DE GRUYTER Open Physics 2022; 20: 259–264

Research Article

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Finite-size effects in one-dimensional Bose–Einstein condensation of photons

https://doi.org/10.1515/phys-2022-0031
received February 18, 2022; accepted March 29, 2022

Abstract: The finite-size effect plays a key role in one-dimensional Bose–Einstein condensation (BEC) of photons since such condensation cannot occur in the thermodynamic limit due to the linear dispersion relation of photons. However, since a divergence difficulty arises, the previous theoretical analysis of the finite-size effect often only gives the leading-order contribution. In this article, by using an analytical continuation method to overcome the divergence difficulty, we give an analytical treatment for the finite-size effect in BEC. We show that the deviation between experiment and theory becomes much smaller by taking into account the next-to-leading correction.

Keywords: Bose–Einstein condensation, photon condensation, finite-size effect, heat kernel expansion

1 Introduction

The Bose–Einstein condensation (BEC) of photons was generally believed to be impossible since the number of photons is not conserved and the extremely weak interaction between photons cannot thermalize the gas. However, the situation changed in recent years. By trapping photons in a dye-filled microcavity, the BEC has been realized in two-dimensional systems [1–4]. In these experiments, the photons are trapped between two curved mirrors. The fixed longitudinal momentum gives an effective mass to the photon and a nonvanishing chemical potential to the photon gas. The repeated absorbtion and emission cycle of the dye molecules thermalizes the photon gas. Recently, a one-dimensional photon condensation is also reported [5]. In the experiment, the photons are confined in a closed Erbium–Ytterbium co-doped fiber with a cutoff wavelength. The existence of the cutoff wavelength gives the photons a nonvanishing chemical potential.

In the experiments of the photon condensation, the finite particle number makes the behavior of the phase transition different from the thermodynamic limit case. In particular, the finite-size effect in the one-dimensional condensation is of special interest since such a condensation cannot occur in thermodynamic limit due to the linear dispersion relation of photons. The finite-size effect has a significant influence in this case and needs to be carefully analyzed.

Many studies have been devoted to the finite-size effect in BEC. However, besides the numerical calculation method [6,7], the previous approximate methods can only give the leading-order correction to the critical temperature and the condensate fraction [8–10]. The main obstacle for accurately studying the finite-size effect is the divergence problem: When taking into account the contribution from the discrete energy levels of the trapped particles accurately, most terms in the expressions of thermodynamic quantities become divergent at the transition point.

To overcome the divergence difficulty, we will use an analytical continuation method [11,12] to give an analytical treatment to the problem of photon condensation. In this way, we will obtain more accurate expressions of critical temperature and condensate fraction with next-to-leading corrections. We will also give the analytical expression of the chemical potential, which is hard to obtain before. Our result shows that the chemical potential is linear in temperature at low temperature, which is quite different from the thermodynamic limit case. The comparison with the numerical solution confirms our result.

In the experiment of one-dimensional photon condensation [5], the deviation of the critical particle number between experiment and theory is about 5.6%. However, according to our result, this deviation is mainly caused by the inaccurate estimate of the finite-size effect in the previous studies. If the finite-size effect is correctly taken into

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account, the deviation between experiment and theory will reduce to about 1.4%, *i.e.*, the agreement is actually very well.

This article is organized as follows. In Section 2, we give an analytical treatment to the finite-size effect of the photon condensation in one dimension. In Section 3, we compare our result with the experiment. Conclusions and some discussions are presented in Section 4.

