Research Article

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Propagation of some new traveling wave patterns of the double dispersive equation

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Abstract: This article aims to address the exact solution of the prestigious partial differential equation, namely, a double dispersive equation. Here, we are obtaining some new traveling wave solutions of the double dispersive equation with the more general mathematical technique, which is a direct algebraic extended method. This proposed technique is more general and integrated. The obtained solutions contain dark, bright, dark–bright, singular, periodic, kink, and rational function solutions. More illustration of traveling wave solutions of the double dispersive equation is given by plotting the two- and three-dimensional graphs with the suitable selection of parameters. This graphical presentation of solutions identifies the pattern of wave propagation. The acquired consequences are new and may play a significant role to examine the physical phenomena of wave propagation, where this model is used.

Keywords: new direct extended algebraic method, traveling wave solutions

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1 Introduction

The first water wave model was introduced by a Swiss Mathematician Leonard Euler in the eighteenth century [1]. Afterward, various scientists and mathematicians derived a variety of models in the fields of Physics and Applied Mathematics. Joseph Boussinesq was the first to present the Boussinesq equation utilizing the Euler equation [2]. The Boussinesq model is utilized to describe the motion of the water waves with a long wavelength and a small amplitude [3]. The Boussinesq model is also suitable for delineating the phenomenon of plasma waves, heat transfer, acoustic waves, and propagation of electrical signals on the dispersive transmission [4]. There are many different Boussinesq-type models that exist in the literature, which have different physical and geometrical structures [5], sudden variation in the motion of the deep water [6], higher-order gradient elasticity, and narrow rod [7]. The two-dimensional (2D) non-linear longitudinal long-wave models have been introduced [8]. The numerous variety of Boussinesq-type models with different dispersive terms are depicted [9,10].

The partial differential equations containing non-linearity have been utilized for the illustration of various noteworthy non-linear physical phenomena in the diversified disciplines of science. Some of them are, solid-state physics, plasma physics, ocean waves, non-linear and linear optics, chemistry, condensed matter physics, atmospheric waves, and mathematical material science [11–14]. A large number of Mathematicians have paid attention to analyze such types of non-linear phenomena and to uncover the complexity of the solutions [15–18]. The exact solutions are imperative for understanding the vital characteristics of the various phenomena. The exact consequences are at the high priority to make difference between the mechanisms of distinct remarkable non-linear models, like non-appearance of steady state under the different constraints, the existence of peaking regimes, *etc.*

Various effective and efficient mathematical techniques have been developed to construct the soliton solutions of Boussinesq-type completely integrable models. To analyze the $(2 + 1)$-dimensional Boussinesq-type model, many N-solitons solution have been produced by using the bilinear method [19]. Many different kinds of solitary solutions have also been produced by $\left(\frac{G'}{G}\right)$ expansion method [20]. Wazwaz has studied the soliton solution and singular soliton solutions of the Boussinesq model [21]. Samsonov investigated the dependence of a traveling wave solution on the general cubic double dispersive and phase variable of double dispersive equation [22]. The traveling and similarity solutions of double dispersive equations for multi-dimensional have been investigated using the classic Lie group method [23]. Jacobi elliptic function has appeared as a solution of the double dispersion model in the non-homogeneous Murnaghan's rod and also introduced non-topological, topological, compound solitons and singular periodic wave solutions [24].

Some latest work is added here. Two integrated schemes have been applied to establish the chirped W-shape bright, dark, and other soliton solutions to perturbed nonlinear Schrödinger's equation with conformable fractional order [25]. Nestor is investigating a series of new optical soliton solutions to the perturbed nonlinear Schrödinger equation [26]. The exact traveling wave solutions to the higher-order nonlinear Schrödinger equation having Kerr non-linearity form are derived utilizing the extended sinh-Gordon expansion and Jacobi-elliptic function method [27]. The nonlinear multicore coupling (coupling with all neighbors) with the parabolic law non-linearity and optical meta-material parameters is done by using the sub-ordinary differential equation scheme [28]. The authors investigate the solitary waves to the discrete nonlinear electrical transmission lines with resonant tunneling diode [29]. The diverse soliton solutions and new chirped bright and dark solitons, trigonometric function solutions, and rational solutions have been obtained by adopting two formal integration methods [30]. The obtained results of the Klein–Gordon–Zakharov equations are diverse and some specific ones emerge as dark, bright, and bell-shape [31]. The modified Sardar sub-equation method is used to construct optical soliton solutions to the coupled nonlinear Schrödinger equation having third-order and fourth-order dispersions [32]. The new extended auxiliary equation method and the generalized Kudryashov method have been applied to secure soliton solutions for the higher order dispersive nonlinear Schrödinger's equation [33].

The semi inverse variational law has been used to find the exact solution of $(1 + 1)$-dimensional Boussinesq-type models [34]. Some remarkable tools are presented here, which are used to construct the exact solution of prestigious evolution non-linear equations like sinh-Gordon equation method [36], F-expansion method [37], Hirota bilinear method [38,39], Jacobi elliptic method [40], $\left(\frac{G'}{G^2}\right)$ expansion method [41], modified Kudryashov method [42,43], Bernoulli sub-equation function method, [44] *etc*. The new extended algebraic equation method is a significant technique, which provides the generalized solution to peruse the soliton solutions of a highly nonlinear and integrable partial differential equation. Thus, it is meaningful to use in a broader way to investigate the physical phenomenon [45]. The considered equation is

$$\phi_{tt} - c^2\phi_{xx} = \frac{\beta}{2\gamma}(\phi^2)_{xx} - \frac{\vartheta(1-\vartheta)R^2}{2}\phi_{xxtt} + \frac{\vartheta c^2R^2}{2}\phi_{xxxx}, \tag{1}$$

where γ represents the material density, c is the wave speed with linearity, the Poisson ratio delineates by ϑ, R is the radius of rod, and β is the constant. This appraised model is non-linear evolution partial differential equation with double dispersion term. It gives rise to the specification of properties of mechanics states with time fluctuation in the quantum mechanics. It has an indispensable significance in mathematics as well as physics and other fields of science in which including optics, molecular biology, fluid mechanics, quantum mechanics, water surface and plasma physics, *etc*. This equation is proficient to utilize as a model of diverse physical problems. In the field of optical fiber with non-linearity, the optical soliton is a strange subject matter because it has extensive applications in telecommunications (short optical pulse) and ultra-fast routing signal system. The attaining of exact consequences in the sort of solitons and solitary waves has become a glowing avenue.

The soliton solutions of the double dispersive equation were obtained by various researchers but the latest work on this model was done with the sine-Gordon expansion method by Yel [1], which was the generalized work till now. But in this work, the used scheme is not much as compared to our used technique, that is why some special types of solitons solutions missed such as rational solution, logarithmic solutions, kink-type solution, and many more. So, to fulfill this curiosity, the direct extended algebraic method has been operated on the double dispersive equation and acquired some novel results that did not exist in the literature. The achieved results have decisive applications in the field of applied mathematics, engineering, and physics.

This article is sorted as Section 2 is dedicated to the illustration of the direct extended algebraic method. The application to the method on the considered model and graphical delineation is given in Section 3. Finally, at the end concluding comments are embellished.

2 Illustration of proposed technique

The proposed scheme is reliable [46].

