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Research Article

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Closed-form solutions and conservation laws of a generalized Hirota-Satsuma coupled KdV system of fluid mechanics

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Abstract: In this article, a generalized Hirota–Satsuma coupled Korteweg–de Vries (KdV) system is investigated from the group standpoint. This system represents an interplay of long waves with distinct dispersion correlations. Using Lie's theory several symmetry reductions are performed and the system is reduced to systems of non-linear ordinary differential equations (NLODEs). Subsequently, the simplest equation method is invoked to find exact solutions of the NLODE systems, which then provides the solitary wave solutions for the system under discussion. Finally, we construct conservation laws of generalized Hirota–Satsuma coupled KdV system with the aid of general multiplier approach.

Keywords: generalized Hirota–Satsuma coupled KdV system, Lie's theory, simplest equation method, conservation laws, multiplier method

1 Introduction

Many physical phenomena in numerous disciplines, for instance, plasma physics, engineering, relativity, fluid and classical mechanics, control theory and geochemistry, are modeled by nonlinear evolution equations (NEEs). For some of the recent work published in the literature, see for example refs [1–11], in which conservation laws, multi-soliton, bright, dark and Gaussons optical solutions are presented. These equations are studied by various

scientists and engineers from different aspects. One of the important aspects is the integrability of such NEEs. Because of the importance of NEEs, in the last several decades, a number of techniques were developed by researchers to establish exact closed-form solutions of such NEEs. Some of the predominant techniques that exist in the literature include the sine-Gordon expansion method [12], Bäcklund transformations [13], tanh–coth technique [14], inverse scattering transform technique [15], Hirota's bilinear technique [16], the homogeneous balance of undetermined coefficient technique [17], Darboux transformation technique [18], simplest equation technique [19,20], extended simplest equation technique [21], Kudryashov's method [22], bifurcation technique [23], the first integral method [24], Lie's theory [25–29] and *F*-expansion method [30].

It is common knowledge that the renowned Korteweg-de Vries (KdV) equation [31]

$$u_t + 6uu_x + u_{xxx} = 0$$

models the attributes of solitary waves. It was first formulated as an equation that governed the shallow water waves in channels. However, subsequently it was discovered that it models an extensive variety of natural phenomena, especially those demonstrating solitons and travelling waves.

A coupled KdV system [32]

$$u_t - \frac{1}{2}u_{xxx} + 3uu_x - 6vv_x = 0,$$
 (1.1a)

$$v_t + v_{xxx} - 3uv_x = 0,$$
 (1.1b)

was introduced by Hirota and Satsuma in 1981. In the literature, this system is called the Hirota–Satsuma coupled KdV system and portrays an interplay of two long waves that has distinct dispersion relations. Subsequently, the authors of ref. [33] introduced a generalized version of (1.1), called the generalized Hirota–Satsuma coupled system of KdV equations (gHS-KdVes) that are given by

$$Q_1 \equiv u_t - 3(vw)_x - \frac{1}{2}(u_{xxx} - 6uu_x) = 0,$$
 (1.2a)

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- 19

$$Q_2 \equiv v_t - 3uv_x + v_{yyy} = 0,$$
 (1.2b)

$$Q_3 \equiv w_t - 3uw_x + w_{xxx} = 0.$$
 (1.2c)

This new system received a great deal of attention from the researchers and has been extensively studied. For instance, the decomposition technique [34], the modified extended tanh function technique [35], the homotopy analysis method [36], the differential transform method [37] and the (G'/G)-expansion method [38] were utilized to study the gHS-KdVes (1.2).

In this study, an entirely distinct approach is used to investigate the gHS-KdVes (1.2). Lie's theory combined with the simplest equation technique is employed to find explicit closed-form solutions of (1.2), Subsequently, conserved quantities, using the general multiplier approach, will be computed.

During the eighteenth and nineteenth century, one of the fundamental problems of differential equations (DEs) was to find their closed-form explicit solutions. Perhaps the first explicit solution was the travelling wave solution of the second-order linear wave equation given by d'Alembert in 1747. Fourier developed the separation of variable method during his work on heat conduction problems. Likewise, several other eminent mathematicians contributed to finding exact solutions to linear and nonlinear DEs of physics. S. Lie (1842-1899) and F. Klein (1849-1925) worked on DEs from the point of view of transformation groups that left the DEs unchanged. However, Lie further went onto developing the theory of continuous transformation groups and its applications to DEs [25–29]. Today, Lie's theory is widely used by scientists to find exact closed-form solutions of DEs that arise in countless fields of research. See for example refs [5-11].