2 Critical temperature and chemical potential

Consider photons in a one-dimensional closed fiber with length L and index of refraction n. The possible frequencies of the photons are restricted by periodic boundary conditions as

$$\omega = m' \frac{2\pi c}{nL} \equiv m'\Delta, \tag{1}$$

where m' is an integer, and we have introduced $\Delta \equiv 2\pi c / nL$ with c as the speed of light in vacuum. If there is a cutoff frequency $\omega_0 = m_0\Delta$, namely, only the photons with a frequency higher than ω_0 can exist in the fiber and the quantum number m' in Eq. (1) must not be less than m_0. For convenience, we shift the energy spectrum to make the ground-state energy vanish. Then, the spectrum of the photons in the fiber becomes

$$\varepsilon_m = m\hbar\Delta, \quad (m = 0, 1, 2, 3, \dots). \tag{2}$$

The photon in such a system has the same energy spectrum as that of nonrelativistic particles in a one-dimensional harmonic trap, and hence, these two kinds of systems should show the same transition behavior.

As we know, the BEC occurs when the number of excited particles N_e equals the total number of particles N at the chemical potential $\mu = 0$. The excited photon number is

$$N_e = \sum_{m=1}^{\infty} \frac{1}{e^{\beta(\varepsilon_m - \mu)} - 1}, \tag{3}$$

where $\beta = 1/k_B T$ with k_B the Boltzmann constant. In the thermodynamic limit, the energy spectrum becomes continuous and the density of states is $\rho(\varepsilon) = 1/\hbar\Delta$, and the summation in Eq. (3) is converted to an integral as follows:

$$N_e = \frac{1}{\hbar\Delta} \int_0^{\infty} \frac{1}{e^{\beta(\varepsilon - \mu)} - 1} d\varepsilon = \frac{1}{\beta\hbar\Delta} g_1(e^{\beta\mu}), \tag{4}$$

where $g_\sigma(z) = \sum_{\ell=1}^{\infty} z^\ell / \ell^\sigma$ is the Bose–Einstein integral, which has the following asymptotic behavior

$$g_\sigma(e^{\beta\mu}) \approx \begin{cases} \zeta(\sigma), & (\sigma > 1) \\ -\ln(-\beta\mu), & (\sigma = 1) \\ \Gamma(-\sigma + 1) \dfrac{1}{(-\beta\mu)^{-\sigma+1}}, & (\sigma < 1) \quad (\mu \to 0). \end{cases} \tag{5}$$

In the thermodynamic limit, N_e in Eq. (4) is divergent at $\mu = 0$. This implies that there is no phase transition (in fact, under the continuous-spectrum condition, the ground-state number is also included in N_e. However, subtracting the ground-state number from N_e cannot avoid the divergence difficulty).

On the other hand, in finite systems, the energy spectrum is discrete and the first excited energy is not 0, the summation in Eq. (3) should be convergent and a finite critical temperature can be obtained. In ref. [5], the summation is approximately converted to an integral similar to Eq. (4), but the lower limit of the integral is replaced by the first excited energy $\hbar\Delta$. Then, the critical particle number can be calculated as follows [5]:

$$N_c^{(0)} = \frac{k_B T}{\hbar\Delta} \ln \frac{k_B T}{\hbar\Delta}. \tag{6}$$

In this treatment, the interval between the ground state and the first excited state is taken into account, but the higher levels are still regarded as continuous. In fact, many previous studies of BEC in finite systems along the similar line. The finite-size effect of the BEC in one-dimensional harmonic trap is also discussed in refs. [8–10]. Although the treatments have some difference, they all depended on similar approximations and can only give the leading-order correction similar to Eq. (6) (may differ by a factor).

Obviously, a more rigorous treatment of Eq. (3) is to perform the summation directly. To do this, one can Taylor expand every term in the summation as follows:

$$\begin{aligned} N_e &= \sum_{m=1}^{\infty} \frac{1}{e^{\beta(\varepsilon_m - \mu)} - 1} \\ &= \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} [e^{-\beta(\varepsilon_m - \mu)}]^\ell \\ &= \sum_{\ell=1}^{\infty} e^{\ell\beta\mu} K(\ell\beta\hbar\Delta), \end{aligned} \tag{7}$$

where

$$K(t) = \sum_{m=1}^{\infty} e^{-mt} = \frac{1}{e^t - 1} \tag{8}$$

is the global heat kernel [13–15]. For small t, the heat kernel (8) can be expanded as a series of t,

$$K(t) = \sum_{k=0}^{\infty} C_k t^{k-1}, \quad (t \to 0^+) \tag{9}$$

with the coefficients

$$\begin{aligned} &C_0 = 1, \quad C_1 = -\frac{1}{2}, \quad C_2 = \frac{1}{12}, \\ &C_3 = 0, \quad C_4 = -\frac{1}{720}, \ \ldots. \end{aligned} \tag{10}$$