Consider non-linear partial differential equation of the form:

$$Q(\phi, \phi_t, \phi_x, \phi_{tt}, \phi_{xx}, \ldots) = 0, \quad (2)$$

which can be converted into non-linear ordinary differential equation:

$$G(\psi, \psi', \psi'', \ldots) = 0, \quad (3)$$

by applying complex transformation:

$$\phi(x, t) = \psi(\chi_1), \quad (4)$$

where $\chi_1 = k_1x + k_2t$ and here prime in Eq. (3) represents the notation of differentiation. Assume the solutions of Eq. (3) is in the form:

$$\psi(\chi_1) = a_0 + \sum_{i=-m}^{m} [a_i(W(\chi_1))^i], \quad (5)$$

where

$$W'(\chi_1) = \ln(\rho)(\mu_1 + \nu_1 W(\chi_1) + \zeta W^2(\chi_1)), \rho \neq 0, 1, \quad (6)$$

while ζ, μ_1, and ν_1 are real constants.

The general solutions with respect to parameters μ_1, ν_1, and ζ of Eq. (6) are as follows:

(Case 1): When $\nu_1^2 - 4\mu_1\zeta < 0$ but $\zeta \neq 0$,

$$W_1(\chi_1) = -\frac{\nu_1}{2\zeta} + \frac{\sqrt{-(\nu_1^2 - 4\mu_1\zeta)}}{2\zeta} \tan_\rho\left(\frac{\sqrt{-(\nu_1^2 - 4\mu_1\zeta)}}{2}\chi_1\right), \quad (7)$$

$$W_2(\chi_1) = -\frac{\nu_1}{2\zeta} - \frac{\sqrt{-(\nu_1^2 - 4\mu_1\zeta)}}{2\zeta} \cot_\rho\left(\frac{\sqrt{-(\nu_1^2 - 4\mu_1\zeta)}}{2}\chi_1\right), \quad (8)$$

$$W_3(\chi_1) = -\frac{\nu_1}{2\zeta} + \frac{\sqrt{-(\nu_1^2 - 4\mu_1\zeta)}}{2\zeta}(\tan_\rho(\sqrt{-(\nu_1^2 - 4\mu_1\zeta)}\chi_1) \pm \sqrt{mn}\sec_\rho(\sqrt{-(\nu_1^2 - 4\mu_1\zeta)}\chi_1)), \quad (9)$$

$$W_4(\chi_1) = -\frac{\nu_1}{2\zeta} + \frac{\sqrt{-(\nu_1^2 - 4\mu_1\zeta)}}{2\zeta}(\cot_\rho(\sqrt{-(\nu_1^2 - 4\mu_1\zeta)}\chi_1) \pm \sqrt{mn}\csc_\rho(\sqrt{-(\nu_1^2 - 4\mu_1\zeta)}\chi_1)), \quad (10)$$

$$W_5(\chi_1) = -\frac{\nu_1}{2\zeta} + \frac{\sqrt{-(\nu_1^2 - 4\mu_1\zeta)}}{4\zeta}\left(\tan_\rho\left(\frac{\sqrt{-(\nu_1^2 - 4\mu_1\zeta)}}{4}\chi_1\right) - \cot_\rho\left(\frac{\sqrt{-(\nu_1^2 - 4\mu_1\zeta)}}{4}\chi_1\right)\right). \quad (11)$$

(Case 2): When $\nu_1^2 - 4\mu_1\zeta > 0$ and $\zeta \neq 0$,

$$W_6(\chi_1) = -\frac{\nu_1}{2\zeta} - \frac{\sqrt{\nu_1^2 - 4\mu_1\zeta}}{2\zeta}\tanh_\rho\left(\frac{\sqrt{\nu_1^2 - 4\mu_1\zeta}}{2}\chi_1\right), \quad (12)$$

$$W_7(\chi_1) = -\frac{\nu_1}{2\zeta} - \frac{\sqrt{\nu_1^2 - 4\mu_1\zeta}}{2\zeta}\coth_\rho\left(\frac{\sqrt{\nu_1^2 - 4\mu_1\zeta}}{2}\chi_1\right), \quad (13)$$

$$W_8(\chi_1) = -\frac{\nu_1}{2\zeta} + \frac{\sqrt{\nu_1^2 - 4\mu_1\zeta}}{2\zeta}(-\tanh_\rho(\sqrt{\nu_1^2 - 4\mu_1\zeta}\chi_1) \pm i\sqrt{mn}\,\mathrm{sech}_\rho(\sqrt{\nu_1^2 - 4\mu_1\zeta}\chi_1)), \quad (14)$$

$$W_9(\chi_1) = -\frac{\nu_1}{2\zeta} + \frac{\sqrt{\nu_1^2 - 4\mu_1\zeta}}{2\zeta}(-\coth_\rho(\sqrt{\nu_1^2 - 4\mu_1\zeta}\varsigma) \pm \sqrt{mn}\,\mathrm{csch}_\rho(\sqrt{\nu_1^2 - 4\mu_1\zeta}\chi_1)), \quad (15)$$

$$W_{10}(\chi_1) = -\frac{\nu_1}{2\zeta} - \frac{\sqrt{\nu_1^2 - 4\mu_1\zeta}}{4\zeta}\left(\tanh_\rho\left(\frac{\sqrt{\nu_1^2 - 4\mu_1\zeta}}{4}\chi_1\right) + \coth_\rho\left(\frac{\sqrt{\nu_1^2 - 4\mu_1\zeta}}{4}\chi_1\right)\right). \quad (16)$$

(Case 3): When $\mu_1\zeta > 0$ and $\nu_1 = 0$,

$$W_{11}(\chi_1) = \sqrt{\frac{\mu_1}{\zeta}}\tan_\rho(\sqrt{\mu_1\zeta}\chi_1), \quad (17)$$

$$W_{12}(\chi_1) = -\sqrt{\frac{\mu_1}{\zeta}}\cot_\rho(\sqrt{\mu_1\zeta}\chi_1), \quad (18)$$

$$W_{13}(\chi_1) = \sqrt{\frac{\mu_1}{\zeta}}(\tan_\rho(2\sqrt{\mu_1\zeta}\chi_1) \pm \sqrt{mn}\sec_\rho(2\sqrt{\mu_1\zeta}\chi_1)), \quad (19)$$

$$W_{14}(\varsigma) = \sqrt{\frac{\mu_1}{\zeta}}(-\cot_\rho(2\sqrt{\mu_1\zeta}\chi_1) \pm \sqrt{mn}\csc_\rho(2\sqrt{\mu_1\zeta}\chi_1)), \quad (20)$$

$$W_{15}(\chi_1) = \frac{1}{2}\sqrt{\frac{\mu_1}{\zeta}}\left(\tan_\rho\left(\frac{\sqrt{\mu_1\zeta}}{2}\chi_1\right) - \cot_\rho\left(\frac{\sqrt{\mu_1\zeta}}{2}\chi_1\right)\right). \quad (21)$$