Conservation laws are pivotal in the investigation of solutions to DEs. These are laws of nature that are expressed as mathematical expressions. For instance, we have conservation of momentum, energy, angular momentum, charge, just to mention a few. Conservation laws have been utilized to determine the existence, uniqueness and moreover, stability of DEs. Conservation laws are exploited in the investigation of numerical techniques and can also be used in the reduction of order and solution process of DEs [39–44].

This article is organized as follows. In Section 2, we calculate symmetries and perform symmetry reductions of gHS-KdVes (1.2) using Lie's theory and according to the optimal systems of one-dimensional Lie subalgebras of (1.2). We then invoke the simplest equation technique to obtain exact explicit solutions of (1.2). Thereafter, with

the aid of general multiplier method, conservation laws are obtained in Section 3. Finally, in Section 4 we put forward concluding remarks.

2 Symmetry reductions and exact solutions

We first compute symmetry group of gHS-KdVes (1.2). This is the classical group of point transformations of the dependent and independent variables that map solutions of (1.2) into new solutions of (1.2). Assume that the symmetry group is generated by the vector field

$$S = \xi^{1} \frac{\partial}{\partial t} + \xi^{2} \frac{\partial}{\partial x} + \eta^{1} \frac{\partial}{\partial u} + \eta^{2} \frac{\partial}{\partial v} + \eta^{3} \frac{\partial}{\partial w}, \quad (2.3)$$

where the infinitesimals ξ^1 , ξ^2 , η^1 , η^2 , η^3 depend on t, x, u, v, w. The generator (2.3) is a point symmetry of (1.2) if Lie's invariance condition

$$S^{[3]}(Q_1)|_{Q_1=Q_2=Q_3=0}=0, \quad S^{[3]}(Q_2)|_{Q_1=Q_2=Q_3=0}=0,$$

 $S^{[3]}(Q_3)|_{Q_1=Q_2=Q_3=0}=0$

holds. Here $S^{[3]}$ denotes third prolongation of S [28]. The above three equations, on expanding, give a system of linear PDEs, whose solution yields the following four symmetries of (1.2):

$$S_{1} = \frac{\partial}{\partial x},$$

$$S_{2} = \frac{\partial}{\partial t},$$

$$S_{3} = -v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w},$$

$$S_{4} = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 4v\frac{\partial}{\partial v}.$$

2.1 Optimal system of Lie subalgebras

We first determine optimal system of one-dimensional Lie subalgebras for (1.2). The adjoint representations are given by ref. [27]

$$Ad(\exp(\varepsilon S_i))S_j = S_j - \varepsilon[S_i, S_j] + \frac{1}{2} \varepsilon^2[S_i, [S_i, S_j]] - \cdots,$$

where $[S_i, S_j]$ denotes the commutator of S_i and S_j and is defined as

$$[S_i, S_i] = S_i S_i - S_i S_i.$$

After computing commutators of all symmetries of (1.2), we display the results in Table 1. We then calculate adjoint representations and display them in Table 2.

Using Tables 1 and 2, and following the method described in ref. [27], we obtain optimal system of Lie subalgebras as

$${S_3 + a_4S_4, a_1S_1 + a_2S_2 + S_3, S_4, S_2, vS_1 + S_2},$$
 (2.4)

where $v = \pm 1$, a_1 , a_2 and a_4 are constants with $a_4 \neq 0$.

2.2 Symmetry reductions of (1.2)

We now use each element of the set (2.4) and reduce gHS-KdVes (1.2) to ordinary differential equation (ODE) systems.

Case 1.
$$S_3 + a_4 S_4$$

The symmetry $S_3 + a_4 S_4$ provides the similarity transformation

$$u = t^{-2/3}E(\rho), \quad v = t^{-(1+4a_4)/3a_4}F(\rho),$$

 $w = t^{1/3a_4}G(\rho),$ (2.5)

with $\rho = t^{-1/3}x$ as an invariant of $S_3 + a_4S_4$. Substituting these values of u, v, w from (2.5) into (1.2), we obtain the ODEs

$$9a_4E(\rho)G'(\rho) - 3a_4G'''(\rho) + a_4\rho G'(\rho) - G(\rho) = 0,$$

$$9a_4E(\rho)F'(\rho) - 3a_4F'''(\rho) + a_4\rho F'(\rho) + F(\rho) + 4a_4F(\rho)$$