Substituting the heat kernel expansion (9) into Eq. (7), we have

$$N_{\mathrm{e}} = \sum_{k=0}^{\infty} C_k (\beta\hbar\Delta)^{k-1} g_{1-k}(e^{\beta\mu}). \tag{11}$$

A similar treatment can also apply to the grand potential and other thermodynamic quantities, and these quantities are also expressed as the series of the Bose–Einstein integrals. The higher-order correction terms can describe the influence of the boundary, the potential, or the topology, depending on the details of specific systems. This heat kernel expansion approach has been applied to various problems in statistical physics [13,16]. However, a serious difficulty arises when considering the problem of BEC phase transition. Due to the asymptotic form of the Bose–Einstein integral Eq. (5), every term in equation (11) is divergent at $\mu \to 0$, and the divergence becomes more severe in the higher orders. This divergence difficulty is the main obstacle for treating the problem of phase transition in finite systems. As mentioned earlier, in ref. [5], the divergence is avoided by replacing the summation of excited states with an integral approximately, but this approach only gives the leading-order correction to the critical temperature. If we want to obtain a more accurate result, the divergence problem in Eq. (11) must be solved. In the following, we will use an analytical continuation method [11,12] based on the heat kernel expansion to overcome the divergence problem.

First, substituting the leading term of the asymptotic expansion of each Bose–Einstein integral (5) into Eq. (11) gives

$$\begin{aligned} N_{\mathrm{e}} = &-C_0 \frac{\ln(-\beta\mu)}{\beta\hbar\Delta} \\ &+ \sum_{k=1}^{\infty} C_k (\beta\hbar\Delta)^{k-1} \Gamma(k) \frac{1}{(-\beta\mu)^k}. \end{aligned} \tag{12}$$

The summation in the second term can be represented by the heat kernel if the gamma function is replaced by the integral form

$$\Gamma(\xi) = \int_0^{\infty} x^{\xi-1+s} e^{-x} \mathrm{d}x, \quad (s \to 0), \tag{13}$$

where we have introduced a small parameter s, which will be taken as 0 at the end of the calculation. Then, Eq. (12) becomes

$$\begin{aligned} N_{\mathrm{e}} &= -C_0 \frac{\ln(-\beta\mu)}{\beta\hbar\Delta} + \int_0^{\infty} \mathrm{d}x e^{-x} x^s \sum_{k=1}^{\infty} C_k (\beta\hbar\Delta)^{k-1} x^{k-1} \frac{1}{(-\beta\mu)^k} \\ &= -C_0 \frac{\ln(-\beta\mu)}{\beta\hbar\Delta} + \int_0^{\infty} \mathrm{d}x e^{-x} x^s \frac{1}{-\beta\mu} \sum_{k=1}^{\infty} C_k \left(\frac{x\hbar\Delta}{-\mu} \right)^{k-1}. \end{aligned} \tag{14}$$

The summation in the last term differs from the heat kernel expansion (9) only by one term corresponding to $k = 0$, and it can be expressed by the heat kernel as follows:

$$\begin{aligned} N_{\mathrm{e}} &= -C_0 \frac{\ln(-\beta\mu)}{\beta\hbar\Delta} + \int_0^{\infty} \mathrm{d}x e^{-x} x^s \frac{1}{-\beta\mu} \left[K\left(\frac{x\hbar\Delta}{-\mu} \right) - C_0 \frac{-\mu}{x\hbar\Delta} \right] \\ &= -C_0 \frac{\ln(-\beta\mu)}{\beta\hbar\Delta} + \frac{\Gamma(1+s)}{-\beta\mu} \sum_{m=1}^{\infty} \left(1 + \frac{m\hbar\Delta}{-\mu} \right)^{-1-s} - C_0 \frac{\Gamma(s)}{\beta\hbar\Delta}. \end{aligned} \tag{15}$$