(Case 4): When $\mu_1\zeta < 0$ and $\nu_1 = 0$,

$$W_{16}(\chi_1) = -\sqrt{-\frac{\mu_1}{\zeta}}\tanh_\rho(\sqrt{-\mu_1\zeta}\chi_1), \quad (22)$$

$$W_{17}(\chi_1) = -\sqrt{-\frac{\mu_1}{\zeta}}\coth_\rho(\sqrt{-\mu_1\zeta}\chi_1), \quad (23)$$

$$W_{18}(\chi_1) = \sqrt{-\frac{\mu_1}{\zeta}}(-\tanh_\rho(2\sqrt{-\mu_1\zeta}\chi_1) \pm i\sqrt{mn}\,\mathrm{sech}_\rho(2\sqrt{-\mu_1\zeta}\chi_1)), \quad (24)$$

$$W_{19}(\chi_1) = \sqrt{-\frac{\mu_1}{\zeta}}(-\coth_\rho(2\sqrt{-\mu_1\zeta}\chi_1) \pm \sqrt{mn}\,\mathrm{csch}_\rho(2\sqrt{-\mu_1\zeta}\chi_1)), \quad (25)$$

$$W_{20}(\chi_1) = -\frac{1}{2}\sqrt{-\frac{\mu_1}{\zeta}}\left(\tanh_\rho\left(\frac{\sqrt{-\mu_1\zeta}}{2}\chi_1\right) + \coth_\rho\left(\frac{\sqrt{-\mu_1\zeta}}{2}\chi_1\right)\right). \quad (26)$$

(Case 5): When $\nu_1 = 0$ but $\mu_1 = \zeta$,

$$W_{21}(\chi_1) = \tan_\rho(\mu_1\chi_1), \quad (27)$$

$$W_{22}(\chi_1) = -\cot_\rho(\mu_1\chi_1), \quad (28)$$

$$W_{23}(\chi_1) = \tan_\rho(2\mu_1\chi_1) \pm \sqrt{mn}\sec_\rho(2\mu_1\chi_1), \quad (29)$$

$$W_{24}(\chi_1) = -\cot_\rho(2\mu_1\chi_1) \pm \sqrt{mn}\csc_\rho(2\mu_1\chi_1), \quad (30)$$

$$W_{25}(\chi_1) = \frac{1}{2}\left(\tan_\rho\left(\frac{\mu_1}{2}\chi_1\right) - \cot_\rho\left(\frac{\mu_1}{2}\chi_1\right)\right). \quad (31)$$

(Case 6): When $\nu_1 = 0$ but $\zeta = -\mu_1$,

$$W_{26}(\chi_1) = -\tanh_\rho(\mu_1\chi_1), \quad (32)$$

$$W_{27}(\chi_1) = -\coth_\rho(\mu_1\chi_1), \quad (33)$$

$$W_{28}(\chi_1) = -\tanh_\rho(2\mu_1\chi_1) \pm i\sqrt{mn}\,\mathrm{sech}_\rho(2\mu_1\chi_1), \quad (34)$$

$$W_{29}(\chi_1) = -\cot_\rho(2\mu_1\chi_1) \pm \sqrt{mn}\,\mathrm{csch}_\rho(2\mu_1\chi_1), \quad (35)$$

$$W_{30}(\chi_1) = -\frac{1}{2}\left(\tanh_\rho\left(\frac{\mu_1}{2}\chi_1\right) + \cot_\rho\left(\frac{\mu_1}{2}\chi_1\right)\right). \quad (36)$$

(Case 7): When $\nu_1^2 = 4\mu_1\zeta$,

$$W_{31}(\chi_1) = \frac{-2\mu_1(\nu_1\chi_1\ln\rho + 2)}{\nu_1^2\chi_1\ln\rho}. \quad (37)$$

(Case 8): When $\nu_1 = p$ and $\mu_1 = pq$, $(q \neq 0)$ but $\zeta = 0$,

$$W_{32}(\chi_1) = \rho^{p\chi_1} - q. \quad (38)$$

(Case 9): When $\nu_1 = \zeta = 0$,

$$W_{33}(\chi_1) = \mu_1\chi_1\ln\rho. \quad (39)$$

(Case 10): When $\nu_1 = \mu_1 = 0$,

$$W_{34}(\chi_1) = \frac{-1}{\zeta\chi_1\ln\rho}. \quad (40)$$

(Case 11): When $\mu_1 = 0$ but $\nu_1 \neq 0$,

$$W_{35}(\chi_1) = -\frac{m\nu_1}{\zeta(\cosh_\rho(\nu_1\chi_1) - \sinh_\rho(\nu_1\chi_1) + m)}, \quad (41)$$

$$W_{36}(\chi_1) = -\frac{\nu_1(\sinh_\rho(\nu_1\chi_1) + \cosh_\rho(\nu_1\chi_1))}{\zeta(\sinh_\rho(\nu_1\chi_1) + \cosh_\rho(\nu_1\chi_1) + n)}. \quad (42)$$

(Case 12): When $\nu_1 = p$, and $\zeta = pq$, $(q \neq 0$ but $\mu_1 = 0)$,

$$W_{37}(\chi_1) = -\frac{m\rho^{p\chi_1}}{m - qn\rho^{p\chi_1}}. \quad (43)$$

$$\sinh_\rho(\chi_1) = \frac{m\rho_1^{X} - n\rho^{-\chi_1}}{2}, \quad \cosh_\rho(\chi_1) = \frac{m\rho_1^{X} + n\rho^{-\chi_1}}{2},$$

$$\tanh_\rho(\chi_1) = \frac{m\rho_1^{X} - n\rho^{-\chi_1}}{m\rho_1^{X} + n\rho^{-\chi_1}}, \quad \coth_\rho(\chi_1) = \frac{m\rho_1^{X} + n\rho^{-\chi_1}}{m\rho_1^{X} - n\rho^{-\chi_1}},$$

$$\mathrm{sech}_\rho(\chi_1) = \frac{2}{m\rho_1^{X} + n\rho^{-\chi_1}}, \quad \mathrm{csch}_\rho(\chi_1) = \frac{2}{m\rho_1^{X} - n\rho^{-\chi_1}},$$

$$\sin_\rho(\chi_1) = \frac{m\rho^{i\chi_1} - n\rho^{-i\chi_1}}{2i}, \quad \cos_\rho(\chi_1) = \frac{m\rho^{i\chi_1} + n\rho^{-i\chi_1}}{2},$$

$$\tan_\rho(\chi_1) = -i\frac{m\rho^{i\chi_1} - n\rho^{-i\chi_1}}{m\rho^{i\chi_1} + n\rho^{-i\chi_1}}, \quad \cot_\rho(\chi_1) = i\frac{m\rho^{i\chi_1} + n\rho^{-i\chi_1}}{m\rho^{i\chi_1} - n\rho^{-i\chi_1}},$$

$$\sec_\rho(\chi_1) = \frac{2}{m\rho_1^{X} + n\rho^{-\chi_1}}, \quad \csc_\rho(\chi_1) = \frac{2i}{m\rho_1^{X} - n\rho^{-\chi_1}}.$$

The deformation parameters m and n are arbitrary constants greater than zero.