$$= 0,$$

$$3E'''(\rho) - 18E(\rho)E'(\rho) + 2\rho E'(\rho) + 4E(\rho) + 18G(\rho)F'(\rho)$$

$$+ 18F(\rho)G'(\rho) = 0.$$

Case 2.
$$a_1S_1 + a_2S_2 + S_3$$

The symmetry generator $a_1S_1 + a_2S_2 + S_3$ provides the similarity transformation

$$u = E(\rho), \quad v = \exp(-t/a_2)F(\rho),$$

 $w = \exp(t/a_2)G(\rho),$ (2.6)

Table 1: Commutators of Lie algebra of (1.2)

	\mathcal{S}_1	\mathcal{S}_{2}	S_3	S4
$\overline{S_1}$	0	0	0	\mathcal{S}_1
\mathcal{S}_2	0	0	0	$3S_2$
\mathcal{S}_3	0	0	0	0
\mathcal{S}_4	$-S_1$	$-3S_2$	0	0

Table 2: Adjoint commutators of Lie algebra of (1.2)

	\mathcal{S}_1	\mathcal{S}_{2}	S_3	S_4
S_1	\mathcal{S}_1	\mathcal{S}_2	\mathcal{S}_3	$-\epsilon S_1 + S_4$
${\cal S}_2$	${\mathcal S}_1$	${\mathcal S}_2$	${\mathcal S}_3$	$-3\epsilon S_2 + S_4$
\mathcal{S}_3	${\mathcal S}_1$	${\mathcal S}_2$	${\cal S}_3$	\mathcal{S}_4
\mathcal{S}_4	$e^{\epsilon}\mathcal{S}_1$	$e^{3\epsilon}\mathcal{S}_2$	\mathcal{S}_3	\mathcal{S}_4

where $\rho = (a_2x - a_1t)/a_2$ is an invariant of $a_1S_1 + a_2S_2 + S_3$ and E, F and G solves

$$3a_{2}E(\rho)F'(\rho) - a_{2}F'''(\rho) + a_{1}F'(\rho) + F(\rho) = 0,$$

$$3a_{2}E(\rho)G'(\rho) - a_{2}G'''(\rho) + a_{1}G'(\rho) - G(\rho) = 0,$$

$$a_{2}E'''(\rho) + 2a_{1}E'(\rho) - 6a_{2}E(\rho)E'(\rho) + 6a_{2}G(\rho)F'(\rho) + 6a_{2}F(\rho)G'(\rho) = 0.$$

Case 3. S_4

The symmetry generator S_4 furnishes us with similarity transformation

$$u = t^{-2/3}E(\rho), \quad v = t^{-4/3}F(\rho), \quad w = G(\rho),$$
 (2.7)

where invariant of S_4 is $\rho = t^{-1/3}x$ and E, F, G solve

$$9E(\rho)G'(\rho) - 3G'''(\rho) + \rho G'(\rho) = 0,$$

$$9E(\rho)F'(\rho) - 3F'''(\rho) + \rho F'(\rho) + 4F(\rho) = 0,$$

$$3E'''(\rho) - 18E(\rho)E'(\rho) + 2\rho E'(\rho) + 4E(\rho) + 18G(\rho)F'(\rho) + 18F(\rho)G'(\rho) = 0.$$

Case 4. S_2

The symmetry generator S_2 provides $\rho = x$ as an invariant, and consequently, similarity transformation is

$$u = E(\rho), \quad v = F(\rho), \quad w = G(\rho),$$
 (2.8)

with E, F, G satisfying

$$\begin{split} E'''(\rho) &- 6E(\rho)E'(\rho) + 6G(\rho)F'(\rho) + 6F(\rho)G'(\rho) = 0, \\ F'''(\rho) &- 3E(\rho)F'(\rho) = 0, \\ G'''(\rho) &- 3E(\rho)G'(\rho) = 0. \end{split}$$