In the last step, the definition of heat kernel (8) has been employed to perform the integral, *i.e.*,

$$\begin{aligned} &\int_0^{\infty} \mathrm{d}x e^{-x} x^s \sum_{m=1}^{\infty} e^{-m\frac{x\hbar\Delta}{-\mu}} \\ &= \sum_{m=1}^{\infty} \Gamma(1+s) \left(1 + \frac{m\hbar\Delta}{-\mu} \right)^{-1-s}. \end{aligned} \tag{16}$$

For $\mu \to 0$, the summation in Eq. (15) becomes

$$\begin{aligned} \sum_{m=1}^{\infty} \left(1 + \frac{m\hbar\Delta}{-\mu} \right)^{-1-s} &\approx \sum_{m=1}^{\infty} \left(\frac{m\hbar\Delta}{-\mu} \right)^{-1-s} \\ &= \zeta(1+s) \frac{(-\mu)^{1+s}}{(\hbar\Delta)^{1+s}}, \end{aligned} \tag{17}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann ζ-function.

Now taking the limit $s \to 0$ in Eq. (15), we have

$$\begin{aligned} N_{\mathrm{e}} &\approx -\frac{\ln(-\beta\mu)}{\beta\hbar\Delta} + \frac{1}{\beta\hbar\Delta} \left(\ln \frac{-\mu}{\hbar\Delta} + \gamma_{\mathrm{E}} \right) \\ &= \frac{1}{\beta\hbar\Delta} \left(\ln \frac{1}{\beta\hbar\Delta} + \gamma_{\mathrm{E}} \right), \end{aligned} \tag{18}$$

where the Euler constant $\gamma_{\mathrm{E}} = 0.577216$. In this result, all the divergent terms of s and μ are canceled, and the expression of the number of excited particles is completely analytical. That is to say, with the help of the idea of analytical continuous, the heat kernel expansion is successfully applied to the phase transition point and the divergence is eliminated.

From Eq. (18), the critical particle number for a given temperature T is obviously

$$N_c = \frac{k_B T}{\hbar\Delta}\left(\ln\frac{k_B T}{\hbar\Delta} + \gamma_E\right). \tag{19}$$

The critical temperature for a fixed particle number N is

$$T_c = \frac{\hbar\Delta}{k_B}\frac{N}{W(Ne^{\gamma_E})}, \tag{20}$$

where $W(z)$ is the Lambert W function, satisfying $z = W(ze^z)$. This critical temperature is lower than the previous result corresponding to the critical particle number (6)

$$T_0 = \frac{\hbar\Delta}{k_B}\frac{N}{W(N)}. \tag{21}$$

According to the asymptotic expansion of the Lambert function $W(x) \approx \ln x - \ln\ln x$ for $x \to \infty$, the critical temperature can be approximated as follows:

$$T_c \approx \frac{\hbar\Delta}{k_B}\frac{N}{\ln N + \gamma_E - \ln(\ln N + \gamma_E)}. \tag{22}$$

We retain the second term in the denominator since for a relative small particle number, *e.g.*, $N \sim 10^4$, $\ln\ln N$ is not much smaller than $\ln N$.

The condensate fraction is straightforward from Eq. (18),

$$\frac{N_0}{N} = 1 - \frac{1}{N}\frac{k_B T}{\hbar\Delta}\left(\ln\frac{k_B T}{\hbar\Delta} + \gamma_E\right). \tag{23}$$

This result is not very accurate, especially near the transition point. The reason is that the chemical potential is taken as 0 below the transition point in the aforementioned calculation, which is of course an approximation. As the temperature tends to the transition point, the deviation of the chemical potential from 0 becomes larger and larger. To describe the phase transition more accurately, we need to find the expression of the chemical potential.

