3\\
&\quad + (408{,}240 + 187{,}920))\\
&\quad \times \left[1 + 2\omega\left(\frac{t^2}{2!} - t\right) + \omega^2\left(\frac{2t^3}{3!} - \frac{3t^2}{2!} + 2t - 1\right)\right.\\
&\quad \left. + \omega^3\left(\frac{t^3}{3!} - \frac{3t^2}{2!} + 3t - 1\right)\right],
\end{aligned}
\tag{19}
$$

and so on. In this way, we obtain the series solution of Example 5.2 as follows:

$$w(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + \cdots .$$

We present 3D plots in Figure 3, the approximate solutions up to first few terms for different values of ω and taking $\kappa = 1.0$. We observe as the fractional order ω approaches to its integer value, the approximate solution tending to the classical order result.

In Figure 4, we provide plots at fixed time and different fractional order.

Example 5.3. Consider another example of FBPDE as follows:

$$
\begin{aligned}
&{}^{\mathrm{CF}}_{0}\mathbf{D}_t^{\omega} w(x,t) + 3w^2(x,t)w_x(x,t) + w_{xx}(x,t)\\
&\quad + \kappa w_{xxx}(x,t) + w_{xxxx}(x,t) = 0,\\
&w(x,0) = \sin x.
\end{aligned}
\tag{20}
$$

Here, $g_0(x) = \sin x$, $f(x,t) = 0$ and $\omega \in (0,1]$. On the proposed method, we compute few terms of solution as follows:

$$
\begin{aligned}
w_0(x,t) &= \sin x,\\
w_1(x,t) &= \frac{1}{\mathcal{H}(\omega)}[(3\sin^2 x + \kappa)(x+\kappa)](1+\omega(t-1)),\\
w_2(x,t) &= \frac{1}{(\mathcal{H}(\omega))^2}[-3\sin 2x(3\sin^2 x + \kappa)\cos x\\
&\quad - \sin x\cos x - 6\sin 2x\cos x + 3\sin 2x\sin x\\
&\quad + 9\sin^2 x\cos x - \kappa\cos x - \kappa(15\sin 2x\cos x\\
&\quad + 12\cos 2x\sin x + 9\sin 2x - 9\sin^3 x\kappa\sin x)\\
&\quad + 42\cos 2x\cos x - 39\sin 2x\sin x + 18\cos 2x\\
&\quad - 27\sin^2 x\cos x + \kappa\cos x]\\
&\quad \times \left[1 + 3\omega(t-1) + 3\omega^2\left(\frac{t^2}{2!} - 2t + 1\right)\right.\\
&\quad \left. + \omega^3\left(\frac{t^3}{3!} - 3\frac{t^2}{2!} + 3t - 1\right)\right],
\end{aligned}
\tag{21}
$$

and so on. In this fashion, the reaming terms can be computed to get the series type solution as follows:

$$w(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + \cdots . \tag{22}$$

We present 3D plots in Figure 5, the approximate solutions up to three first terms for different values of ω and taking $\kappa = 0.5$. We observe as the fractional order ω approaches to its integer value, the approximate solution tending to the classical order result of the considered problem.

In Figure 6, we provide plots at fixed time and different fractional order.

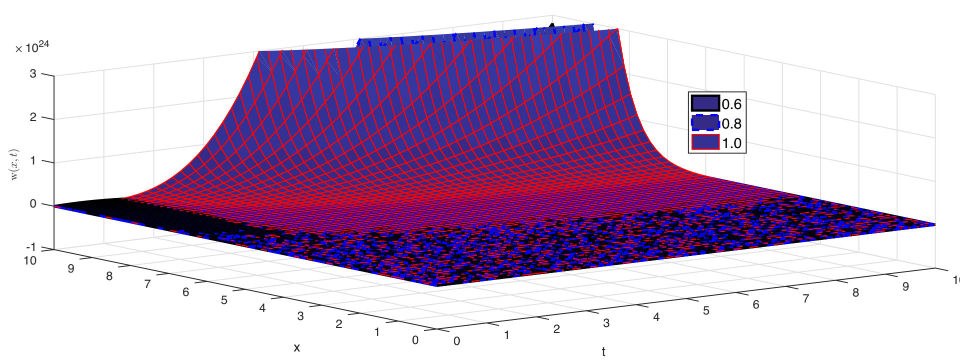

Figure 3: Plot of first three terms approximate solution at various values of ω for Example 5.2 with $\kappa = 0.5$.

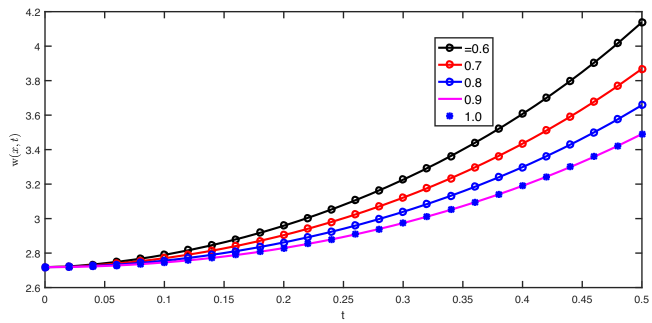

Figure 4: Plot of first three terms approximate solution at various values of ω for Example 5.2 with $\kappa = 0.5$ and $t = 10$.