Case 5. $vS_1 + S_2$

Finally, the symmetry $vS_1 + S_2$ gives $\rho = x - vt$ as an invariant and hence the invariant solution is

$$u = E(\rho), \quad v = F(\rho), \quad w = G(\rho),$$
 (2.9)

where *E*, *F*, *G* solve

$$E'''(\rho) + 2\nu E'(\rho) - 6E(\rho)E'(\rho) + 6G(\rho)F'(\rho) + 6F(\rho)G'(\rho) = 0,$$
(2.10a)

$$3E(\rho)F'(\rho) - F'''(\rho) + \nu F'(\rho) = 0,$$
 (2.10b)

$$3E(\rho)G'(\rho) - G'''(\rho) + \nu G'(\rho) = 0.$$
 (2.10c)

2.3 Exact solutions via the simplest equation method

The simplest equation technique is an effective and robust technique, which can be used to construct closed-form solutions of DEs. It was introduced and used by the Russian mathematician Kudryashov [45,46]. Here we invoke this technique and use it on the reduced ODE system (2.10). The technique involves the use of a well-known ODE whose solution exists in the closed form. In our work, we shall make use of two famous ODEs, namely Bernoulli and Riccati equations.

We consider Bernoulli's equation

$$\mathcal{H}'(\rho) = r\mathcal{H}(\rho) + s\mathcal{H}^2(\rho) \tag{2.11}$$

with r and s constants, whose solution is

$$\mathcal{H}(\rho) = r \left\{ \frac{\cosh[(\rho + C)r] + \sinh[(\rho + C)r]}{1 - s\cosh[(\rho + C)r] - s\sinh[(\rho + C)r]} \right\},\,$$

where C is an arbitrary integration constant. For Riccati's equation

$$\mathcal{H}'(\rho) = r\mathcal{H}^{2}(\rho) + s\mathcal{H}(\rho) + c \tag{2.12}$$

(*r*, *s*, *c* constants) the two solutions we use are

$$\mathcal{H}(\rho) = -\frac{s}{2r} - \frac{\theta}{2r} \tanh \left[\frac{1}{2} \vartheta(\rho + C) \right]$$

and

$$\begin{split} \mathcal{H}(\rho) &= -\frac{s}{2r} - \frac{\vartheta}{2r} \tanh\left(\frac{1}{2}\vartheta\rho\right) \\ &+ \frac{\mathrm{sech}\left(\frac{\vartheta\rho}{2}\right)}{C\cosh\left(\frac{\vartheta\rho}{2}\right) - \frac{2r}{\vartheta}\sinh\left(\frac{\vartheta\rho}{2}\right)}, \end{split}$$

with $\theta^2 = s^2 - 4rc > 0$ and *C* an integration constant.

For the system of three ODEs (2.10), we consider its solutions in the form

$$E(\rho) = \sum_{\mu=0}^{K} \mathcal{A}_{\mu}(\mathcal{H}(\rho))^{\mu},$$

$$F(\rho) = \sum_{\mu=0}^{K} \mathcal{B}_{\mu}(\mathcal{H}(\rho))^{\mu},$$

$$G(\rho) = \sum_{\mu=0}^{K} C_{\mu}(\mathcal{H}(\rho))^{\mu},$$
(2.13)

where $\mathcal{H}(\rho)$ solves Bernoulli or Riccati equation, K>0 is an integer and \mathcal{H}_{μ} , \mathcal{B}_{μ} and C_{μ} ($\mu=0,1,...,K$) are unknown constants.

Solutions of (1.2) with Bernoulli's equation as simplest equation

From (2.10) by balancing the highest order derivatives with the nonlinear terms [45] yields K = 2, so (2.13) takes the form

$$E(\rho) = \mathcal{A}_0 + \mathcal{A}_1 \mathcal{H} + \mathcal{A}_2 \mathcal{H}^2, \qquad (2.14a)$$

$$F(\rho) = \mathcal{B}_0 + \mathcal{B}_1 \mathcal{H} + \mathcal{B}_2 \mathcal{H}^2, \qquad (2.14b)$$

$$G(\rho) = C_0 + C_1 \mathcal{H} + C_2 \mathcal{H}^2$$
. (2.14c)

