Research Article

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Multiple velocity composition in the standard synchronization

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Abstract: Mansouri and Sexl (MS) presented a general framework for coordinate transformations between inertial frames, presupposing a preferred reference frame the space-time of which is isotropic. The relative velocity between inertial frames in the standard synchronization is shown to be determined by the first row of the transformation matrix based on the MS framework. Utilizing this fact, we investigate the relativistic velocity addition. To effectively deal with it, we employ a diagram of velocity that consists of nodes and arrows. Nodes, which are connected to each other by arrows with relative velocities, represent inertial frames. The velocity composition law of special relativity has been known to be inconsistent with the reciprocity principle of velocity, through the investigation of a simple case where the inertial frames of interest are connected *via* a single node. When they are connected through more than one node, many inconsistencies including the violation of the reciprocity principle are found, as the successive coordinate transformation is not reduced to a Lorentz transformation. These inconsistencies can be cured by introducing a reference node such that the velocity composition is made in conjunction with it. The reference node corresponds to the preferred frame. The relativistic velocity composition law has no inconsistencies under the uniqueness of the isotropic frame.

Keywords: velocity composition, standard synchronization, successive coordinate transformation, Mocanu paradox, preferred reference frame

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1 Introduction

Special relativity (SR) was formulated based on the principle of relativity and the constancy of the speed of light [1], which lead to the Lorentz transformation (LT). Usually the LT has been utilized for a direct coordinate transformation from one inertial frame S_i into another S_j when the relative velocity of S_j with respect to S_i is known. In such a situation, the results of the experiments to test the validity of SR have been known to be in agreement with its predictions [2–5]. In accordance with the relativity principle, the reverse velocity, the velocity of S_i relative to S_j, should be of equal magnitude and in the direction opposite to the forward one. When considering an intermediate frame between S_i and S_j, however, the Mocanu paradox is raised so that the reciprocity relation is not satisfied [6]. The velocity composition of SR is neither commutative nor associative, which causes inconsistencies in the composition. Some alternative approaches have been presented that aim at overcoming these inconsistencies [7–12]. The Mocanu paradox has been explained mostly by resorting to the Thomas rotation [13–15]. It seems to be generally accepted that the paradox is resolved by the Thomas rotation.

The relativistic velocity composition has been investigated, as in the case of the Mocanu paradox, under a single intermediate frame, but such an investigation would not provide sufficient information on the consistency in the velocity composition law of SR. Considering the case where S_i and S_j are connected *via* more than one inertial frame, this article deals with the problem of the relativistic velocity composition. To this end, we introduce a diagram of velocity where inertial frames are represented as nodes that are connected by arrows indicating relative velocities. The diagram is useful to see the relationships between the relative velocities of the nodes.

Though all inertial frames are equivalent according to the relativity principle in SR, sometimes a preferred (or privileged) frame that can provide absolute simultaneity has been attracted and investigated [11,16–21], mainly in terms of clock synchronization. Presupposing a preferred reference frame the space-time of which is isotropic, Mansouri and Sexl (MS) presented a general framework for the transformation between the preferred frame and an arbitrary inertial frame to investigate the role of convention in various definitions of clock synchronization and simultaneity [21]. The formulation is general in that it has been derived from fundamental kinematics and the isotropy of the preferred frame, and can be applied to various synchronizations.

It is shown based on the MS general framework that when the standard synchronization is adopted, the relative velocity of S_j with respect to S_i can be found from the first row of the transformation matrix from the latter to the former. Exploiting this fact together with the velocity diagram allows us to readily approach the problems of the relativistic velocity composition, extending them to the case of multiple connecting nodes. Associativity-related properties in the velocity composition may have been known to be difficult to prove because of seemingly high mathematical complexity involved. It is stated in ref. [14] (p. 72) that "A direct proof ... is lengthy and, hence, requires the use of computer algebra." The properties are easily derived using the relationship between the relative velocity and the first row of the transformation matrix.

The inconsistency associated with the reciprocity of velocity results from the non-commutativity of the velocity composition. When S_i and S_j are connected through more than one node, many inconsistencies in the composition law of SR are raised in addition to the violation of the velocity reciprocity since the composition operation is neither commutative nor associative. As a result, the velocity of S_j with respect to S_i, depending on the connecting nodes, has not only multiple directions but also multiple magnitudes. Therefore the proper time (PT) in S_j is not uniquely determined though it should have the same value from frame to frame regardless of synchronization schemes. To cure these inconsistencies and contradictions, a reference node is introduced so that the composition law is applied in conjunction with it. The reference node corresponds to the unique isotropic frame. In the preferred frame theory (PFT), which refers to a theory based on the uniqueness of the isotropic frame [22], there are no inconsistencies and no contradictions.

Throughout the article, the following notations are used. We represent vectors with lower-case boldface letters and matrices with upper-case boldface letters. Superscript T stands for transpose. Matrix $\boldsymbol{I}$ is an identity matrix of appropriate size. For a real vector $\boldsymbol{v}$, $|\boldsymbol{v}|$ denotes its magnitude. The vector with hat represents its unit vector, *i.e.*, $\hat{\boldsymbol{v}} = \boldsymbol{v}/|\boldsymbol{v}|$. For convenience we introduce a partitioned matrix to be used in place of transformation matrices:

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}|_{1r} \\ \boldsymbol{A}|_{2r} \end{bmatrix} = [\boldsymbol{A}|_{1c} \;\; \boldsymbol{A}|_{2c}], \tag{1}$$

where $\boldsymbol{A}_{mn}$, $\boldsymbol{A}|_{mr}$, and $\boldsymbol{A}|_{nc}$ denote the (m, n)-entry, mth row, and nth column, respectively, and A_{11} is a single quantity. The preferred reference frame is denoted as S where the speed of light is a constant c regardless of the propagation direction. We use imaginary time $\tau = ict$, where t designates time.

2 General framework for inertial transformation

The matrices that make the transformation of coordinates between inertial frames include the information on relative velocities. Hence we can discover the velocities from them. This section addresses this matter in general transformations, employing the MS general framework. In the general framework, a preferred reference frame S is presumed whose space-time is isotropic. An inertial frame S_l is in uniform linear motion at a velocity $\boldsymbol{v}_l$ with respect to S and its normalized velocity is $\boldsymbol{\beta}_l = \boldsymbol{v}_l/c$. Representing time as an imaginary number, the space-time coordinate vector of S is expressed as $\boldsymbol{p} = [\tau, \;\; \boldsymbol{x}^T]^T$, where $\boldsymbol{x}$ is a spatial vector. Similarly the coordinate vector of S_l is expressed as $\boldsymbol{p}_{(l)} = [\tau_{(l)}, \;\; \boldsymbol{x}_{(l)}^T]^T$. The coordinate vector $\boldsymbol{p}$ of S is transformed into S_l and the resulting coordinate vector is written as:

$$\boldsymbol{p}_{(l)} = \boldsymbol{T}(\boldsymbol{\beta}_l)\boldsymbol{p}, \tag{2}$$

where $\boldsymbol{T}(\boldsymbol{\beta}_l)$ is a transformation matrix, which is given in the MS framework as [21–23]:

$$\boldsymbol{T}_{\mathrm{MS}}(\boldsymbol{\beta}_l) = \begin{bmatrix} g_l & i\boldsymbol{\rho}_l^T \\ ib_l\boldsymbol{\beta}_l & \boldsymbol{M}(\boldsymbol{\beta}_l) \end{bmatrix}, \tag{3}$$

with

$$g_l = a_l - b_l\boldsymbol{\varepsilon}_l^T\boldsymbol{\beta}_l, \tag{4a}$$

$$\boldsymbol{\rho}_l = (b_l - d_l)(\boldsymbol{\varepsilon}_l^T\hat{\boldsymbol{\beta}}_l)\hat{\boldsymbol{\beta}}_l + d_l\boldsymbol{\varepsilon}_l, \tag{4b}$$

$$\boldsymbol{M}(\boldsymbol{\beta}_l) = (b_l - d_l)\hat{\boldsymbol{\beta}}_l\hat{\boldsymbol{\beta}}_l^T + d_l\boldsymbol{I}. \tag{4c}$$

The transformation coefficients a_l, b_l, and d_l can depend on $\boldsymbol{\beta}_l$. When the standard synchronization is employed, they are given as Eq. (16) below. As $|\boldsymbol{\beta}_l|$ approaches zero, $\boldsymbol{p}_{(l)}$ tends to $\boldsymbol{p}$. Then, $\boldsymbol{M}(\boldsymbol{\beta}_l)$ is reduced to an identity matrix so that a_l, b_l, and d_l all become equal to one. The vector $\boldsymbol{\varepsilon}_l$ is a synchronization vector, which is determined by a clock synchronization in S_l. Indeed, the spatial vector $\boldsymbol{x}_{(l)}$ of S_l is irrelevant to $\boldsymbol{\varepsilon}_l$. One can confirm it from Eq. (2). The last three rows of $\boldsymbol{T}_{\text{MS}}(\boldsymbol{\beta}_l)$ are independent of $\boldsymbol{\varepsilon}_l$, which leads to the irrelevance of $\boldsymbol{x}_{(l)}$ to $\boldsymbol{\varepsilon}_l$.