3 Application to Eq. (1)

To find the solution of Eq. (1), we introduced a linear traveling wave transformation:

$$\phi = \phi(x,t), \quad \text{where } \phi(x,t) = \psi(\chi_1) = \psi(\varrho(x - kt)). \quad (44)$$

By the implementation of Eq. (44) on the Eq. (1), we will obtain:

$$(\vartheta R^2\varrho^2\gamma((1-\vartheta)k^2 - c^2))\psi'' + 2\gamma(k^2 - c^2)\psi - \beta\psi^2 = 0. \quad (45)$$

By setting of homogeneous balancing constant for Eq. (45), we assume the solution of the form:

$$\psi(\chi_1) = a_0 + a_1(W(\chi_1)) + a_2(W(\chi_1)^2), \quad (46)$$

where

$$W'(\chi_1) = \ln(\rho)(\mu_1 + \nu_1 W(\chi_1) + \zeta W^2(\chi_1)), \rho \neq 0, 1. \quad (47)$$

After plugging Eq. (46) in Eq. (45) and then equating the coefficients of distinct powers for $W(\chi_1)$, we obtain the algebraic system. After solving the obtained system by using `Mathematica`, we obtain following sets of solutions:

Set 1:

$$a_0 = \frac{2\gamma c^2 \varrho^2 \vartheta^2 R^2(\nu_1^2 + 2\zeta\mu_1)\log[\rho]^2}{d(-2 + \varrho^2\vartheta R^2(1 - \vartheta)\Pi\log[\rho^2])}, \quad (48)$$

$$a_1 = \frac{12\gamma c^2 \varrho^2 \vartheta^2 R^2 \nu_1\zeta \log[\rho]^2}{d(-2 + \varrho^2\vartheta R^2(1 - \vartheta)\Pi\log[\rho^2])}, \quad (49)$$

$$a_2 = \frac{12\gamma c^2 \varrho^2 \vartheta^2 R^2 \zeta^2 \log[\rho]^2}{d(-2 + \varrho^2\vartheta R^2(1 - \vartheta)\Pi\log[\rho^2])}, \quad (50)$$

$$k = \pm\frac{\sqrt{c^2(-2 + \varrho^2\vartheta R^2\Pi\log[\rho]^2)}}{\sqrt{-2 + \varrho^2\vartheta R^2(1 - \vartheta)\Pi\log[\rho]^2}}. \quad (51)$$

The general solution of Eq. (1) after plugging Eq. (48), Eq. (49), and Eq. (50) in Eq. (46) we obtain:

$$\phi(x, t) = \frac{2\gamma c^2 \varrho^2 \vartheta^2 R^2(\nu_1^2 + 2\zeta\mu_1)\log[\rho]^2}{d(-2 + \varrho^2\vartheta R^2(1 - \vartheta)\Pi\log[\rho^2])} + \frac{12\gamma c^2 \varrho^2 \vartheta^2 R^2 \nu_1\zeta \log[\rho]^2}{d(-2 + \varrho^2\vartheta R^2(1 - \vartheta)\Pi\log[\rho^2])}[W_i(\mu_1(x - kt))] + \frac{12\gamma c^2 \varrho^2 \vartheta^2 R^2 \zeta^2 \log[\rho]^2}{d(-2 + \varrho^2\vartheta R^2(1 - \vartheta)\Pi\log[\rho^2])} \times [W_i(\mu_1(x - kt))]^2. \quad (52)$$

Set 2:

$$a_0 = -\frac{12\gamma c^2 \varrho^2 \vartheta^2 R^2 \zeta\mu_1 \log[\rho]^2}{d(2 + \varrho^2\vartheta R^2(1 - \vartheta)\Pi\log[\rho^2])}, \quad (53)$$

$$a_1 = -\frac{12\gamma c^2 \varrho^2 \vartheta^2 R^2 \nu_1\zeta \log[\rho]^2}{d(2 + \varrho^2\vartheta R^2(1 - \vartheta)\Pi\log[\rho^2])}, \quad (54)$$

$$a_2 = -\frac{12\gamma c^2 \varrho^2 \vartheta^2 R^2 \zeta^2 \log[\rho]^2}{d(2 + \varrho^2\vartheta R^2(1 - \vartheta)\Pi\log[\rho^2])}, \quad (55)$$

$$k = \pm\frac{\sqrt{c^2(2 + \varrho^2\vartheta R^2\Pi\log[\rho]^2)}}{\sqrt{2 + \varrho^2\vartheta R^2(1 - \vartheta)\Pi\log[\rho]^2}}. \quad (56)$$

The general solution of Eq. (1) by plugging Eq. (53), Eq. (54), and Eq. (55) in Eq. (46) is as follows:

$$\phi(x, t) = -\frac{2\gamma c^2 \varrho^2 \vartheta^2 R^2 \zeta\mu_1 \log[\rho]^2}{d(2 + \varrho^2\vartheta R^2(1 - \vartheta)\Pi\log[\rho^2])} - \frac{12\gamma c^2 \varrho^2 \vartheta^2 R^2 \nu_1\zeta \log[\rho]^2}{d(2 + \varrho^2\vartheta R^2(1 - \vartheta)\Pi\log[\rho^2])}[W_i(\mu_1(x - kt))] - \frac{12\gamma c^2 \varrho^2 \vartheta^2 R^2 \zeta^2 \log[\rho]^2}{d(2 + \varrho^2\vartheta R^2(1 - \vartheta)\Pi\log[\rho^2])} \times [W_i(\mu_1(x - kt))^2]. \quad (57)$$

3.1 Traveling wave patterns of Eq. (1)

Set 1:

Now, we will obtain many different solutions by taking W_i from (7)–(43), respectively.

(Case 1): When $\nu_1^2 - 4\mu_1\zeta < 0$ and $\zeta \neq 0$,

$$\phi_1(x, t) = (\nu_1^2 + 2\zeta\mu_1)\Upsilon + 6\nu_1\zeta\Upsilon\left(-\frac{\nu_1}{2\zeta} + \frac{\sqrt{-\Pi}}{2\zeta}\tan_\rho\left(\frac{\sqrt{-\Pi}}{2}(\varrho(x - kt))\right)\right) + 6\zeta^2\Upsilon\left(-\frac{\nu_1}{2\zeta} + \frac{\sqrt{-\Pi}}{2\zeta}\tan_\rho\left(\frac{\sqrt{-\Pi}}{2}(\varrho(x - kt))\right)\right)^2, \quad (58)$$

$$\phi_2(x, t) = (\nu_1^2 + 2\zeta\mu_1)\Upsilon + 6\nu_1\zeta\Upsilon\left(-\frac{\nu_1}{2\zeta} - \frac{\sqrt{-\Pi}}{2\zeta}\cot_\rho\left(\frac{\sqrt{-\Pi}}{2}(\varrho(x - kt))\right)\right) + 6\zeta^2\Upsilon\left(-\frac{\nu_1}{2\zeta} - \frac{\sqrt{-\Pi}}{2\zeta}\cot_\rho\left(\frac{\sqrt{-\Pi}}{2}(\varrho(x - kt))\right)\right)^2,$$

$$\phi_3(x, t) = (\nu_1^2 + 2\zeta\mu_1)\Upsilon + 6\nu_1\zeta\Upsilon\left(-\frac{\nu_1}{2\zeta} + \frac{\sqrt{-\Pi}}{2\zeta}(\tan_\rho(\sqrt{-\Pi}(\varrho(x - kt))) \pm \sqrt{mn}\sec_\rho\sqrt{-\Pi}(\varrho(x - kt)))\right) + 6\zeta^2\Upsilon\left(-\frac{\nu_1}{2\zeta} + \frac{\sqrt{-\Pi}}{2\zeta}(\tan_\rho(\sqrt{-\Pi}(\varrho(x - kt))) \pm \sqrt{mn}\sec_\rho(\sqrt{-\Pi}(\varrho(x - kt))))\right)^2, \quad (59)$$