We substitute (2.14) into (2.10) and invoke (2.11) and then equate the coefficients of like powers of \mathcal{H} to zero. This gives a system of algebraic equations in \mathcal{A}_i , \mathcal{B}_i , C_i (i = 0, 1, 2). Using Mathematica to solve the above system, one obtains

$$\mathcal{A}_{0} = \frac{1}{3}(r^{2} - v),$$

$$\mathcal{A}_{1} = 4rs,$$

$$\mathcal{A}_{2} = 4s^{2},$$

$$\mathcal{B}_{2} = \frac{s\mathcal{B}_{1}}{r},$$

$$C_{0} = \frac{2}{3\mathcal{B}_{1}^{2}}(-6r^{2}s^{2}\mathcal{B}_{0} + r^{3}s\mathcal{B}_{1} - 4rsv\mathcal{B}_{1}),$$

$$C_{1} = \frac{4rs^{3}}{\mathcal{B}_{2}},$$

$$C_{2} = \frac{sC_{1}}{r}.$$

As a result, we obtain a solution of (1.2) as

u(t, x)

$$= \mathcal{A}_{0} + \mathcal{A}_{1}r \left\{ \frac{\cosh\{(\rho + C)r\} + \sinh\{(\rho + C)r\}}{1 - s \cosh\{(\rho + C)r\} - s \sinh\{(\rho + C)r\}} \right\} + \mathcal{A}_{2}r^{2} \left\{ \frac{\cosh\{(\rho + C)r\} + \sinh\{(\rho + C)r\}}{1 - s \cosh\{(\rho + C)r\} - s \sinh\{(\rho + C)r\}} \right\}^{2},$$
(2.15a)

$$v(t, x) = \mathcal{B}_{0} + \mathcal{B}_{1}r$$

$$\times \left\{ \frac{\cosh\{(\rho + C)r\} + \sinh\{(\rho + C)r\}}{1 - s\cosh\{(\rho + C)r\} - s\sinh\{(\rho + C)r\}} \right\}$$

$$+ \mathcal{B}_{2}r^{2} \left\{ \frac{\cosh\{r(\rho + C)r\} + \sinh\{(\rho + C)r\}}{1 - s\cosh\{(\rho + C)r\} - s\sinh\{(\rho + C)r\}} \right\}^{2},$$
(2.15b)

$$w(t, x) = C_0 + C_1 r$$

$$\times \left\{ \frac{\cosh\{(\rho + C)r\} + \sinh\{(\rho + C)r\}}{1 - s\cosh\{(\rho + C)r\} - s\sinh\{(\rho + C)r\}} \right\} + C_2 r^2 \left\{ \frac{\cosh\{(\rho + C)r\} + \sinh\{(\rho + C)r\}}{1 - s\cosh\{(\rho + C)r\} - s\sinh\{(\rho + C)r\}} \right\}^2,$$
(2.15c)

with $\rho = x - vt$ and C an integration constant.

Solutions of (1.2) with the Riccati equation as simplest equation

As before, for this case, K = 2 and so

$$E(\rho) = \mathcal{A}_0 + \mathcal{A}_1 \mathcal{H} + \mathcal{A}_2 \mathcal{H}^2, \qquad (2.16a)$$

$$F(\rho) = \mathcal{B}_0 + \mathcal{B}_1 \mathcal{H} + \mathcal{B}_2 \mathcal{H}^2, \qquad (2.16b)$$

$$G(\rho) = C_0 + C_1 \mathcal{H} + C_2 \mathcal{H}^2.$$
 (2.16c)

Following the same procedure as above, but using (2.12), we obtain

$$\mathcal{A}_{0} = \frac{1}{3}(s^{2} + 8rc - v),$$

$$\mathcal{A}_{1} = 4rs,$$

$$\mathcal{A}_{2} = 4r^{2},$$

$$\mathcal{B}_{2} = \frac{r\mathcal{B}_{1}}{s},$$

$$C_{0} = \frac{2}{3\mathcal{B}_{2}^{2}}(-6r^{4}\mathcal{B}_{0} + r^{3}b\mathcal{B}_{1} + 8r^{3}c\mathcal{B}_{2} - 4r^{2}v\mathcal{B}_{2}),$$

$$C_{1} = \frac{4r^{3}s}{\mathcal{B}_{2}},$$

$$C_{2} = \frac{rC_{1}}{s}.$$

Thus, the two solutions of (1.2) are

$$u_{1}(t,x) = \mathcal{A}_{0} + \mathcal{A}_{1} \left\{ -\frac{s}{2r} - \frac{\vartheta}{2r} \tanh \left[\frac{1}{2} \vartheta(\rho + C) \right] \right\}$$