The transformation between arbitrary inertial frames S_i and S_j is written from Eq. (2) as:

$$\boldsymbol{p}_{(j)} = \boldsymbol{T}_{\text{MS}}(\boldsymbol{\beta}_j, \ \boldsymbol{\beta}_i)\boldsymbol{p}_{(i)}, \tag{5}$$

where

$$\boldsymbol{T}_{\text{MS}}(\boldsymbol{\beta}_j, \ \boldsymbol{\beta}_i) = \boldsymbol{T}_{\text{MS}}(\boldsymbol{\beta}_j)\boldsymbol{T}_{\text{MS}}^{-1}(\boldsymbol{\beta}_i). \tag{6}$$

Using Eqs. (3), (4), and (6), $\boldsymbol{T}_{\text{MS}}(\boldsymbol{\beta}_j, \ \boldsymbol{\beta}_i)$ is expressed as [22,23]:

$$\boldsymbol{T}_{\text{MS}}(\boldsymbol{\beta}_j, \ \boldsymbol{\beta}_i) = \begin{bmatrix} a_i^{-1}(g_j + \boldsymbol{\rho}_j^T\boldsymbol{\beta}_i) & -ig_ja_i^{-1}\boldsymbol{\varepsilon}_i^T + i\boldsymbol{\rho}_j^T\boldsymbol{M}'(\boldsymbol{\beta}_i) \\ ia_i^{-1}(b_j\boldsymbol{\beta}_j - \boldsymbol{M}(\boldsymbol{\beta}_j)\boldsymbol{\beta}_i) & a_i^{-1}b_j\boldsymbol{\beta}_j\boldsymbol{\varepsilon}_i^T + \boldsymbol{M}(\boldsymbol{\beta}_j)\boldsymbol{M}'(\boldsymbol{\beta}_i) \end{bmatrix}, \tag{7}$$

where

$$\boldsymbol{M}'(\boldsymbol{\beta}_l) = \left(\frac{1}{b_l} - \frac{1}{d_l}\right)\hat{\boldsymbol{\beta}}_l\hat{\boldsymbol{\beta}}_l^T - \frac{1}{a_l}\boldsymbol{\beta}_l\boldsymbol{\varepsilon}_l^T + \frac{1}{d_l}\boldsymbol{I}. \tag{8}$$

The transformation between S_i and S_j is dependent on both $\boldsymbol{\beta}_i$ and $\boldsymbol{\beta}_j$. Given $\boldsymbol{\beta}_i$ and $\boldsymbol{\beta}_j$, the velocity of S_j relative to S_i is given by [5,22,23]:

$$\boldsymbol{\beta}_{ji} = \frac{1}{a_j\Gamma(\boldsymbol{\beta}_i, \boldsymbol{\beta}_j)}(b_i\,[\hat{\boldsymbol{\beta}}_i(\hat{\boldsymbol{\beta}}_i^T\boldsymbol{\beta}_j) - \boldsymbol{\beta}_i] + d_i[\boldsymbol{\beta}_j - \hat{\boldsymbol{\beta}}_i(\hat{\boldsymbol{\beta}}_i^T\boldsymbol{\beta}_j)]), \tag{9}$$

where

$$\Gamma(\boldsymbol{\beta}_i, \boldsymbol{\beta}_j) = \frac{a_i}{a_j} + \frac{b_i}{a_j}(\boldsymbol{\varepsilon}_i^T\,[\hat{\boldsymbol{\beta}}_i(\hat{\boldsymbol{\beta}}_i^T\boldsymbol{\beta}_j) - \boldsymbol{\beta}_i]) + \frac{d_i}{a_j}(\boldsymbol{\varepsilon}_i^T\,[\boldsymbol{\beta}_j - \hat{\boldsymbol{\beta}}_i(\hat{\boldsymbol{\beta}}_i^T\boldsymbol{\beta}_j)]). \tag{10}$$

It should be noted that the direction of $\boldsymbol{\beta}_{ji}$ is independent of $\boldsymbol{\varepsilon}_i$ and $\boldsymbol{\varepsilon}_j$ whereas its magnitude is dependent on $\boldsymbol{\varepsilon}_i$.

PT, which is the time by the clock of an observer, is well known to be independent of the synchronization of clocks. We use a subscript "$\circ$" at PT, say $\tau_{(l)^\circ}$, to distinguish it from the time adjusted through a synchronization procedure. For simplicity $\tau_{(l)^\circ}$ is referred to as PT, though $\tau_{(l)^\circ}$ is different from the clock time of O_l since $\tau_{(l)^\circ} = ict_{(l)^\circ}$. Let $\boldsymbol{A} = \boldsymbol{T}_{\text{MS}}(\boldsymbol{\beta}_j, \ \boldsymbol{\beta}_i)$. The quantity A_{11}, which represents the time dilation factor, is given by [22,23]:

$$A_{11} = \Gamma(\boldsymbol{\beta}_j, \boldsymbol{\beta}_i). \tag{11}$$

The differential coordinate vector of an observer O_j who is at rest in S_j can be represented as $\mathrm{d}\mathbf{p}_{(j)} = [\mathrm{d}\tau_{(j)^\circ}, \ \mathbf{0}]^T$. Substituting the differential vector of O_j in Eq. (5) with subscripts i and j interchanged, we have:

$$\mathrm{d}\mathbf{p}_{(i)} = \mathrm{d}\tau_{(j)^\circ}\boldsymbol{T}_{\text{MS}}(\boldsymbol{\beta}_i, \ \boldsymbol{\beta}_j)|_{1c}, \tag{12a}$$

$$= \mathrm{d}\tau_{(j)^\circ}[\Gamma(\boldsymbol{\beta}_i, \boldsymbol{\beta}_j), \ (\boldsymbol{T}_{\text{MS}}(\boldsymbol{\beta}_i, \ \boldsymbol{\beta}_j)|_{21})^T]^T. \tag{12b}$$

The matrix $\boldsymbol{T}_{\text{MS}}(\boldsymbol{\beta}_i, \ \boldsymbol{\beta}_j)$ is the inverse of $\boldsymbol{A}$. Since the normalized velocity of O_j is $i\mathrm{d}\boldsymbol{x}_{(i)}/\mathrm{d}\tau_{(i)} = \boldsymbol{\beta}_{ji}$ in S_i, the differential vector $\mathrm{d}\boldsymbol{p}_{(i)}$ can be generally expressed as:

$$\mathrm{d}\boldsymbol{p}_{(i)} = \mathrm{d}\tau_{(i)}[1, \ \mathrm{d}\boldsymbol{x}_{(i)}^T/\mathrm{d}\tau_{(i)}]^T, \tag{13a}$$

$$= \mathrm{d}\tau_{(i)}[1, \ -i\boldsymbol{\beta}_{ji}^T]^T. \tag{13b}$$

Comparing Eqs. (12b) and (13b), we see that the differential PT of O_j is related to $\mathrm{d}\tau_{(i)}$ by

$$\mathrm{d}\tau_{(j)^\circ} = \frac{\mathrm{d}\tau_{(i)}}{\Gamma(\boldsymbol{\beta}_i, \boldsymbol{\beta}_j)}. \tag{14}$$

From Eq. (12a) $\boldsymbol{T}_{\text{MS}}(\boldsymbol{\beta}_i, \ \boldsymbol{\beta}_j)|_{1c} = \mathrm{d}\boldsymbol{p}_{(i)}/\mathrm{d}\tau_{(j)^\circ}$, which is written using Eqs. (13b) and (14) as

$$\boldsymbol{T}_{\text{MS}}(\boldsymbol{\beta}_i, \ \boldsymbol{\beta}_j)|_{1c} = \Gamma(\boldsymbol{\beta}_i, \boldsymbol{\beta}_j)[1, \ -i\boldsymbol{\beta}_{ji}^T]^T. \tag{15}$$

The relative velocity can be expressed as $\boldsymbol{\beta}_{ji} = i\boldsymbol{T}_{\text{MS}}(\boldsymbol{\beta}_i, \ \boldsymbol{\beta}_j)|_{21}/\Gamma(\boldsymbol{\beta}_i, \boldsymbol{\beta}_j)$. The first column of the matrix that transforms coordinates from S_j to S_i contains the information on the velocity of S_j relative to S_i so that the relative velocity can be discovered from it. This fact holds regardless of the transformation coefficients and the synchronization vectors.

3 Relativistic velocity composition

As mentioned in Section 1, the predictions of SR have been in agreement with numerous experimental results. The transformation coefficients in Eq. (2) are employed in accordance with SR and are given by:

$$a_l = \gamma_l^{-1}, \ \ b_l = \gamma_l, \ \ d_l = 1, \tag{16}$$

where $\gamma_l = (1 - |\boldsymbol{\beta}_l|^2)^{-1/2}$. In case the coefficients of Eq. (16) are used, the two-way speed of light in S_l becomes c irrespective of the synchronization gauge, $\boldsymbol{\varepsilon}_l$. We adopt the standard synchronization, which leads to $\boldsymbol{\varepsilon}_l = -\boldsymbol{\beta}_l$. Then, Eqs. (4a) and (4b) are written as $g_l = \gamma_l$ and $\boldsymbol{\rho}_l = -\gamma\boldsymbol{\beta}_l$. When employing the coefficients of (16) in the standard synchronization, the $\boldsymbol{T}_{\text{MS}}(\boldsymbol{\beta}_l)$ is expressed as:

$$\boldsymbol{T}_{\text{MS}}(\boldsymbol{\beta}_l) = \begin{bmatrix} \gamma_l & -i\gamma_l\boldsymbol{\beta}_l^T \\ i\gamma_l\boldsymbol{\beta}_l & (\gamma_l - 1)\hat{\boldsymbol{\beta}}_l\hat{\boldsymbol{\beta}}_l^T + \boldsymbol{I} \end{bmatrix}. \tag{17}$$

It is identical with the LT matrix when the relative velocity is $\boldsymbol{\beta}_l$. One can confirm from Eq. (17) that $\boldsymbol{T}_{\mathrm{MS}}(\boldsymbol{\beta}_l)$ tends to an identity matrix as $|\boldsymbol{\beta}_l|$ goes to zero.