$$\begin{aligned}\phi_4(x,t) &= (\nu_1^2+2\zeta\mu_1)\Upsilon + 6\nu_1\zeta\Upsilon\left(-\frac{\nu_1}{2\zeta}+\frac{\sqrt{-\Pi}}{2\zeta}(\cot_\rho(\sqrt{-\Pi}(\varrho(x-kt))) \pm \sqrt{mn}\csc_\rho(\sqrt{-\Pi}(\varrho(x-kt))))\right)\\ &\quad + 6\zeta^2\Upsilon\left(-\frac{\nu_1}{2\zeta}+\frac{\sqrt{-\Pi}}{2\zeta}(\cot_\rho(\sqrt{-\Pi}(\varrho(x-kt))) \pm \sqrt{mn}\csc_\rho(\sqrt{-\Pi}(\varrho(x-kt))))\right)^2,\end{aligned} \tag{60}$$

$$\begin{aligned}\phi_5(x,t) &= (\nu_1^2+2\zeta\mu_1)\Upsilon + 6\nu_1\zeta\Upsilon\left(-\frac{\nu_1}{2\zeta}+\frac{\sqrt{-\Pi}}{4\zeta}\left(\tan_\rho\left(\frac{\sqrt{-\Pi}}{4}(\varrho(x-kt))\right)-\cot_\rho\left(\frac{\sqrt{-\Pi}}{4}(\varrho(x-kt))\right)\right)\right)\\ &\quad + 6\zeta^2\Upsilon\left(-\frac{\nu_1}{2\zeta}+\frac{\sqrt{-\Pi}}{4\zeta}\left(\tan_\rho\left(\frac{\sqrt{-\Pi}}{4}(\varrho(x-kt))\right)-\cot_\rho\left(\frac{\sqrt{-\Pi}}{4}(\varrho(x-kt))\right)\right)\right)^2.\end{aligned} \tag{61}$$

(Case 2): When $\nu_1^2-4\mu_1\zeta>0$ and $\zeta\neq 0$,

$$\begin{aligned}\phi_6(x,t) &= (\nu_1^2+2\zeta\mu_1)\Upsilon + 6\nu_1\zeta\Upsilon\left(-\frac{\nu_1}{2\zeta}-\frac{\sqrt{\Pi}}{2\zeta}\tanh_\rho\left(\frac{\sqrt{\Pi}}{2}(\varrho(x-kt))\right)\right)\\ &\quad + 6\zeta^2\Upsilon\left(-\frac{\nu_1}{2\zeta}-\frac{\sqrt{\Pi}}{2\zeta}\tanh_\rho\left(\frac{\sqrt{\Pi}}{2}(\varrho(x-kt))\right)\right)^2,\end{aligned} \tag{62}$$

$$\phi_7(x,t) = (\nu_1^2+2\zeta\mu_1)\Upsilon + 6\nu_1\zeta\Upsilon\left(-\frac{\nu_1}{2\zeta}-\frac{\sqrt{\Pi}}{2\zeta}\coth_\rho\left(\frac{\sqrt{\Pi}}{2}\chi_1\right)\right) + 6\zeta^2\Upsilon\left(-\frac{\nu_1}{2\zeta}-\frac{\sqrt{\Pi}}{2\zeta}\coth_\rho\left(\frac{\sqrt{\Pi}}{2}\chi_1\right)\right)^2, \tag{63}$$

$$\begin{aligned}\phi_8(x,t) &= (\nu_1^2+2\zeta\mu_1)\Upsilon + 6\nu_1\zeta\Upsilon\left(-\frac{\nu_1}{2\zeta}+\frac{\sqrt{\Pi}}{2\zeta}(-\tanh_\rho(\sqrt{\Pi}(\varrho(x-kt))) \pm i\sqrt{mn}\,\mathrm{sech}_\rho(\sqrt{\Pi}(\varrho(x-kt))))\right)\\ &\quad + 6\zeta^2\Upsilon\left(-\frac{\nu_1}{2\zeta}+\frac{\sqrt{\Pi}}{2\zeta}(-\tanh_\rho(\sqrt{\Pi}(\varrho(x-kt))) \pm i\sqrt{mn}\,\mathrm{sech}_\rho(\sqrt{\Pi}(\varrho(x-kt))))\right)^2,\end{aligned} \tag{64}$$

$$\begin{aligned}\phi_9(x,t) &= (\nu_1^2+2\zeta\mu_1)\Upsilon + 6\nu_1\zeta\Upsilon\left(-\frac{\nu_1}{2\zeta}+\frac{\sqrt{\Pi}}{2\zeta}(-\coth_\rho(\sqrt{\Pi}(\varrho(x-kt))) \pm \sqrt{mn}\,\mathrm{csch}_\rho(\sqrt{\Pi}(\varrho(x-kt))))\right)\\ &\quad + 6\zeta^2\Upsilon\left(-\frac{\nu_1}{2\zeta}+\frac{\sqrt{\Pi}}{2\zeta}(-\coth_\rho(\sqrt{\Pi}(\varrho(x-kt))) \pm \sqrt{mn}\,\mathrm{csch}_\rho(\sqrt{\Pi}(\varrho(x-kt))))\right)^2,\end{aligned} \tag{65}$$

$$\begin{aligned}\phi_{10}(x,t) &= (\nu_1^2+2\zeta\mu_1)\Upsilon + 6\nu_1\zeta\Upsilon\left(-\frac{\nu_1}{2\zeta}-\frac{\sqrt{\Pi}}{4\zeta}\left(\tanh_\rho\left(\frac{\sqrt{\Pi}}{4}(\varrho(x-kt))\right)+\coth_\rho\left(\frac{\sqrt{\Pi}}{4}(\varrho(x-kt))\right)\right)\right)\\ &\quad + 6\zeta^2\Upsilon\left(-\frac{\nu_1}{2\zeta}-\frac{\sqrt{\Pi}}{4\zeta}\left(\tanh_\rho\left(\frac{\sqrt{\Pi}}{4}(\varrho(x-kt))\right)+\coth_\rho\left(\frac{\sqrt{\Pi}}{4}(\varrho(x-kt))\right)\right)\right)^2.\end{aligned} \tag{66}$$

(Case 3): When $\mu_1\zeta>0$ and $\nu_1=0$,

$$\phi_{11}(x,t) = 2\zeta\mu_1\Upsilon + 6\zeta^2\Upsilon\left(\sqrt{\frac{\mu_1}{\zeta}}\tan_\rho(\sqrt{\mu_1\zeta}(\varrho(x-kt)))\right)^2, \tag{67}$$

$$\phi_{12}(x,t) = 2\zeta\mu_1\Upsilon + 6\zeta^2\Upsilon\left(-\sqrt{\frac{\mu_1}{\zeta}}\cot_\rho(\sqrt{\mu_1\zeta}(\varrho(x-kt)))\right)^2, \tag{68}$$

$$\phi_{13}(x,t) = 2\zeta\mu_1\Upsilon + 6\zeta^2\Upsilon\left(\sqrt{\frac{\mu_1}{\zeta}}(\tan_\rho(2\sqrt{\mu_1\zeta}(\varrho(x-kt))) \pm \sqrt{mn}\sec_\rho(2\sqrt{\mu_1\zeta}(\varrho(x-kt))))\right)^2, \tag{69}$$

$$\phi_{14}(x,t)=2\zeta\mu_1\Upsilon+6\zeta^2\Upsilon\left(\sqrt{\frac{\mu_1}{\zeta}}(-\cot_\rho(2\sqrt{\mu_1\zeta}(\varrho(x-kt)))\pm\sqrt{mn}\csc_\rho(2\sqrt{\mu_1\zeta}(\varrho(x-kt))))\right)^2, \tag{70}$$

$$\phi_{15}(x,t)=2\zeta\mu_1\Upsilon+6\zeta^2\Upsilon\left(\frac{1}{2}\sqrt{\frac{\mu_1}{\zeta}}\left(\tan_\rho\left(\frac{\sqrt{\mu_1\zeta}}{2}(\varrho(x-kt))\right)-\cot_\rho\left(\frac{\sqrt{\mu_1\zeta}}{2}(\varrho(x-kt))\right)\right)\right)^2. \tag{71}$$