The chemical potential μ can be addressed by the help of the analytical result of the number of excited particles (18). For a small but nonzero chemical potential μ, the ground-state particle number $1/(-\beta\mu)$ is not zero at the phase transition point. Then, the total particle number N should contain the contributions from both the ground state and the excited states:

$$N = \frac{1}{-\beta\mu} + N_e. \tag{24}$$

Here, the number of excited particles N_e takes the same form as equation (15), but in the summation (17), an extra term that is proportional to μ should be added. Similar to the aforementioned procedure, we can obtain

$$\begin{aligned} N_0 = &\frac{1}{-\beta\mu} - \zeta(2)\frac{-\beta\mu}{(\beta\hbar\Delta)^2} \\ &- \frac{1}{2} + \frac{1}{2}\frac{-\beta\mu}{\beta\hbar\Delta}, \end{aligned} \tag{25}$$

where N_0 has been given in Eq. (23). In the right-hand side of Eq. (25), the last two terms are small. After neglecting these two terms, the chemical potential can be solved as follows:

$$\begin{aligned} \mu = -\frac{\sqrt{6}}{\pi}\hbar\Delta\Bigg[&\sqrt{1 + \left(\frac{\sqrt{3}}{\sqrt{2}\pi}\frac{T_0}{T}\frac{N_0}{N/\ln N}\right)^2} \\ &- \frac{\sqrt{3}}{\sqrt{2}\pi}\frac{T_0}{T}\frac{N_0}{N/\ln N}\Bigg]. \end{aligned} \tag{26}$$

An interesting feature of this result is that at low temperature $T \ll T_c$, the chemical potential is expressed as follows:

$$\mu \approx -\hbar\Delta\frac{1}{\ln N}\frac{T}{T_0}, \quad (T \ll T_c), \tag{27}$$

which is linearly related to the temperature. This is different from the thermodynamic limit result

$$\mu = -k_B T e^{-\frac{\hbar\Delta}{k_B T}N}, \quad (T \ll T_c), \tag{28}$$

which is exponentially small and leads to an unreasonable large particle number in the ground state at low temperature. In Figure 1, we compare the chemical potential in Eq. (26) with the thermodynamic limit result and the exact numerical solution, and it confirms the aforementioned low-temperature behavior.

3 Comparison with the experiment

In Eq. (19), we present the critical particle number of BEC in the one-dimensional photon system. Compared with the previous result [5] given in Eq. (6), the leading-order term is the same, but our result also gives a new next-to-leading correction term. This next-to-leading correction leads to a relative deviation as follows:

$$\frac{N_c - N_c^{(0)}}{N_c^{(0)}} = \frac{\gamma_E}{\ln\frac{k_B T}{\hbar\Delta}} \sim \frac{\gamma_E}{\ln N_c^{(0)}}. \tag{29}$$

It indicates that the previous treatment in which the excited states are regarded as continuous gives a lower prediction of the order of $1/\ln N$, which is usually not very small in realistic systems.

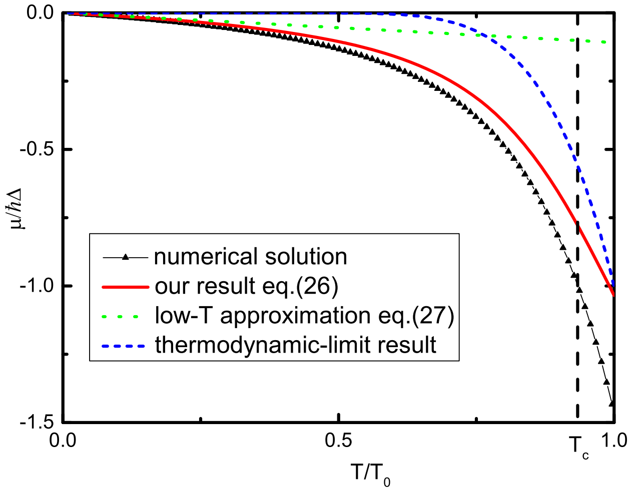

Figure 1: The chemical potential below the critical temperature for total particle number $N = 10^4$. Our result of critical temperature T_c is lower than the previous result T_0. At low temperature, the chemical potential is approximately linearly related to the temperature.