The transformation matrix from S_i to S_j is given as Eq. (6) with the $\boldsymbol{T}_{\mathrm{MS}}(\boldsymbol{\beta}_l)$, $l = i,\ j$, of Eq. (17). As $\boldsymbol{T}_{\mathrm{MS}}^{-1}(\boldsymbol{\beta}_l) = \boldsymbol{T}_{\mathrm{MS}}^{T}(\boldsymbol{\beta}_l)$, the inverse of $\boldsymbol{A}\,(=\boldsymbol{T}_{\mathrm{MS}}(\boldsymbol{\beta}_j,\ \boldsymbol{\beta}_i))$ also becomes equal to its transpose. Recalling that the transpose of $\boldsymbol{A}$ is the same as $\boldsymbol{T}_{\mathrm{MS}}(\boldsymbol{\beta}_i,\ \boldsymbol{\beta}_j)$ and using the relationship of $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$ and Eq. (15), we have [22,23]:

$$A_{11} = \gamma_{ji} = \gamma_{ij}, \tag{18a}$$

$$\gamma_{ji} = (1 - |\boldsymbol{\beta}_{ji}|^2)^{-1/2}. \tag{18b}$$

Then, the first row of $\boldsymbol{A}$ is written as:

$$\boldsymbol{A}\,|_{1r} = (\boldsymbol{T}_{\mathrm{MS}}(\boldsymbol{\beta}_i,\ \boldsymbol{\beta}_j)|_{1c})^T = \gamma_{ji}[1,\ -i\boldsymbol{\beta}_{ji}^T]. \tag{19}$$

It should be noted that in the standard synchronization, the relative velocity can be found from the first row of the transformation matrix. The $\boldsymbol{\beta}_{ji}$ is calculated from Eqs. (9), (16), and (18) as [22,23]:

$$\boldsymbol{\beta}_{ji} = \gamma_j\gamma_{ij}^{-1}[\gamma_i(\hat{\boldsymbol{\beta}}_i(\hat{\boldsymbol{\beta}}_i^T\boldsymbol{\beta}_j) - \boldsymbol{\beta}_i) + \boldsymbol{\beta}_j - \hat{\boldsymbol{\beta}}_i(\hat{\boldsymbol{\beta}}_i^T\boldsymbol{\beta}_j)], \tag{20}$$

and γ_{ij} can be expressed as $\gamma_{ij} = \gamma_i\gamma_j(1 - \boldsymbol{\beta}_j^T\boldsymbol{\beta}_i)$ in terms of $\boldsymbol{\beta}_i$ and $\boldsymbol{\beta}_j$, which confirms that $\gamma_{ji} = \gamma_{ij}$. Eq. (20) can also be expressed as Eq. (36) below by using the velocity composition of SR. If $\boldsymbol{\beta}_i$ and $\boldsymbol{\beta}_j$ are collinear, the reverse velocity $\boldsymbol{\beta}_{ij}$ is identical to $-\boldsymbol{\beta}_{ji}$. In the event that the absolute velocities are non-collinear, however it is not, though the magnitudes of $\boldsymbol{\beta}_{ij}$ and $\boldsymbol{\beta}_{ji}$ are the same.

3.1 In SR

When making the analysis of velocity composition, it is convenient to employ a diagram of velocity, as in Figures 1–5, where inertial frames are designated by nodes so that node i, for example, represents S_i and $\boldsymbol{\beta}_{mn}$ denotes the velocity of S_m relative to S_n. Consider a case where S_i and S_j are connected *via* an inertial frame S_k, as shown in Figure 1. In SR, the coordinate transformation from S_i to S_j *via* S_k is given by:

$$\bar{\boldsymbol{p}}_{(j)/k} = \boldsymbol{T}_{LL}(\boldsymbol{\beta}_{jk},\ \boldsymbol{\beta}_{ki})\boldsymbol{p}_{(i)}, \tag{21}$$

where $\boldsymbol{T}_{LL}(\boldsymbol{\beta},\ \boldsymbol{\delta}) = \boldsymbol{T}_L(\boldsymbol{\beta})\boldsymbol{T}_L(\boldsymbol{\delta})$ with $\boldsymbol{T}_L(\boldsymbol{\beta})$ representing the LT matrix for velocity $\boldsymbol{\beta}$. The $\boldsymbol{T}_{LL}(\boldsymbol{\beta},\ \boldsymbol{\delta})$ denotes a successive LT with relative velocities $\boldsymbol{\delta}$ and $\boldsymbol{\beta}$. Generally, the coordinate vector in S_j calculated by the right side of Eq. (21) will be different from $\boldsymbol{p}_{(j)}$ of Eq. (5) with the coefficients of (16) and thus, a different notation $\bar{\boldsymbol{p}}_{(j)/k}$ is used in Eq. (21). The forward velocities $\boldsymbol{\beta}_{ki}$ and $\boldsymbol{\beta}_{jk}$ are initially given, and then the

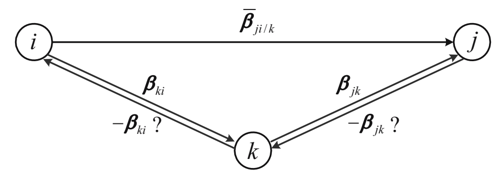

Figure 1: Connection configuration without reference frame when frames S_i and S_j are connected through a single frame. Question marks are put beside the velocities with minus sign as the actual reverse velocities are not given by the reciprocity principle.

reverse velocities are dependently determined by the reciprocity principle in SR [24] so that they, having the same magnitude as the respective forward ones, are in the opposite directions like $-\boldsymbol{\beta}_{ki}$ and $-\boldsymbol{\beta}_{jk}$. Reflecting the reciprocity principle, Figure 1 has been drawn just for the analysis of relative velocity in SR, though the correct reverse velocities under the standard synchronization may be different from them. Hence question marks are put beside the velocities given by the reciprocity principle. It is well known that according to the velocity composition law of SR, the velocity of S_j relative to S_i is expressed as:

$$\bar{\boldsymbol{\beta}}_{ji/k} = \frac{1}{1 + \boldsymbol{\beta}_{ki}^T\boldsymbol{\beta}_{jk}}\left[\boldsymbol{\beta}_{ki} + \frac{\boldsymbol{\beta}_{jk}}{\gamma_{ki}} + \frac{\gamma_{ki}(\boldsymbol{\beta}_{ki}^T\boldsymbol{\beta}_{jk})\boldsymbol{\beta}_{ki}}{1 + \gamma_{ki}}\right], \quad \equiv \boldsymbol{\beta}_{ki} \oplus \boldsymbol{\beta}_{jk} \tag{22a}$$

and the reverse velocity $\bar{\boldsymbol{\beta}}_{ij/k}$ is given by

$$\bar{\boldsymbol{\beta}}_{ij/k} = (-\boldsymbol{\beta}_{jk}) \oplus (-\boldsymbol{\beta}_{ki}). \tag{22b}$$

The $\bar{\boldsymbol{\beta}}_{ij/k}$ should be equal to $-\ \bar{\boldsymbol{\beta}}_{ji/k}$ in accordance with the reciprocity principle, but unless $\boldsymbol{\beta}_i$ and $\boldsymbol{\beta}_j$ are collinear it

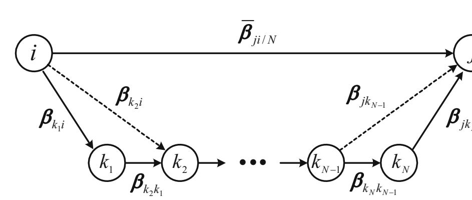

Figure 2: Connection configuration without reference frame when frames S_i and S_j are connected through N frames.

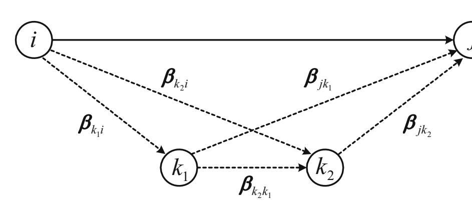

Figure 3: Connection configuration for frames S_i and S_j with two added frames.

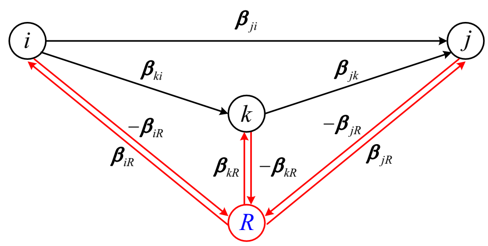

Figure 4: Connection configuration with reference frame when frames S_i and S_j are connected through a single frame.