(Case 4): When $\mu_1\zeta<0$ and $\nu_1=0$,

$$\phi_{16}(x,t)=2\zeta\mu_1\Upsilon+6\zeta^2\Upsilon\left(-\sqrt{-\frac{\mu_1}{\zeta}}\tanh_\rho(\sqrt{-\mu_1\zeta}(\varrho(x-kt)))\right)^2, \tag{72}$$

$$\phi_{17}(x,t)=2\zeta\mu_1\Upsilon+6\zeta^2\Upsilon\left(-\sqrt{-\frac{\mu_1}{\zeta}}\coth_\rho(\sqrt{-\mu_1\zeta}(\varrho(x-kt)))\right)^2, \tag{73}$$

$$\phi_{18}(x,t)=2\zeta\mu_1\Upsilon+6\zeta^2\Upsilon\left(\sqrt{-\frac{\mu_1}{\zeta}}(-\tanh_\rho(2\sqrt{-\mu_1\zeta}(\varrho(x-kt)))\pm i\sqrt{mn}\,\mathrm{sech}_\rho(2\sqrt{-\mu_1\zeta}(\varrho(x-kt))))\right)^2, \tag{74}$$

$$\phi_{19}(x,t)=2\zeta\mu_1\Upsilon+6\zeta^2\Upsilon\left(\sqrt{-\frac{\mu_1}{\zeta}}(-\coth_\rho(2\sqrt{-\mu_1\zeta}(\varrho(x-kt)))\pm\sqrt{mn}\,\mathrm{csch}_\rho(2\sqrt{-\mu_1\zeta}(\varrho(x-kt))))\right)^2, \tag{75}$$

$$\phi_{20}(x,t)=2\zeta\mu_1\Upsilon+6\zeta^2\Upsilon\left(-\frac{1}{2}\sqrt{-\frac{\mu_1}{\zeta}}\left(\tanh_\rho\left(\frac{\sqrt{-\mu_1\zeta}}{2}(\varrho(x-kt))\right)+\coth_\rho\left(\frac{\sqrt{-\mu_1\zeta}}{2}(\varrho(x-kt))\right)\right)\right)^2. \tag{76}$$

(Case 5): When $\nu_1=0$ but $\mu_1=\zeta$,

$$\phi_{21}(x,t)=2\mu_1^2\Upsilon+6\mu_1^2\Upsilon(\tan_\rho(\mu_1(\varrho(x-kt))))^2, \tag{77}$$

$$\phi_{22}(x,t)=2\mu_1^2\Upsilon+6\mu_1^2\Upsilon(-\cot_\rho(\mu_1(\varrho(x-kt))))^2, \tag{78}$$

$$\phi_{23}(x,t)=2\mu_1^2\Upsilon+6\mu_1^2\Upsilon(\tan_\rho(2\mu_1(\varrho(x-kt)))\pm\sqrt{mn}\sec_\rho(2\mu_1(\varrho(x-kt))))^2, \tag{79}$$

$$\phi_{24}(x,t)=2\mu_1^2\Upsilon+6\mu_1^2\Upsilon(-\cot_\rho(2\mu_1(\varrho(x-kt)))\pm\sqrt{mn}\csc_\rho(2\mu_1(\varrho(x-kt))))^2, \tag{80}$$

$$\phi_{25}(x,t)=2\mu_1^2\Upsilon+6\mu_1^2\Upsilon\left(\frac{1}{2}\left(\tan_\rho\left(\frac{\mu_1}{2}(\varrho(x-kt))\right)-\cot_\rho\left(\frac{\mu_1}{2}(\varrho(x-kt))\right)\right)\right)^2. \tag{81}$$

(Case 6): When $\nu_1=0$ but $\zeta=-\mu_1$,

$$\phi_{26}(x,t)=-2\mu_1^2\Upsilon+6\mu_1^2\Upsilon(-\tanh_\rho(\mu_1(\varrho(x-kt))))^2, \tag{82}$$

$$\phi_{27}(x,t)=-2\mu_1^2\Upsilon+6\mu_1^2\Upsilon(-\coth_\rho(\mu_1(\varrho(x-kt))))^2, \tag{83}$$

$$\phi_{28}(x,t)=-2\mu_1^2\Upsilon+6\mu_1^2\Upsilon(-\tanh_\rho(2\mu_1(\varrho(x-kt)))\pm i\sqrt{mn}\,\mathrm{sech}_\rho(2\mu_1(\varrho(x-kt))))^2, \tag{84}$$

$$\phi_{29}(x,t)=-2\mu_1^2\Upsilon+6\mu_1^2\Upsilon(-\cot_\rho(2\mu_1(\varrho(x-kt)))\pm\sqrt{mn}\,\mathrm{csch}_\rho(2\mu_1(\varrho(x-kt))))^2, \tag{85}$$

$$\phi_{30}(x,t)=-2\mu_1^2\Upsilon+6\mu_1^2\Upsilon\left(-\frac{1}{2}\tanh_\rho\left(\frac{\mu_1}{2}(\varrho(x-kt))\right)+\cot_\rho\left(\frac{\mu_1}{2}(\varrho(x-kt))\right)\right)^2. \tag{86}$$

(Case 7): When $\nu_1^2=4\mu_1\zeta$,

$$\phi_{31}(x,t)=\frac{3\nu_1^2}{2}\Upsilon-3\nu_1\Upsilon\left(\frac{(\nu_1(\varrho(x-kt))\ln\rho+2)}{(\varrho(x-kt))\ln\rho}\right)+\Upsilon\left(\frac{(\nu_1(\varrho(x-kt))\ln\rho+2)}{(\varrho(x-kt))\ln\rho}\right)^2. \tag{87}$$

(Case 8): When $\nu_1 = p$, $\mu_1 = pq$, and $q \neq 0$ but $\zeta = 0$,

$$\phi_{32}(x,t) = p^2. \tag{88}$$

(Case 9): When $\nu_1 = \zeta = 0$,

It is the trivial solution.