The following are the relevant experimental parameters [5]. The length of the fiber $L = 27$ m, the refraction coefficient $n = 1.444$, the critical temperature $T = 296$ K, and the cutoff wavelength $\lambda_0 = 1{,}568$ nm. Then, the critical particle numbers given by Eqs. (6) and (19) are as follows:

$$N_c^{(0)} = 1.09 \times 10^7, \quad N_c = 1.14 \times 10^7. \tag{30}$$

Our prediction of N_c is about 4.2% higher than $N_c^{(0)}$ given in ref. [5].

In the experiment [5], the measured quantity is the pump power, which is proportional to the photon number, and the measurement result is $P_c^{\text{exp}} = 9.5\,\mu\text{W}$. Compared with the theoretical prediction $P_c^{(0)} = 9.0\,\mu\text{W}$ [5], the experimental result is about 5.6% higher. This is not a large deviation, but according to the aforementioned analysis, most of the deviation is caused by the inaccurate theoretical prediction. Our result shows that the actual deviation of the critical particle number is only about 1.4%. Consequently, including the next-to-leading contribution of the finite-size effect greatly improves the agreement between experiment and theory.

4 Conclusion and discussion

In this article, we give a more systematic and accurate discussion on the finite-size effect in one-dimensional BEC of photons. By using an analytical continuous method based on the heat kernel expansion, we overcome the divergence difficulty and obtain the next-to-leading order finite-size corrections on thermodynamic quantities. In the experiment of one-dimensional photon BEC [5], the measurement value of the critical particle number is about 5.6% higher than the previous theoretical prediction. However, our result shows that the most part of the deviation arise from the inaccurate analysis of the finite-size effect. When taking into account of the next-to-leading correction, the deviation between experiment and theory reduces to about 1.4%. Moreover, the chemical potential at the low temperature given by our approach is also consistent with the exact solution, while the thermodynamic-limit result is physically unreasonable since it may lead to a too large ground-state particle number.

The magnitude of the finite-size effect is closely related to the spatial dimension of the system. In fact, the most important factor determining the statistical properties is the density of states, and the density of states is strongly affected by the spatial dimension. In some specific systems, the density of states has different behavior in different energy scales, which may significantly affect the critical temperature of BEC [17]. When considering the finite-size effect in a two-dimensional harmonic trap, the leading term of the finite-size correction to the critical temperature is of the order of $\ln N/\sqrt{N}$, and the next-to-leading correction has the order of $1/\sqrt{N}$ [12], which is often negligible. However, in the one-dimensional case, the leading correction is about $N/\ln N$, and our calculation gives the next-to-leading term of the order of $1/\ln N$, which is much larger than the two-dimensional case. In the thermodynamic limit, photon BEC cannot occur in one dimension, so the correction caused by the finite-size effect in one-dimensional system must be significant. The same behavior also appears in similar systems, *e.g.*, nonrelativistic particles in one-dimensional harmonic traps or in two-dimensional boxes.

The method used in this article is based on the heat kernel expansion. We know that the heat kernel expansion is a short-wavelength (high-energy) expansion. In principle, it is only applicable to the high-temperature and low-density case. When applying the heat kernel expansion to the problem of phase transition, the divergence problem arises indeed. In this article, however, we show that with the help of the analytical continuation method, the application range of heat kernel expansion can be extended to below the transition point, and the thermodynamic quantities can also be obtained analytically.

Funding information: The authors state no funding involved.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: Authors state no conflict of interest.

Data availability statement: The data that support the finding of this study are available from the corresponding author upon request.

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