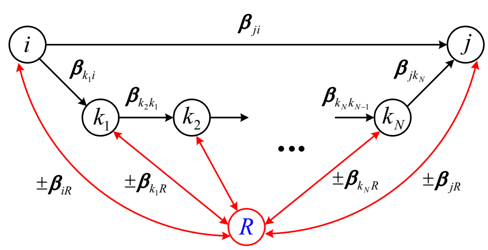

Figure 5: Connection configuration with reference frame when frames S_i and S_j are connected through N frames.

is not, which represents the Mocanu paradox. According to the principle of relativity, the frame S is nothing more than one of an infinite number of isotropic frames. Nonetheless, the $\bar{\boldsymbol{\beta}}_{ji/k}$ is not identical to the $\boldsymbol{\beta}_{ji}$ of Eq. (20) either. These inconsistencies result from the fact that $\boldsymbol{T}_L(\bar{\boldsymbol{\beta}}_{ji/k})$ is not the same as $\boldsymbol{T}_{LL}(\boldsymbol{\beta}_{jk},\ \boldsymbol{\beta}_{ki})$:

$$\boldsymbol{T}_L(\bar{\boldsymbol{\beta}}_{ji/k}) \neq \boldsymbol{T}_{LL}(\boldsymbol{\beta}_{jk},\ \boldsymbol{\beta}_{ki})\ (\neq \boldsymbol{T}_L(\boldsymbol{\beta}_{ji})). \tag{23}$$

To resolve the non-equality, a Thomas rotation [13–15] is employed in such a way that

$$\boldsymbol{R}(\boldsymbol{\beta}_{jk},\ \boldsymbol{\beta}_{ki})\boldsymbol{T}_L(\bar{\boldsymbol{\beta}}_{ji/k}) = \boldsymbol{T}_{LL}(\boldsymbol{\beta}_{jk},\ \boldsymbol{\beta}_{ki}), \tag{24}$$

where the spatial rotation matrix $\boldsymbol{R}(\boldsymbol{\beta}_{jk},\ \boldsymbol{\beta}_{ki})$ is determined such that the equality is fulfilled. However, the rotation, which is a function of $\boldsymbol{\beta}_{jk}$ and $\boldsymbol{\beta}_{ki}$, is not uniquely given since it depends on the intermediate node k. Suppose that $\bar{\boldsymbol{\beta}}_{ji/k}$ is the correct relative velocity. Following the path for SR [1] that Einstein walked to discover the transformation of coordinates from S_i to S_j will lead to the LT $\boldsymbol{T}_L(\bar{\boldsymbol{\beta}}_{ji/k})$, not $\boldsymbol{R}(\boldsymbol{\beta}_{jk},\ \boldsymbol{\beta}_{ki})\boldsymbol{T}_L(\bar{\boldsymbol{\beta}}_{ji/k})$. As the successive transformation depends on the connecting node, in general:

$$\boldsymbol{T}_{LL}(\boldsymbol{\beta}_{jk_1},\ \boldsymbol{\beta}_{k_1i}) \neq \boldsymbol{T}_{LL}(\boldsymbol{\beta}_{jk_2},\ \boldsymbol{\beta}_{k_2i}), \tag{25}$$

where $k_1 \neq k_2$. Spatial rotations may explain the non-equality (23), but cannot resolve the non-equality (25), which is irrelevant to them.

Besides the inconsistencies mentioned above, numerous inconsistencies can be found when the connection between the two inertial frames of interest is made *via* more than one node. As illustrated in Figure 2, nodes i and j are connected through N nodes. In the case of $N = 1$, Figure 2 is reduced to Figure 1 for forward velocities. Given the forward velocities, say $\boldsymbol{\beta}_{k_1i}$ and $\boldsymbol{\beta}_{k_2k_1}$, the reverse velocities with respect to them are given as $-\boldsymbol{\beta}_{k_1i}$ and $-\boldsymbol{\beta}_{k_2k_1}$ by the reciprocity principle. The forward velocities are assumed to be non-collinear. The successive transformation matrix $\boldsymbol{T}_{L/N}$ from node i to j is written as:

$$\boldsymbol{T}_{L/N} = \boldsymbol{T}_L(\boldsymbol{\beta}_{jk_N})\boldsymbol{T}_L(\boldsymbol{\beta}_{k_Nk_{N-1}}) \cdots \boldsymbol{T}_L(\boldsymbol{\beta}_{k_2k_1})\boldsymbol{T}_L(\boldsymbol{\beta}_{k_1i}). \tag{26}$$

Then, the relative velocities between nodes i and j are calculated as

$$\bar{\boldsymbol{\beta}}_{ji/N} = \boldsymbol{\beta}_{k_1i} \oplus (\boldsymbol{\beta}_{k_2k_1} \oplus \cdots \oplus(\boldsymbol{\beta}_{k_Nk_{N-1}} \oplus \boldsymbol{\beta}_{jk_N})\cdots), \tag{27a}$$

$$\bar{\boldsymbol{\beta}}_{ij/N} = (-\boldsymbol{\beta}_{jk_N}) \oplus ((-\boldsymbol{\beta}_{k_Nk_{N-1}}) \oplus \cdots \oplus ((-\boldsymbol{\beta}_{k_2k_1}) \oplus (-\boldsymbol{\beta}_{k_1i}))\cdots). \tag{27b}$$

In Eq. (27), the velocity addition is performed from right to left, rather than from left to right. For example, $\bar{\boldsymbol{\beta}}_{ji/3}$ is computed as $\bar{\boldsymbol{\beta}}_{ji/3} = \boldsymbol{\beta}_{k_1i} \oplus (\boldsymbol{\beta}_{k_2k_1} \oplus (\boldsymbol{\beta}_{k_3k_2} \oplus \boldsymbol{\beta}_{jk_3}))$. Refer Appendix for the derivation of Eq. (27).

The composition operation is not associative and has the following properties for arbitrary velocities $\boldsymbol{u}_l$, $l = 1, \ \cdots, \ 3$ [13,14]:

$$\boldsymbol{u}_1 \oplus (\boldsymbol{u}_2 \oplus \boldsymbol{u}_3) = (\boldsymbol{u}_1 \oplus \boldsymbol{u}_2) \oplus \boldsymbol{A}_{22}^T\boldsymbol{u}_3, \tag{28a}$$

$$(\boldsymbol{u}_1 \oplus \boldsymbol{u}_2) \oplus \boldsymbol{u}_3 = \boldsymbol{u}_1 \oplus (\boldsymbol{u}_2 \oplus \boldsymbol{A}_{22}\boldsymbol{u}_3), \tag{28b}$$

where $\boldsymbol{A}$, which has a form of Eq. (1), is a block diagonal matrix and is given by

$$\boldsymbol{A} = \boldsymbol{T}_L(\boldsymbol{u}_n)\boldsymbol{T}_L(\boldsymbol{u}_m)\boldsymbol{T}_L^{-1}(\boldsymbol{u}_m \oplus \boldsymbol{u}_n), \tag{29}$$

with $m = 1$ and $n = 2$. If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are collinear, the velocity composition $\boldsymbol{u}_1 \oplus \boldsymbol{u}_2$ is equal to $\boldsymbol{u}_2 \oplus \boldsymbol{u}_1$ and $\boldsymbol{A}_{22}$ is reduced to an identity matrix. The associativity-related properties can be readily derived, as shown in Appendix.

The relative velocities Eqs. (27a) and (27b) each should remain the same for every N and arbitrary connecting nodes in order that SR becomes consistent. At first glance, the composition law (27a) for the forward velocities appears to be consistently applied because $\bar{\boldsymbol{\beta}}_{ji/N}$ is the same for every N. Clearly $\bar{\boldsymbol{\beta}}_{ji/N}$ is equal to $\bar{\boldsymbol{\beta}}_{ji/N-1}$ since $\boldsymbol{\beta}_{jk_{N-1}} = \boldsymbol{\beta}_{k_Nk_{N-1}} \oplus \boldsymbol{\beta}_{jk_N}$. However, there exist many inconsistencies. Let us first consider simple configurations of connection with two added nodes as shown in Figure 3. One can see two connection configurations for

$N = 1$: $i \to k_1 \to j$ and $i \to k_2 \to j$. The relative velocities for the connections are:

$$\bar{\boldsymbol{\beta}}_{ji/k_1} = \boldsymbol{\beta}_{k_1i} \oplus \boldsymbol{\beta}_{jk_1}, \tag{30a}$$

$$\bar{\boldsymbol{\beta}}_{ji/k_2} = \boldsymbol{\beta}_{k_2i} \oplus \boldsymbol{\beta}_{jk_2}. \tag{30b}$$

In SR, $\boldsymbol{\beta}_{jk_1} = \boldsymbol{\beta}_{k_2k_1} \oplus \boldsymbol{\beta}_{jk_2}$ and $\boldsymbol{\beta}_{k_2i} = \boldsymbol{\beta}_{k_1i} \oplus \boldsymbol{\beta}_{k_2k_1}$. Substituting these in Eqs. (30a) and (30b), we have [22]:

$$\bar{\boldsymbol{\beta}}_{ji/k_1} = \boldsymbol{\beta}_{k_1i} \oplus (\boldsymbol{\beta}_{k_2k_1} \oplus \boldsymbol{\beta}_{jk_2}), \tag{31a}$$

$$\bar{\boldsymbol{\beta}}_{ji/k_2} = (\boldsymbol{\beta}_{k_1i} \oplus \boldsymbol{\beta}_{k_2k_1}) \oplus \boldsymbol{\beta}_{jk_2}. \tag{31b}$$