(Case 10): When $\nu_1 = 0$ and $\mu_1 = 0$,

$$\phi_{33}(x,t) = 6\zeta^2\Upsilon\left(\frac{-1}{\zeta(\varrho(x-kt))\ln\rho}\right)^2. \tag{89}$$

(Case 11): When $\mu_1 = 0$ but $\nu_1 \neq 0$,

$$\begin{aligned}\phi_{34}(x,t) = \nu_1^2\Upsilon + 6\nu_1\zeta\Upsilon\left(-\frac{m\nu_1}{\zeta(\cosh_\rho(\nu_1(\varrho(x-kt))) - \sinh_\rho(\nu_1(\varrho(x-kt))) + m)}\right)\\ + 6\zeta^2\Upsilon\left(-\frac{m\nu_1}{\zeta(\cosh_\rho(\nu_1(\varrho(x-kt))) - \sinh_\rho(\nu_1(\varrho(x-kt))) + m)}\right)^2,\end{aligned} \tag{90}$$

$$\begin{aligned}\phi_{35}(x,t) = \nu_1^2\Upsilon + 6\nu_1\zeta\Upsilon\left(-\frac{\nu_1(\sinh_\rho(\nu_1(\varrho(x-kt))) + \cosh_\rho(\nu_1(\varrho(x-kt))))}{\zeta(\sinh_\rho(\nu_1(\varrho(x-kt))) + \cosh_\rho(\nu_1(\varrho(x-kt))) + n)}\right)\\ + 6\zeta^2\Upsilon\left(-\frac{\nu_1(\sinh_\rho(\nu_1(\varrho(x-kt))) + \cosh_\rho(\nu_1(\varrho(x-kt))))}{\zeta(\sinh_\rho(\nu_1(\varrho(x-kt))) + \cosh_\rho(\nu_1(\varrho(x-kt))) + n)}\right)^2.\end{aligned} \tag{91}$$

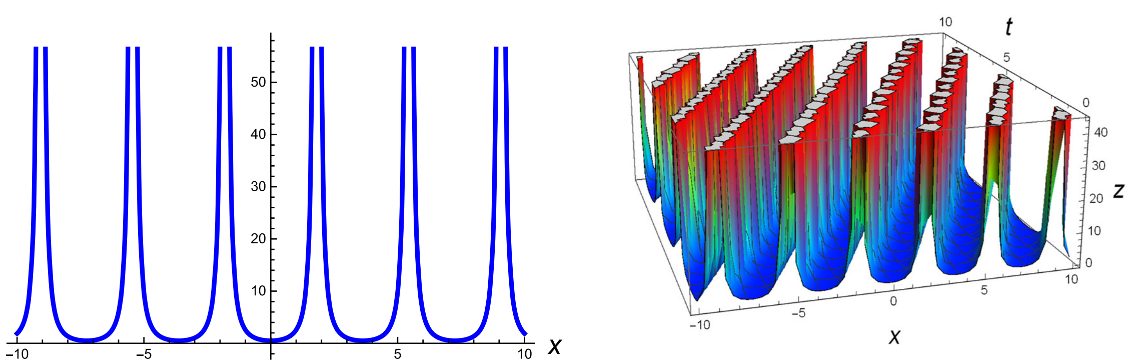

Figure 1: 2D and 3D graphical illustration of $\phi_1(x,t)$ at $\rho = e$, $\gamma = 1$, $c = 1$, $d = -3$, $\varrho = 1$, $R = 1$, $\vartheta = 1$, $\nu_1 = 1$, $\zeta = 1$, and $\mu_1 = 1$.

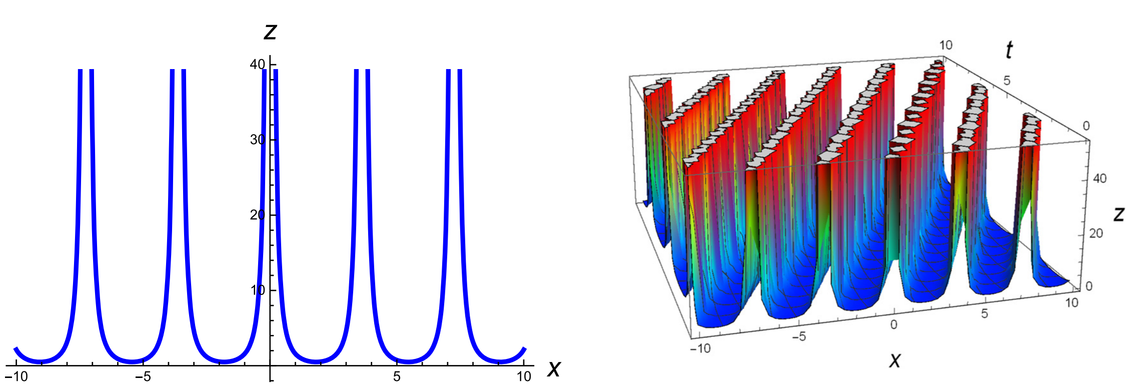

Figure 2: 2D and 3D graphical illustration of $\phi_5(x,t)$ at $\rho = e$, $\gamma = 1$, $c = 1$, $d = -3$, $\varrho = 1$, $R = 1$, $\vartheta = 1$, $\nu_1 = 1$, $\zeta = 1$, and $\mu_1 = 1$.

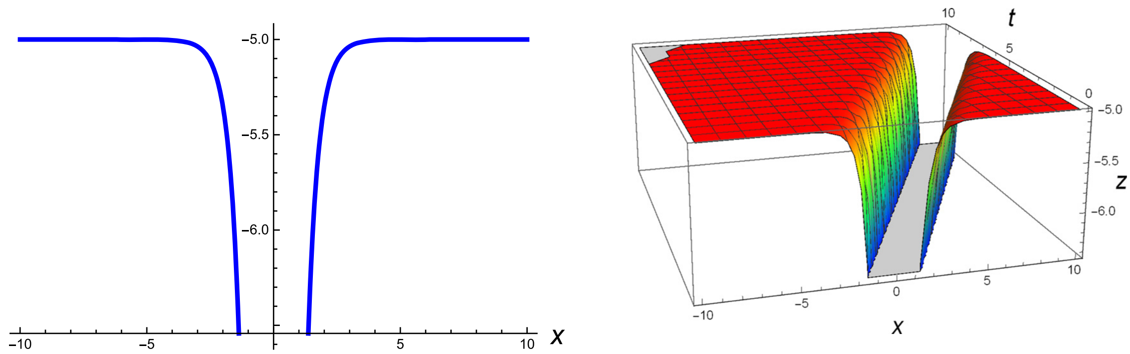

Figure 3: 2D and 3D graphical illustration of $\phi_{10}(x,t)$ at $\rho = e$, $\gamma = 1$, $c = 1$, $d = -1$, $\varrho = 1$, $R = 1$, $\vartheta = 1$, $v_1 = 1$, $\zeta = 1$ and $\mu_1 = -1$.

(Case 12): When $v_1 = p$, $\zeta = pq$, and $q \neq 0$, but $\mu_1 = 0$,

$$\phi_{36}(x,t) = p^2\Upsilon + 6p^2q\Upsilon\left(-\frac{m\rho^{p(\varrho(x-kt))}}{m - qn\rho^{p(\varrho(x-kt))}}\right) + 6(pq)^2\Upsilon\left(-\frac{m\rho^{p(\varrho(x-kt))}}{m - qn\rho^{p(\varrho(x-kt))}}\right)^2, \tag{92}$$

where

$$\Upsilon = \frac{2\gamma c^2\varrho^2\vartheta^2R^2\log[\rho]^2}{d(-2 + \varrho^2\vartheta R^2(1-\vartheta)\Pi\log[\rho^2])} \text{ and } \Pi = v_1^2 - 4\mu_1\zeta. \tag{93}$$