+
$$\mathcal{A}_{2} \left\{ -\frac{s}{2r} - \frac{\vartheta}{2r} \tanh \left[\frac{1}{2} \vartheta(\rho + C) \right] \right\}^{2},$$
 (2.17a)

$$v_{1}(t, x) = \mathcal{B}_{0} + \mathcal{B}_{1} \left\{ -\frac{s}{2r} - \frac{\vartheta}{2r} \tanh \left[\frac{1}{2} \vartheta(\rho + C) \right] \right\} + \mathcal{B}_{2} \left\{ -\frac{s}{2r} - \frac{\vartheta}{2r} \tanh \left[\frac{1}{2} \vartheta(\rho + C) \right] \right\}^{2},$$
(2.17b)

$$w_{1}(t,x) = C_{0} + C_{1} \left\{ -\frac{s}{2r} - \frac{9}{2r} \tanh \left[\frac{1}{2} 9(\rho + C) \right] \right\} + C_{2} \left\{ -\frac{s}{2r} - \frac{9}{2r} \tanh \left[\frac{1}{2} 9(\rho + C) \right] \right\}^{2}$$
(2.17c)

and

$$u_{2}(t,x) = \mathcal{A}_{0} + \mathcal{A}_{1} \left\{ -\frac{s}{2r} - \frac{\vartheta}{2r} \tanh\left(\frac{1}{2}\vartheta\rho\right) + \frac{\operatorname{sech}\left(\frac{\vartheta\rho}{2}\right)}{C \cosh\left(\frac{\vartheta\rho}{2}\right) - \frac{2r}{\vartheta} \sinh\left(\frac{\vartheta\rho}{2}\right)} \right\} + \mathcal{A}_{2} \left\{ -\frac{s}{2r} - \frac{\vartheta}{2r} \tanh\left(\frac{1}{2}\vartheta\rho\right) \right\} + \mathcal{A}_{2} \left\{ -\frac{s}{2r} - \frac{\vartheta}{2r} \tanh\left(\frac{1}{2}\vartheta\rho\right) \right\} + \frac{\operatorname{sech}\left(\frac{\vartheta\rho}{2}\right)}{C \cosh\left(\frac{\vartheta\rho}{2}\right) - \frac{2r}{\vartheta} \sinh\left(\frac{\vartheta\rho}{2}\right)} \right\} + \frac{\operatorname{sech}\left(\frac{\vartheta\rho}{2}\right)}{C \cosh\left(\frac{\vartheta\rho}{2}\right) - \frac{2r}{\vartheta} \sinh\left(\frac{\vartheta\rho}{2}\right)} \right\} ,$$

with $\rho = x - vt$ and *C* an integration constant. Solution (2.18) is sketched in Figure 1.

3 Conservation laws

We now determine conservation laws for gHS-KdVes (1.2) by employing the general multiplier technique [27,40–42]. Here we seek second-order multipliers Λ^1 , Λ^2 and Λ^3 that

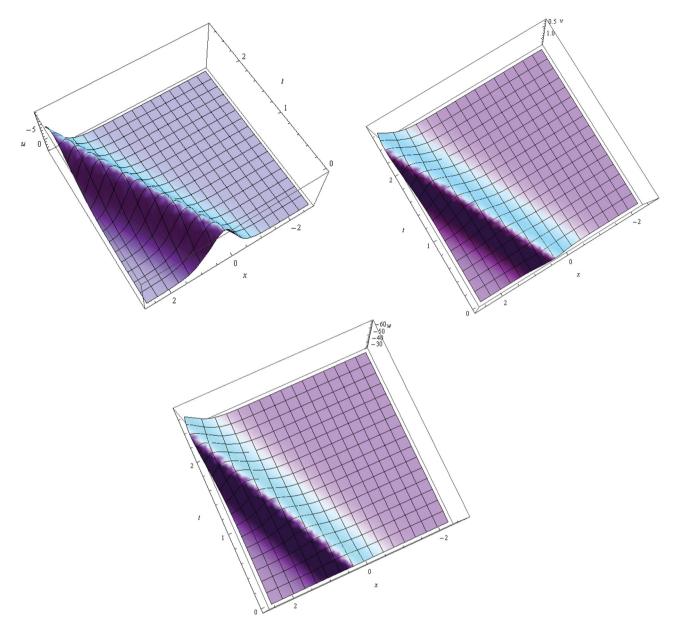


Figure 1: Profile of solitary wave solution (2.18).