The composition operation is not associative and thus generally $\bar{\boldsymbol{\beta}}_{ji/k_1} \neq \bar{\boldsymbol{\beta}}_{ji/k_2}$. In the case of $N = 2$, one can confirm that the $\bar{\boldsymbol{\beta}}_{ji/2}$ is the same as $\bar{\boldsymbol{\beta}}_{ji/1}$, which is identical to Eq. (31a). If the order of connection is changed so that node i is first connected to node k_2 and node k_1 is connected to node j, then the $\bar{\boldsymbol{\beta}}_{ji/2}$ is given as $\bar{\boldsymbol{\beta}}_{ji/2} = \boldsymbol{\beta}_{k_2i} \oplus (-\boldsymbol{\beta}_{k_2k_1} \oplus \boldsymbol{\beta}_{jk_1}) = \boldsymbol{\beta}_{k_2i} \oplus \boldsymbol{\beta}_{jk_2}$, which is equal to Eq. (31b) but different from the previous one. As more nodes are added, the inconsistency comes to be more deepened since the relative velocity is dependent on more nodes. Given N intermediate nodes as in Figure 2, there are $N!$ permutations taking account of the order of connection. If SR is consistent in the composition law, $\bar{\boldsymbol{\beta}}_{ji/N}$ should be constant for all the permutations. The composition operation should be commutative and associative in order that the relative velocity remains the same for the permutations. However, it is neither commutative nor associative and the velocity, depending on the arrangement of intermediate nodes, varies with the permutation.

Now let us turn our eyes to reverse velocities. The reverse velocity composition law Eq. (27b) itself is inconsistent since $\bar{\boldsymbol{\beta}}_{ij/N} \neq \bar{\boldsymbol{\beta}}_{ij/N-1}$. For $N = 1$ and 2, the reverse velocities can be written as:

$$\bar{\boldsymbol{\beta}}_{ij/1} = (-\boldsymbol{\beta}_{jk_1}) \oplus (-\boldsymbol{\beta}_{k_1i})(\neq((-\boldsymbol{\beta}_{jk_2}) \oplus (-\boldsymbol{\beta}_{k_2k_1})) \oplus (-\boldsymbol{\beta}_{k_1i})), \tag{32a}$$

$$\bar{\boldsymbol{\beta}}_{ij/2} = (-\boldsymbol{\beta}_{jk_2}) \oplus ((-\boldsymbol{\beta}_{k_2k_1}) \oplus (-\boldsymbol{\beta}_{k_1i})). \tag{32b}$$

The non-equality in (32a) results from the non-equality of $\boldsymbol{\beta}_{jk_1} \neq -((-\boldsymbol{\beta}_{jk_2}) \oplus (-\boldsymbol{\beta}_{k_2k_1}))$, which indicates that the reciprocity principle is not satisfied in SR. The $\bar{\boldsymbol{\beta}}_{ij/1}$ is not identical to the $\bar{\boldsymbol{\beta}}_{ij/2}$.

3.2 From inconsistency to consistency

We can make the velocity composition consistent by putting some constraints on it. A reference frame R, as shown in Figure 4, is introduced such that the composition law is always applied in conjunction with R. The reverse velocity that is the opposite of the corresponding forward one according to the reciprocity principle is allowed only between R and other nodes, but not between two nodes neither of which is R. For example, $\boldsymbol{\beta}_{kR}$ is the velocity of node k relative to R and then the velocity in the opposite direction from k to R is $\boldsymbol{\beta}_{Rk} = -\boldsymbol{\beta}_{kR}$, but $\boldsymbol{\beta}_{ki} \neq -\boldsymbol{\beta}_{ik}$ since neither of the nodes i and k is R. The validity of the constrained velocity composition law will be discussed later.

Let us illustrate the consistency of it, selecting the isotropic frame S as R. Then, $\boldsymbol{\beta}_{k_1R}$ and $\boldsymbol{\beta}_{Ri}$ are equal to $\boldsymbol{\beta}_{k_1}$ and $-\boldsymbol{\beta}_i$. Suppose the absolute velocity $\boldsymbol{\beta}_i$ is available. Given $\boldsymbol{\beta}_m$ and $\boldsymbol{\beta}_{nm}$, the velocity $\boldsymbol{\beta}_n$ is written as [4]

$$\begin{aligned}\boldsymbol{\beta}_n &= \boldsymbol{\beta}_m + \frac{1}{\gamma_m(1 + \boldsymbol{\beta}_{nm}^T\boldsymbol{\beta}_m)} \\ &\times \left(\boldsymbol{\beta}_{nm} + \left(\frac{1}{\gamma_m} - 1\right)(\boldsymbol{\beta}_{nm}^T\hat{\boldsymbol{\beta}}_m)\hat{\boldsymbol{\beta}}_m\right).\end{aligned} \tag{33}$$

If $\boldsymbol{\beta}_i$, $\boldsymbol{\beta}_{ki}$, and $\boldsymbol{\beta}_{jk}$ are known, $\boldsymbol{\beta}_k$ is obtained as Eq. (33) with $m = i$ and $n = k$, and then $\boldsymbol{\beta}_j$ as (33) with $m = k$ and $n = j$. The velocities can also be discovered by using the constrained composition law. Referring to Figure 4, we compute $\boldsymbol{\beta}_k$ and $\boldsymbol{\beta}_j$ as:

$$\boldsymbol{\beta}_k = \boldsymbol{\beta}_i \oplus \boldsymbol{\beta}_{ki}, \tag{34}$$

$$\boldsymbol{\beta}_j = \boldsymbol{\beta}_k \oplus \boldsymbol{\beta}_{jk} = (\boldsymbol{\beta}_i \oplus \boldsymbol{\beta}_{ki}) \oplus \boldsymbol{\beta}_{jk}. \tag{35}$$

Then, $\boldsymbol{\beta}_{ji}$ is obtained as:

$$\boldsymbol{\beta}_{ji} = (-\boldsymbol{\beta}_i) \oplus \boldsymbol{\beta}_j. \tag{36}$$

The velocity addition operation is not associative and thus the $\boldsymbol{\beta}_i$-related terms in Eq. (36) with the substitution of Eq. (35) are not cancelled. Consequently $\boldsymbol{\beta}_{ji}$ becomes a function of the absolute velocity $\boldsymbol{\beta}_i$ and the relative velocities $\boldsymbol{\beta}_{ki}$ and $\boldsymbol{\beta}_{jk}$. It is straightforward to see that Eqs. (36) and (34) are equal to Eqs. (20) and (33) with $m = i$ and $n = k$, respectively. The reverse velocity is given by $\boldsymbol{\beta}_{ij} = (-\boldsymbol{\beta}_j) \oplus \boldsymbol{\beta}_i$, which is not the same as $-\boldsymbol{\beta}_{ji}$. Even if $\boldsymbol{\beta}_{ji}$ is calculated through node l other than node k, the resultant is identical to Eq. (36) since

$$\begin{aligned}&\boldsymbol{T}_{\mathrm{MS}}(\boldsymbol{\beta}_j, \boldsymbol{\beta}_k)\boldsymbol{T}_{\mathrm{MS}}(\boldsymbol{\beta}_k, \boldsymbol{\beta}_i) \\ &\quad = \boldsymbol{T}_{\mathrm{MS}}(\boldsymbol{\beta}_j, \boldsymbol{\beta}_l)\boldsymbol{T}_{\mathrm{MS}}(\boldsymbol{\beta}_l, \boldsymbol{\beta}_i)(=\boldsymbol{T}_{\mathrm{MS}}(\boldsymbol{\beta}_j, \boldsymbol{\beta}_i)).\end{aligned} \tag{37}$$

In PFT, in which the isotropic frame is unique, the transformation between S_i and S_j is independent of the intermediate frame, so are the relative velocities between nodes i and j. The relationship Eq. (37) results in the consistency of the constrained composition law. As $\boldsymbol{\beta}_j$ is independent of intermediate nodes, we have

$$(\boldsymbol{\beta}_i \oplus \boldsymbol{\beta}_{ki}) \oplus \boldsymbol{\beta}_{jk} = (\boldsymbol{\beta}_i \oplus \boldsymbol{\beta}_{li}) \oplus \boldsymbol{\beta}_{jl}. \tag{38}$$

The equality of Eq. (38) results from Eq. (37).