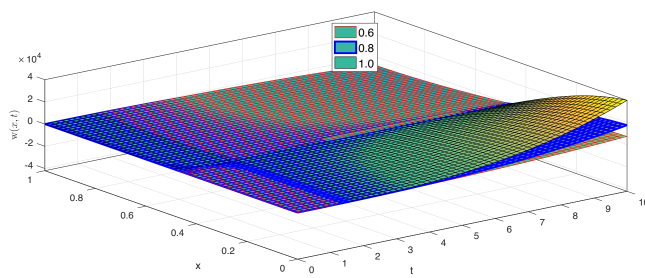

Figure 5: Plot of first three terms approximate solution at various values of ω for Example 5.3 with $\kappa = 0.5$.

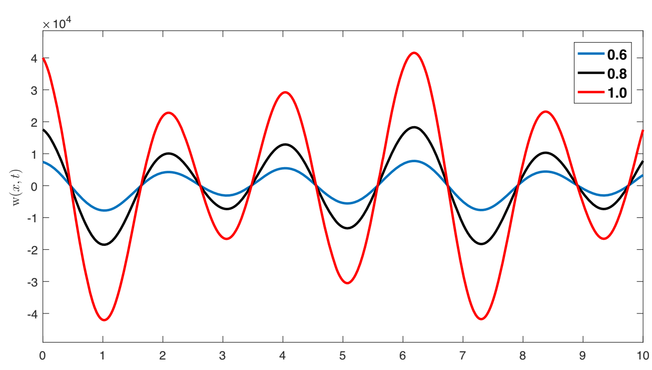

Figure 6: Plot of first three terms approximate solution at various values of ω for Example 5.3 with $\kappa = 0.5$ and $t = 10$.

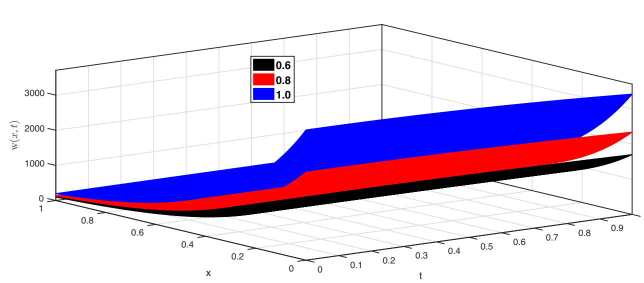

Figure 7: Plot of first three terms approximate solution at various values of ω for Example 5.4 with $\kappa = 0.5$.

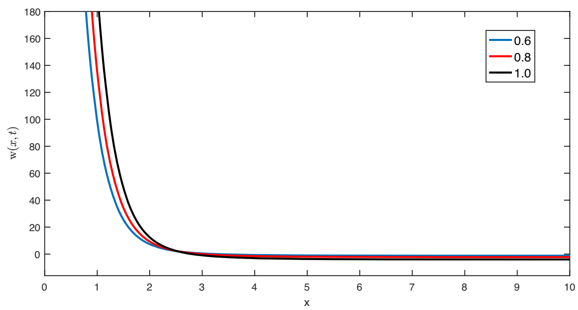

Figure 8: Plot of first three terms approximate solution at various values of ω for Example 5.3 with $\kappa = 0.5$ and $t = 10$.

Example 5.4. Consider another example of FBPDE as follows:

$$\begin{aligned} &{}^{\mathrm{CF}}_{0}\mathbf{D}^{\omega}_{t}w(x,t) + 3w^2(x,t)w_x(x,t) + w_{xx}(x,t) \\ &\quad + \kappa w_{xxx}(x,t) + w_{xxxx}(x,t) = 0, \\ &w(x,0) = \exp(x). \end{aligned} \tag{23}$$

Here, $g_0(x) = \exp(x)$, $f(x,t) = 0$ and $\omega \in (0, 1]$. Upon utilizing the proposed method, we obtain the following few terms:

$$\begin{aligned} w_0(x,t) &= \exp(-x), \\ w_2(x,t) &= \frac{1}{\mathcal{H}(\omega)}[2\exp(-x) + 3\exp(-x) \\ &\quad + \kappa\exp(-x)](1 + \omega(t-1)), \\ w_3(x,t) &= \frac{2}{(\mathcal{H}(\omega))^2}[6\exp(-x) + 759\exp(-3x) \\ &\quad + 4\kappa\exp(-x) + 12\exp(-2x) \\ &\quad - 81\kappa\exp(-3x)] \\ &\quad \times \left(1 + \omega^2\left(1 - \frac{t^2}{2!}\right) + 2\omega(t-1)\right), \end{aligned} \tag{24}$$

and so on. In this manner, all other terms can be computed to get the required series solution. In Figure 7, we provide a 3D plots for the approximate solutions at different fractional order. We observe as the fractional order ω approaches to its integer value, the approximate solution tending to the classical order result.

In Figure 8, we provide plots at fixed time and different fractional order.

6 Concluding remarks

In this work, we have applied a novel semi-analytical method to compute series solutions to FOBEs of second and third order successfully. The respective method is an hybrid method, which generates solutions in the form of infinite power series, which is rapidly convergent. Here, we have for the first times applied the mentioned method for the mentioned nonlinear equations under CFFD. From the visualizations of solutions through plots, we observed that the mentioned method can be successfully extended to nonlinear CFFODEs.

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