depend on $(t, x, u, v, w, u_x, v_x, w_x, u_{xx}, v_{xx}, w_{xx})$. The multipliers are obtained from the equations

$$\frac{\delta}{\delta u} [\Lambda^{1}Q_{1} + \Lambda^{2}Q_{2} + \Lambda^{3}Q_{3}] = 0,$$

$$\frac{\delta}{\delta v} [\Lambda^{1}Q_{1} + \Lambda^{2}Q_{2} + \Lambda^{3}Q_{3}] = 0,$$

$$\frac{\delta}{\delta w} [\Lambda^{1}Q_{1} + \Lambda^{2}Q_{2} + \Lambda^{3}Q_{3}] = 0,$$
(3.19)

where $\delta/\delta u^l$ is the Euler operator

$$\frac{\delta}{\delta u^l} = \frac{\partial}{\partial u^l} - D_j \frac{\partial}{\partial u^l_i} + D_j D_k \frac{\partial}{\partial u^l_{ik}} - \cdots, \quad l = 1, 2, 3,$$

with $u^1 = u$, $u^2 = v$, $u^3 = w$ and

$$D_{j} = \frac{\partial}{\partial x^{j}} + u_{j}^{l} \frac{\partial}{\partial u^{l}} + u_{jk}^{l} \frac{\partial}{\partial u_{k}^{l}} + \cdots, \quad j = 1, 2,$$

is the total differentiation operator. Expanding (3.19) and after some calculations, we obtain five conservation law multipliers

$$\begin{split} &\Lambda_1 = -\frac{3}{2}C_1u^2 + C_1vw + (C_3t + C_4)u + \frac{1}{2}C_1u_{xx} - \frac{C_3x}{3} + C_5, \\ &\Lambda_2 = C_1uw - (C_3t + C_4)w - C_1w_{xx} - C_2w_x, \\ &\Lambda_3 = C_1uv - (C_3t + C_4)v - C_1v_{xx} + C_2v_x, \end{split}$$

where C_i , i = 1, 2,..., 5 are constants. Thus, the corresponding five local conserved vectors of (1.2) are as follows [27,44,47]

$$T_{1}^{t} = \frac{1}{4} \{u_{xx}u - 2v_{xx}w - 2w_{xx}v + 4uvw - 2u^{3}\},$$

$$T_{1}^{x} = \frac{1}{8} \{16v_{x}w_{x}u + 8v_{xx}uw + 8w_{xx}uv - 8u_{x}v_{x}w - 8u_{x}w_{x}v - 4u_{xx}vw + 6u_{xx}u^{2} - 2uu_{tx} + 4wv_{tx} + 4vw_{tx} + 12u^{2}vw - 9u^{4} - 12v^{2}w^{2} + 2u_{t}u_{x} - 4v_{t}w_{x} - 4w_{t}v_{x} - u_{xx}^{2} - 8v_{xx}w_{xx}\};$$

$$T_{2}^{t} = \frac{1}{2} \{v_{x}w - w_{x}v\},$$

$$T_{3}^{x} = \frac{1}{6} \{3tu^{2} - 2xu - 6tvw\},$$

$$T_{3}^{x} = \frac{1}{12} \{-6tu_{xx}u - 12tv_{xx}w - 12tw_{xx}v + 12tu^{3} - 6xu^{2} + 12xvw + 3tu_{x}^{2} + 12tv_{x}w_{x} - 2u_{x} + 2xu_{xx}\};$$

$$T_{4}^{t} = \frac{1}{2} \{u^{2} - 2vw\},$$

$$T_{5}^{x} = u,$$

$$T_{5}^{x} = \frac{1}{2} \{3u^{2} - 6vw - u_{xx}\}.$$

Remark. As far as the physical meaning of the conservation laws derived above are concerned, we observe that the first four of them are purely mathematical, whereas the fifth describes mass density and the flux representing conserved currents for the mass. Also, note that by raising the order of multipliers, higher-order conserved vectors of gHS-KdVes (1.2) can be derived.

4 Concluding remarks

In this article, we investigated the generalized system of Hirota–Satsuma coupled KdV equations (1.2) from the group standpoint. The symmetries of the system were found and then used to build optimal system of one-dimensional Lie subalgebras. With the assistance of these subalgebras, system (1.2) was reduced to systems of ODEs and thereafter the simplest equation technique was invoked to manufacture closed-form solutions of (1.2). Moreover, conserved vectors were derived for system (1.2) using the general multiplier method. Five multipliers were computed and accordingly five conservation laws were constructed. The advantages of conservation laws were mentioned in Section 1.

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