Now, it is time to examine the validity of the constrained composition law. From $\boldsymbol{p}_{(k)} = \boldsymbol{A}\boldsymbol{p}_{(i)}$, where $\boldsymbol{A} = \boldsymbol{T}_{\mathrm{MS}}(\boldsymbol{\beta}_k, \boldsymbol{\beta}_i)$, the coordinate vector $\boldsymbol{p}_{(k)}$ is written as $\boldsymbol{p}_{(k)} = \boldsymbol{A}\boldsymbol{T}_L(\boldsymbol{\beta}_i)\boldsymbol{p}$. The velocity composition law of SR is derived from the successive LTs. If $\boldsymbol{A}$ has a form of LT, $\boldsymbol{\beta}_k$ can be obtained by the composition law, but $\boldsymbol{A}$ is different from $\boldsymbol{T}_L(\boldsymbol{\beta}_{ki})$. Nonetheless we can discover $\boldsymbol{\beta}_k$ by it, as explained in the following. The first rows of $\boldsymbol{A}$ and $\boldsymbol{T}_L(\boldsymbol{\beta}_{ki})$ are equal, so are the first rows of $\boldsymbol{A}\boldsymbol{T}_L(\boldsymbol{\beta}_i)$ and $\boldsymbol{T}_L(\boldsymbol{\beta}_{ki})\boldsymbol{T}_L(\boldsymbol{\beta}_i)$. When the standard synchronization is employed, the relative velocity is determined only by the first row of the transformation matrix, as shown in Eq. (19). Hence, the velocity of S_k is given by Eq. (34). The transformation from S to S_j is $\boldsymbol{A}\boldsymbol{T}_L(\boldsymbol{\beta}_k)$, where $\boldsymbol{A} = \boldsymbol{T}_{\mathrm{MS}}(\boldsymbol{\beta}_j, \boldsymbol{\beta}_k)$, and similarly $\boldsymbol{\beta}_j$ is calculated as Eq. (35). The constrained composition law can be successively applied.

When nodes i and j are connected through N nodes as in Figure 5, we can calculate $\boldsymbol{\beta}_j$ and $\boldsymbol{\beta}_{ji}$ from $\boldsymbol{\beta}_i$ and the relative velocities by successively applying the composition law. The velocity $\boldsymbol{\beta}_j$ is given by

$$\boldsymbol{\beta}_j = ((\cdots((\boldsymbol{\beta}_i \oplus \boldsymbol{\beta}_{k_1 i}) \oplus \boldsymbol{\beta}_{k_2 k_1}) \oplus \cdots \oplus \boldsymbol{\beta}_{k_N k_{N-1}}) \oplus \boldsymbol{\beta}_{jk_N}). \tag{39}$$

The multiple composition proceeds from left to right. Substituting Eq. (39) in Eq. (36) yields:

$$\boldsymbol{\beta}_{ji} = (-\boldsymbol{\beta}_i) \oplus ((\cdots((\boldsymbol{\beta}_i \oplus \boldsymbol{\beta}_{k_1 i}) \oplus \boldsymbol{\beta}_{k_2 k_1}) \oplus \cdots \oplus \boldsymbol{\beta}_{k_N k_{N-1}}) \oplus \boldsymbol{\beta}_{jk_N}). \tag{40a}$$

The velocities $\boldsymbol{\beta}_{ik_1}$, $\boldsymbol{\beta}_{k_l k_{l+1}}$, $l = 1, \cdots, N-1$, $\boldsymbol{\beta}_{k_N j}$, and $\boldsymbol{\beta}_j$ are needed to obtain the reverse velocity $\boldsymbol{\beta}_{ij}$ and can be found from $\boldsymbol{\beta}_i$, $\boldsymbol{\beta}_{k_1 i}$, $\boldsymbol{\beta}_{k_{l+1} k_l}$, and $\boldsymbol{\beta}_{jk_N}$. If they are available, then the reverse velocity is calculated as:

$$\boldsymbol{\beta}_{ij} = (-\boldsymbol{\beta}_j) \oplus ((\cdots((\boldsymbol{\beta}_j \oplus \boldsymbol{\beta}_{k_N j}) \oplus \boldsymbol{\beta}_{k_{N-1} k_N}) \oplus \cdots \oplus \boldsymbol{\beta}_{k_1 k_2}) \oplus \boldsymbol{\beta}_{ik_1}). \tag{40b}$$

Each relative velocity from Eqs. (40a) and (40b), which are dependent only on $\boldsymbol{\beta}_i$ and $\boldsymbol{\beta}_j$, remains the same for every N and arbitrary intermediate nodes in virtue of the equality (37). The constrained velocity composition law is consistent.

3.3 Discussion

If SR is consistent, its velocity composition should also satisfy both Eqs. (40a) and (40b) because the LT is employed for coordinate transformations between S and inertial frames. However, the $\bar{\boldsymbol{\beta}}_{ji/N}$ of Eq. (27a), which is not a function of $\boldsymbol{\beta}_i$, is other than the $\boldsymbol{\beta}_{ji}$ of Eq. (40a) depending on $\boldsymbol{\beta}_i$. The former is dependent on connecting nodes and so it is not uniquely determined, varying with them. As a result, PT in SR is not uniquely determined either. The PT of an observer O_j at rest in S_j is absolute in that it has the same value from frame to frame irrespective of the synchronization of clocks. Since the velocity of S_j in S_i by the composition law is dependent on the connecting nodes, so is its magnitude, which leads the PT to depend on them as well so that it is not uniquely determined. The absoluteness of PT holds under the uniqueness of the isotropic frame. From Eqs. (2) and (17) with $l = j$, the differential PT of O_j is given by [22]:

$$\mathrm{d}\tau_{(j)\circ} = \frac{\mathrm{d}\tau}{\gamma_j}, \tag{41a}$$

and from Eqs. (14), (11), and (18a),

$$\mathrm{d}\tau_{(j)\circ} = \frac{\mathrm{d}\tau_{(i)}}{\gamma_{ji}}. \tag{41b}$$

Note that Eq. (41b) is valid even if i and j are interchanged despite $\gamma_{ij} = \gamma_{ji}$. Eq. (5) is a different representation of the same $\boldsymbol{p}_{(j)}$ expressed as Eq. (2) with $l = j$ and Eq. (41b) is a different representation of the same $\mathrm{d}\tau_{(j)^\circ}$ expressed as Eq. (41a). Under the uniqueness of the isotropic frame, PT is absolute, having the same value as Eq. (41a) from frame to frame regardless of the intermediate frames.

The non-equality Eq. (23) in SR, which results in Eq. (25), shows that the principle of relativity is not satisfied. It causes inconsistencies and paradoxes. The origin of the relativity principle is generally recognized to be attributed to Galileo [18]. In the Galilean transformation (GaT), if the velocity of S_j relative to S_i is $\boldsymbol{\beta}_{ji}$, the velocity of S_i relative to S_j is $-\boldsymbol{\beta}_{ji}$ since $\boldsymbol{T}_{\mathrm{Ga}}^{-1}(\boldsymbol{\beta}) = \boldsymbol{T}_{\mathrm{Ga}}(-\boldsymbol{\beta})$ where $\boldsymbol{T}_{\mathrm{Ga}}(\boldsymbol{\beta})$ is the GaT matrix when the relative velocity is $\boldsymbol{\beta}$, namely:

$$\boldsymbol{T}_{\mathrm{Ga}}(\boldsymbol{\beta}) = \begin{bmatrix} 1 & \mathbf{0} \\ i\boldsymbol{\beta} & \boldsymbol{I} \end{bmatrix}. \tag{42}$$

It is obvious that

$$\boldsymbol{T}_{\mathrm{Ga}}(\boldsymbol{\beta}_{ji}) = \boldsymbol{T}_{\mathrm{Ga}}(\boldsymbol{\beta}_{jk})\boldsymbol{T}_{\mathrm{Ga}}(\boldsymbol{\beta}_{ki}) = \boldsymbol{T}_{\mathrm{Ga}}(\boldsymbol{\beta}_{jl})\boldsymbol{T}_{\mathrm{Ga}}(\boldsymbol{\beta}_{li}), \tag{43}$$

where $\boldsymbol{\beta}_{ji} = \boldsymbol{\beta}_{ki} + \boldsymbol{\beta}_{jk}$. In Figure 4, given $\boldsymbol{\beta}_{ki}$ and $\boldsymbol{\beta}_{jk}$, the relative velocities between nodes i and j are clearly given by:

$$\boldsymbol{\beta}_{ji} = \boldsymbol{\beta}_{ki} + \boldsymbol{\beta}_{jk} = -\boldsymbol{\beta}_{iR} + \boldsymbol{\beta}_{jR}, \tag{44a}$$

$$\boldsymbol{\beta}_{ij} = -\boldsymbol{\beta}_{ki} + (-\boldsymbol{\beta}_{jk}) = \boldsymbol{\beta}_{iR} + (-\boldsymbol{\beta}_{jR}). \tag{44b}$$

Given $\boldsymbol{\beta}_{k_1 i}$, $\boldsymbol{\beta}_{k_{l+1} k_l}$, $l = 1, \cdots, N-1$, and $\boldsymbol{\beta}_{jk_N}$ as in Figure 5, they are written as:

$$\boldsymbol{\beta}_{ji} = \boldsymbol{\beta}_{k_1 i} + \boldsymbol{\beta}_{k_2 k_1} + \cdots + \boldsymbol{\beta}_{k_N k_{N-1}} + \boldsymbol{\beta}_{j k_N} = -\boldsymbol{\beta}_{iR} + \boldsymbol{\beta}_{jR}, \quad (45a)$$

$$\boldsymbol{\beta}_{ij} = \boldsymbol{\beta}_{k_N j} + \boldsymbol{\beta}_{k_{N-1} k_N} + \cdots + \boldsymbol{\beta}_{k_1 k_2} + \boldsymbol{\beta}_{i k_1} = -\boldsymbol{\beta}_{jR} + \boldsymbol{\beta}_{iR}, \quad (45b)$$

where $\boldsymbol{\beta}_{k_N j} = -\boldsymbol{\beta}_{j k_N}$, $\boldsymbol{\beta}_{i k_1} = -\boldsymbol{\beta}_{k_1 i}$, and $\boldsymbol{\beta}_{k_l k_{l+1}} = -\boldsymbol{\beta}_{k_{l+1} k_l}$. Both Eqs. (45a) and (45b) are satisfied for every N and arbitrary node R. The consistency in GaT results from the equality (43). In PFT, each relative velocity given by Eqs. (40a) and (40b) remains the same for every N and all the $N!$ permutations of the connecting nodes. The consistency in PFT results from the equality (37).