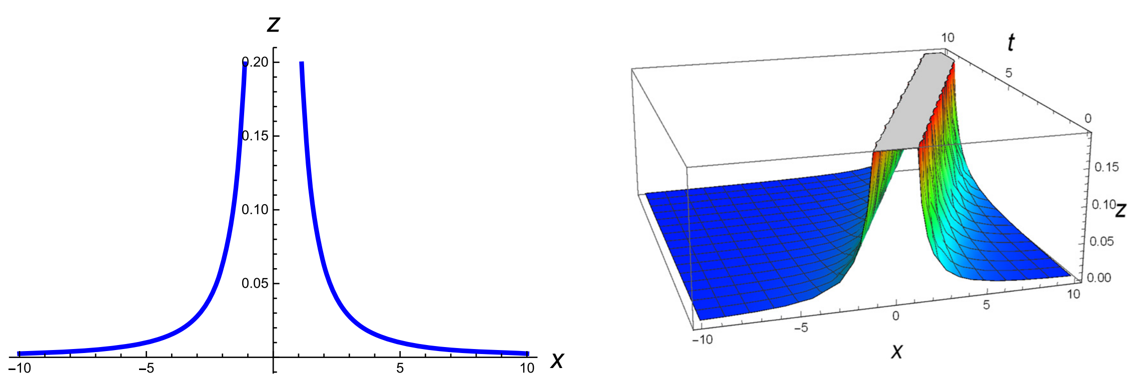

Figure 4: 2D and 3D graphical illustration of $\phi_{31}(x,t)$ at $\rho = e$, $\gamma = 1$, $c = 1$, $d = -6$, $\varrho = 1$, $R = 1$, $\vartheta = 1$, $v_1 = 2$, $\zeta = 1$, and $\mu_1 = 1$.

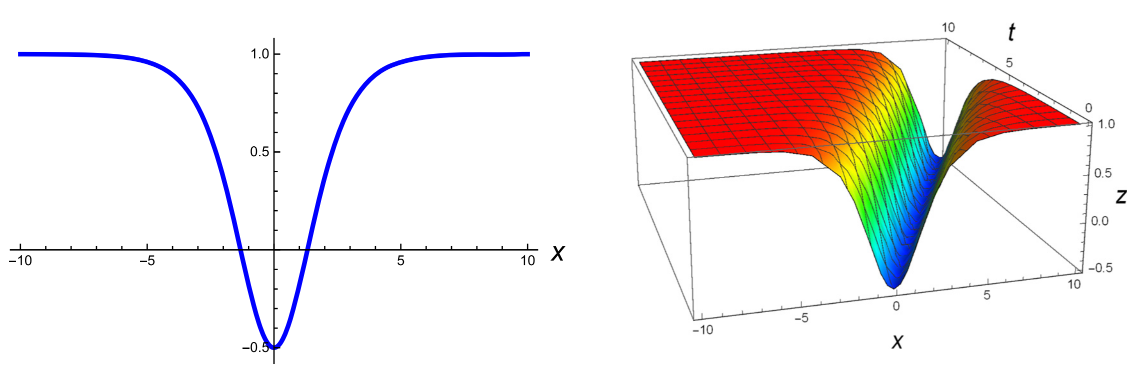

Figure 5: 2D and 3D graphical illustration of $\phi_{34}(x,t)$ at $\rho = e$, $\gamma = 1$, $c = 1$, $d = -1$, $\varrho = 1$, $R = 1$, $\vartheta = 1$, $m = 1$, $v_1 = 1$, $\zeta = 1$, and $\mu_1 = 0$.

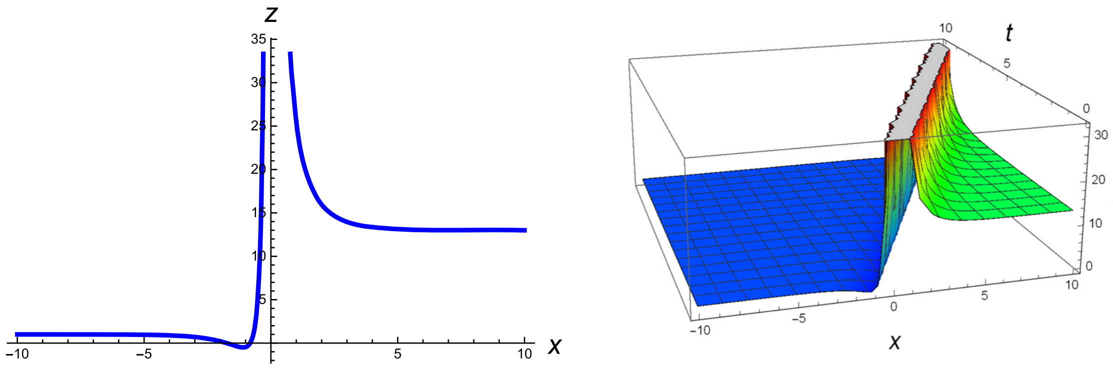

Figure 6: 2D and 3D graphical illustration of $\phi_{36}(x,t)$ at $\rho = e, \gamma = 1, c = 1, d = -1, \varrho = 1, R = 1, \vartheta = 1, m = 1, n = 1, p = 1, q = 1, v_1 = 1, \zeta = 1$, and $\mu_1 = 0$.

Set 2: We will adopt the same pattern as for set 1 to obtain the solutions for set 2.

4 Graphical interpretation

In this section, the physical descriptions for some of the obtained solutions are graphically explained: the 3D and 2D graphical explanation of the explored solutions to the double dispersive equation assuming the particular values of the free parameters in these solutions. These graphics are visualized to display the spatio-temporal distribution of the acquired.

Figure 1 is 2D and 3D graphical representation of $\phi_1(x,t)$ with appropriate free parametric values, which is displaying the dark-periodic wave (the bell shape soliton) solution and propagating along x-axis.

Figure 2 is 2D and 3D graphical representation of $\phi_5(x,t)$ with appropriate free parametric values, displaying the dark-periodic wave (the bell shape soliton) solution and propagating along x-axis.

Figure 3 is 2D and 3D graphical representation of $\phi_{10}(x,t)$ with appropriate free parametric values, displaying the dark-singular wave solution (the bell shape soliton) and propagating along x-axis.

Figure 4 is 2D and 3D graphical representation of $\phi_{31}(x,t)$ with appropriate free parametric values, displaying the bright singular wave solution (the anti-bell shape soliton) and propagating along x-axis.

Figure 5 is 2D and 3D graphical representation of $\phi_{34}(x,t)$ with appropriate free parametric values, displaying the dark wave (the bell shape soliton) solution that is stable. It is passing through origin and symmetric about vertical axis and propagating along x-axis.

Figure 6 is 2D and 3D graphical representation of $\phi_{36}(x,t)$ with appropriate free parametric values, displaying the kink-singular wave solution, and it is passing through origin and propagating along x-axis.

5 Conclusion

In this present work, the extended direct algebraic technique has been successfully applied to the integrable and highly non-linear double dispersive partial differential equations. This proposed method is amalgamation and generalization of the various tools. We have investigated the traveling wave solutions and observed that the acquired solutions have satisfied the governing wave propagation model. In the acquired soliton solutions, the dark soliton, bright solitons, dark–bright, kink, dark singular, bright singular, periodic and rational solitons *etc.*, are incorporated. The functions of such type are beneficial because it permits us to comfortably remark on the physical functioning of waves in any case of a range of graphs for function. All obtained solutions are entirely new and useful. These solutions are obtained with the help of mathematical computational program packets. These solutions are reported by 2D and 3D graphics. The graphical illustration provides a more comprehensive way to understand the physical phenomenon and importance of solutions into experimental, theoretical physics, and engineering. The effectuate consequence dispenses the reports about productivity, mastery, competence, and potential of the applied tool.

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