Though the composition law is not associative, the investigation under the assumption of the associativity would be helpful for deepening the understanding of its inconsistency. Suppose that the relativistic velocity addition is associative, but not commutative. Then, the parentheses in Eq. (27a) can be eliminated, and the terms of $\boldsymbol{\beta}_i$ in Eq. (40a) are canceled so that the resulting $\boldsymbol{\beta}_{ji}$ becomes equal to the $\bar{\boldsymbol{\beta}}_{ji/N}$. If the velocity addition was associative, the relative velocity could be obtained by successively applying it without the absolute velocity, but Eq. (27a) does not remain the same with respect to the permutations because it is not commutative. As for the reverse velocity, Eq. (27b) itself is still inconsistent on account of $\bar{\boldsymbol{\beta}}_{ij/N} \neq \bar{\boldsymbol{\beta}}_{ij/N-1}$, as can be seen from Eq. (32), even though the associativity is assumed. The inconsistency results from the non-commutativity of the composition. The associativity is not sufficient to hold the principle of relativity. In order to fulfil it so that each velocity in Eqs. (27a) and (27b) is the same for every N and all the permutations, the velocity addition should be associative and commutative, as in GaT.

SR is based on the principle of relativity and the isotropy of the speed of light. The inconsistency in the velocity composition law of SR results from the non-equality (23), which indicates that the principle of relativity is infeasible under the light speed constancy. The anisotropy of the speed of light has been observed in the experiments of the Sagnac effect. Recently, under the uniqueness of the isotropic frame, the generalized Sagnac effect [23,25–27], which involves linear motion as well as circular motion, has been comprehensively analyzed based on the MS general framework [23] and by using the transformation under constant light speed (TCL) [26,27]. The analysis results show that the speed of light is anisotropic in inertial frames as well as in rotating ones. TCL provides a coordinate transformation between the rotating and the inertial frames, holding the constancy of the two-way speed of light also in the rotating one. Though the rotating system is in motion with acceleration, it can be regarded as locally inertial. Accordingly, a transformation for inertial frames can be derived from TCL through the limit operation of circular motion to linear motion [26,27]. The derived inertial transformation, the predictions of which are in agreement with the results of the experiments for the validation of SR, indicates the anisotropy of the speed of light as well. Moreover, even if the LT is employed for the transformation of inertial frames, the actual speed of light, with respect to proper time, is shown to be anisotropic in inertial frames [26]. TCL and the inertial transformation, which are consistent with each other, are consistent with PFT.

4 Conclusion

It has been known that the velocity composition of SR is not consistent with the reciprocity principle, in a situation where two nodes i and j of interest are connected through a single intermediate node k, as illustrated in Figure 1. When they are connected *via* multiple nodes, besides the violation of the reciprocity principle, a large number of inconsistencies have been found as the velocity by the composition law is dependent on the connecting nodes. Though the velocity of S_j relative to S_i should remain the same regardless of them, it does not so because of the non-equality Eq. (23) or Eq. (25). As the number of connecting nodes increases, the inconsistency is more deepened since the velocity comes to depend on more nodes. When the inertial frames S_i and S_j are connected *via* N nodes as in Figure 2, the forward and reverse velocities are expressed as Eqs. (27a) and (27b), respectively. If SR is consistent, both velocities should be independent of the connecting nodes such that they do not vary with respect to the $N!$ permutations of the nodes. However neither of them remains the same for the permutations since the velocity addition is not commutative or associative. Moreover, Eq. (27b) itself is inconsistent because $\bar{\boldsymbol{\beta}}_{ij/N} \neq \bar{\boldsymbol{\beta}}_{ij/N-1}$. As a result of these inconsistencies, PT also becomes dependent on the connecting nodes so that it is not uniquely determined.

The problems can be cured by introducing a reference node R such that the velocity composition is carried out in conjunction with it. In fact, the reference node is the unique isotropic frame. In the PFT, the relative velocity between the inertial frames, which depends only on their absolute velocities, is independent of the connecting nodes and remains the same for the permutation. According to the postulates of SR, the frame R is nothing more than one of an infinite number of isotropic frames. Hence, Eqs. (45a) and (45b) with '+' replaced by '$\oplus$' when performed from right to left should be satisfied for every R and arbitrary connecting nodes, which is mathematically infeasible. If the isotropic frame is unique, the velocity composition law is consistent and PT is absolute, not depending on connecting nodes. As far as kinematics is concerned, the equivalence of inertial frames conforms to the GaT, which is, however, incompatible with time dilation. The isotropic frame is unique [22]. Nature itself reveals the uniqueness of the isotropic frame.

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References

[1] Einstein A. On the electrodynamics of moving bodies. Ann Phys Leipzig. 1905;322(10):891–921.
[2] Gwinner G. Experimental tests of time dilation in special relativity. Mod Phys Lett A. 2005;20(11):791–805.
[3] MacArthur DW. Special relativity: Understanding experimental tests and formulations. Phys Rev D. 1986;33(1):1–5.
[4] Will CM. Clock synchronization and isotropy of the one-way speed of light. Phys Rev D. 1992;45(2):403–11.
[5] Kretzschmar M. Doppler spectroscopy on relativistic particle beams in the light of a test theory of special relativity. Z Phys A. 1992;342(4):463–9.
[6] Mocanu CI. Some difficulties within the framework of relativistic electrodynamics. Arch Elektrotech. 1986;69(2):97–110.
[7] Koczan GM. Relative binary, ternary and pseudo-binary 4D velocities in the special relativity: Part I. Covariant Lorentz transformation. arXiv:2108.10725 [Preprint]; 2021. https://arxiv.org/abs/2108.10725.
[8] Koczan GM. Relativistic relative velocity and relativistic acceleration. Acta Physica Polonica A. 2021;139(4):401–6.
[9] Oziewicz Z. Ternary relative velocity; astonishing conflict of the Lorentz group with relativity. arXiv:1104.0682 [Preprint]; 2011. https://arxiv.org/abs/1104.0682.
[10] Oziewicz Z. How do you add relative velocities? Group theoretical methods in physics. Vol. 185. Conference Series. CRC Press; 2004. p. 439–44. Available from https://www.researchgate.net/publication/253628364_How_do_you_add_relative_velocities/link/56fc9e8108aef6d10d91c11b/download.
[11] Oziewicz Z. Æther needs relativitity group: Lorentz group if and only if Æther. Academia; 2006. https://www.academia.edu/12338869/%C3%86ther_Needs_Relativity_Group_Lorentz_Group_if_and_only_if_%C3%86ther.
[12] Fernández-Guasti M. Alternative realization for the composition of relativistic velocities. Proc SPIE. 2011;8121:812108.
[13] Ungar AA. The relativistic velocity composition paradox and the Thomas rotation. Found Phys. 1989;19(11):1385–96.
[14] Ungar AA. The Thomas rotation and the parameterization of the Lorentz group transformation group. Found Phys Lett. 1988;1(1):57–89.
[15] Kholmetskii AL, Yarman T. Thomas-Wigner rotation and Thomas precession in covariant ether theories: novel approach to experimental verification of special relativity. Can J Phys. 2015;93(5):1232–40.
[16] Tangherlini FR. The velocity of light in uniformly moving frames. [Ph.D. thesis]. USA: Stanford University; 1958.
[17] Selleri F. Noninvariant one-way velocity of light. Found Phys. 1996;26(7):641–64.
[18] de Abreu R, Guerra V. The principle of relativity and the indeterminacy of special relativity. Eur J Phys. 2008;29(1):33–52.
[19] Szostek K, Szostek R. The derivation of the general form of kinematics with the universal reference system. Results in Phys. 2018;8:429–37.
[20] Szostek R. Derivation of all linear transformations that meet the results of Michelson-Morley's experiment and discussion of the relativity basics. Moscow Univ Phys Bull. 2020;75(6):684–704.
[21] Mansouri R, Sexl RU. A test theory of special relativity: I. Simultaneity and clock synchronization. Gen Relativ Gravit. 1977;8(7):497–513.
[22] Choi Y-H. Uniqueness of the isotropic frame and usefulness of the Lorentz transformation. J Kor Phys Soc. 2018;72(10):1110–20.
[23] Choi Y-H. Theoretical analysis of generalized Sagnac effect in the standard synchronization. Can J Phys. 2017;95(8):761–6.
[24] Berzi V, Gorini V. Reciprocity principle and the Lorentz transformations. J Math Phys. 1969;10(8):1518–24.
[25] Wang R, Zheng Y, Yao A. Generalized Sagnac effect. Phys Rev Lett. 2004;93(14):143901.1–143901.3.
[26] Choi Y-H. Consistent coordinate transformation for relativistic circular motion and speeds of light. J Kor Phys Soc. 2019;75(3):176–87.
[27] Choi Y-H. Coordinate transformation between rotating and inertial systems under the constant two-way speed of light. Eur Phys J Plus. 2016;131(9):296.

Appendix

If a coordinate transformation matrix $\boldsymbol{C}$ is known in the standard synchronization, the relative velocity can be obtained from its first row, as shown in Eq. (19). We denote the relative velocity from $\boldsymbol{C}$ by $\boldsymbol{\xi}(\boldsymbol{C})$. According to the designation, $\boldsymbol{\xi}[\boldsymbol{T}_L(\boldsymbol{u})] = \boldsymbol{u}$. A partitioned matrix $\boldsymbol{A}$ is a block diagonal matrix with $A_{11} = 1$, $\boldsymbol{A}_{12} = \mathbf{0}$, and $\boldsymbol{A}_{21} = \mathbf{0}$. The following fundamental properties are important for the derivation of Eq. (28):

$$\boldsymbol{\xi}[\boldsymbol{A}\boldsymbol{T}_L(\boldsymbol{u})\boldsymbol{B}] = \boldsymbol{\xi}[\boldsymbol{T}_L(\boldsymbol{u})\boldsymbol{B}], \tag{A1a}$$

$$\boldsymbol{\xi}[\boldsymbol{T}_L(\boldsymbol{u})\boldsymbol{A}\boldsymbol{B}] = \boldsymbol{\xi}[\boldsymbol{T}_L(\boldsymbol{A}_{22}^T\boldsymbol{u})\boldsymbol{B}], \tag{A1b}$$

where $\boldsymbol{B}$ is an identity matrix or an arbitrary matrix associated with coordinate transformation. As the relative velocity is determined by the first row of the transformation matrix, the equalities in Eq. (A1) mean that the first rows of the total matrices in square brackets on one side are equal to the respective ones on the other side. It is straightforward to see that the first rows of $\boldsymbol{A}\boldsymbol{T}_L(\boldsymbol{u})$ and $\boldsymbol{T}_L(\boldsymbol{u})$ are identical and so are those of $\boldsymbol{A}\boldsymbol{T}_L(\boldsymbol{u})\boldsymbol{B}$ and $\boldsymbol{T}_L(\boldsymbol{u})\boldsymbol{B}$. Similarly the equality (A1b) is shown since the first rows of $\boldsymbol{T}_L(\boldsymbol{u})\boldsymbol{A}$ and $\boldsymbol{T}_L(\boldsymbol{A}_{22}^T\boldsymbol{u})$ are identical.

The successive transformation $\boldsymbol{T}_L(\boldsymbol{u}_n)\boldsymbol{T}_L(\boldsymbol{u}_m)$ can be written as:

$$\boldsymbol{T}_L(\boldsymbol{u}_n)\boldsymbol{T}_L(\boldsymbol{u}_m) = \boldsymbol{A}\boldsymbol{T}_L(\boldsymbol{u}_m \oplus \boldsymbol{u}_n), \tag{A2}$$

where $\boldsymbol{A}$ is calculated as Eq. (29). Using Eqs. (A1a) and (A2), we have

$$\begin{aligned}\boldsymbol{\xi}(\boldsymbol{C}) &= \boldsymbol{\xi}[\boldsymbol{A}\boldsymbol{T}_L(\boldsymbol{u}_2 \oplus \boldsymbol{u}_3)\boldsymbol{T}_L(\boldsymbol{u}_1)]\\ &= \boldsymbol{\xi}[\boldsymbol{T}_L(\boldsymbol{u}_2 \oplus \boldsymbol{u}_3)\boldsymbol{T}_L(\boldsymbol{u}_1)]\\ &= \boldsymbol{u}_1 \oplus (\boldsymbol{u}_2 \oplus \boldsymbol{u}_3),\end{aligned} \tag{A3a}$$

where $\boldsymbol{C} = \boldsymbol{T}_L(\boldsymbol{u}_3)\boldsymbol{T}_L(\boldsymbol{u}_2)\boldsymbol{T}_L(\boldsymbol{u}_1)$ and $\boldsymbol{A}$ is given by Eq. (29) with $m = 2$ and $n = 3$. It follows from (A1) and (A2) that

$$\begin{aligned}\boldsymbol{\xi}(\boldsymbol{C}) &= \boldsymbol{\xi}[\boldsymbol{T}_L(\boldsymbol{u}_3)\boldsymbol{A}\boldsymbol{T}_L(\boldsymbol{u}_1 \oplus \boldsymbol{u}_2)]\\ &= \boldsymbol{\xi}[\boldsymbol{T}_L(\boldsymbol{A}_{22}^T\boldsymbol{u}_3)\boldsymbol{T}_L(\boldsymbol{u}_1 \oplus \boldsymbol{u}_2)]\\ &= (\boldsymbol{u}_1 \oplus \boldsymbol{u}_2) \oplus \boldsymbol{A}_{22}^T\boldsymbol{u}_3,\end{aligned} \tag{A3b}$$

where $\boldsymbol{A}$ is given by Eq. (29) with $m = 1$ and $n = 2$. Eqs. (A3a) and (A3b) lead to Eq. (28a).

Using (A2), we have

$$\boldsymbol{\xi}(\mathbf{C}) = \boldsymbol{\xi}[\mathbf{T}_L(\mathbf{u}_3)\mathbf{T}_L(\mathbf{u}_1 \oplus \mathbf{u}_2)] = (\mathbf{u}_1 \oplus \mathbf{u}_2) \oplus \mathbf{u}_3, \tag{A4a}$$

where $\boldsymbol{C} = \boldsymbol{T}_L(\boldsymbol{u}_3)\boldsymbol{A}^{-1}\boldsymbol{T}_L(\boldsymbol{u}_2)\boldsymbol{T}_L(\boldsymbol{u}_1)$ and $\boldsymbol{A}$ is given by Eq. (29) with $m = 1$ and $n = 2$. Since $\boldsymbol{A}$ is a block diagonal matrix, $\boldsymbol{A}^{-1}$ is also a block diagonal matrix and its (2,2)-entry is $\boldsymbol{A}_{22}^{-1}$. It is easy to see by using Eq. (29) that $\boldsymbol{A}^{-1} = \boldsymbol{A}^T$, which results in $\boldsymbol{A}_{22}^{-1} = \boldsymbol{A}_{22}^T$. From Eqs. (A1) and (A2)

$$\boldsymbol{\xi}(\boldsymbol{C}) = \boldsymbol{\xi}[\boldsymbol{T}_L(\boldsymbol{A}_{22}^{-T}\boldsymbol{u}_3)\boldsymbol{T}_L(\boldsymbol{u}_2)\boldsymbol{T}_L(\boldsymbol{u}_1)] = \boldsymbol{u}_1 \oplus (\boldsymbol{u}_2 \oplus \boldsymbol{A}_{22}\boldsymbol{u}_3). \tag{A4b}$$

Eqs. (A4a) and (A4b) lead to Eq. (28b).

Eq. (27a) can be derived by repeatedly applying Eq. (A1a). It is clear that

$$\begin{aligned}\boldsymbol{\xi}(\boldsymbol{T}_{L/N}) &= \boldsymbol{\xi}[\boldsymbol{A}\boldsymbol{T}_L(\boldsymbol{\beta}_{k_Nk_{N-1}} \oplus \boldsymbol{\beta}_{jk_N})\boldsymbol{T}_L(\boldsymbol{\beta}_{k_{N-1}k_{N-2}}) \cdots\\ &\quad \times \boldsymbol{T}_L(\boldsymbol{\beta}_{k_2k_1})\boldsymbol{T}_L(\boldsymbol{\beta}_{k_1i})]\\ &= \boldsymbol{\xi}[\boldsymbol{T}_L(\boldsymbol{\beta}_{k_Nk_{N-1}} \oplus \boldsymbol{\beta}_{jk_N})\boldsymbol{T}_L(\boldsymbol{\beta}_{k_{N-1}k_{N-2}}) \cdots\\ &\quad \times \boldsymbol{T}_L(\boldsymbol{\beta}_{k_2k_1})\boldsymbol{T}_L(\boldsymbol{\beta}_{k_1i})]\\ &= \boldsymbol{\xi}[\boldsymbol{T}_L(\boldsymbol{\beta}_{k_{N-1}k_{N-2}} \oplus (\boldsymbol{\beta}_{k_Nk_{N-1}}\\ &\quad \oplus \boldsymbol{\beta}_{jk_N}))\boldsymbol{T}_L(\boldsymbol{\beta}_{k_{N-2}k_{N-2}}) \cdots \boldsymbol{T}_L(\boldsymbol{\beta}_{k_2k_1})\boldsymbol{T}_L(\boldsymbol{\beta}_{k_1i})].\end{aligned} \tag{A5}$$

Continuing such a computation leads to Eq. (27a). Similarly Eq. (27b) can be obtained. The inverse of $\boldsymbol{T}_{L/N}$ is written as

$$\boldsymbol{T}_{L/N}^{-1} = \boldsymbol{T}_L(-\boldsymbol{\beta}_{k_1i})\boldsymbol{T}_L(-\boldsymbol{\beta}_{k_2k_1}) \cdots \boldsymbol{T}_L(-\boldsymbol{\beta}_{k_Nk_{N-1}})\boldsymbol{T}_L(-\boldsymbol{\beta}_{jk_N}). \tag{A6}$$

It is straightforward to see that $\boldsymbol{\xi}(\boldsymbol{T}_{L/N}^{-1})$ is given as Eq. (27b).