Research Article

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Lump, lump-one stripe, multiwave and breather solutions for the Hunter–Saxton equation

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Abstract: The aim of this article was to address the lump, lump-one stripe, multiwave and breather solutions for the Hunter–Saxton equation with the aid of Hirota bilinear technique. This model concerns in a massive nematic liquid crystal director field. By choosing the function f in Hirota bilinear form, as the general quadratic function, trigonometric function and exponential function along with appropriate set of parameters, we find the lump, lump-one stripe, multiwave and breather solutions successfully. We also interpreted some three-dimensional and contour profiles to anticipate the wave dynamics. These newly obtained solutions have some arbitrary constants and so can be applicable to explain diversity in qualitative features of wave phenomena.

Keywords: lump solitons, lump-one stripe, multiwaves, breathers, Hunter–Saxton equation, solitary wave solutions

1 Introduction

In waves theory, nonlinear partial differential equations (NPDEs), which explain nonlinear aspects, appear in an extensive diversity of scientific and engineering applications, for example, plasma physics, fluid dynamics, hydrodynamics, acoustics, solid-state physics, hydrodynamics and theory of turbulence, optics, optical fibers, chemical physics, chaos theory and many other applications. The study of NPDEs becomes increasingly significant because of their prominent features. A main attachment of scientific work has been perceived in the last few decades on NPDEs such as efficient integration method [1], improved modified Kudryashov method [2], asymptotic method [3], geometric singular perturbation [4], Lie symmetry analysis [5], Painleve expansion procedure [6], $\left(\frac{G'}{G}\right)$ expansion approach, highly optical solitons, homotopy perturbation method, the semi-inverse method [7–10], simple-equation method, logistic function, Backlund transformation technique, asymptotic method, integrability method [11–15], extended mapping method, non-perturbative method, nonlinearity and conservation, Hirota bilinear method [16–19], extended and modified direct algebraic method, extended mapping method and Seadawy techniques [20–27] and so on.

There are many renowned models, such as Vakhnenko dynamical equation [35], nonlinear Schrodinger equation [36], KdV equation [37], Camassa-Holm equation [38], sine-Gordon equation [39] and Biswas-Milovic equation [40], but here we will obtain the exact solutions of the Hunter–Saxton (HS) equation [41],

$$u_{txx} - 2ku_x + 2u_x u_{xx} + uu_{xxx} = 0. \tag{1}$$

This equation is used for propagation of orientation waves in a massive nematic liquid crystal director field. The HS equation can be used as a short wave limit of the Camassa–Holm equation:

$$m_t + 2mu_x + um_x = 0, \quad m = k + u - u_{xx}. \tag{2}$$

The content of this article is organized as follows: in Section 2, we evaluate the lump solutions via some three-dimensional (3D) and contour shapes. In Section 3, we find out lump-one stripe interactional solutions and some physical 3D shapes. In Section 4, the brief discussion of multiwave solutions for the proposed model is given. In Section 5, we find breather solutions. In Section 6, there are results and discussion about our newly obtained solutions and comparison with already published work, and in Section 7, we give concluding remarks.

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2 Lump solution

In the direction to find lump solutions of equation (1), we apply the transformation [42],

$$u = m + 2b(\ln f)_x, \tag{3}$$

which transforms equation (1),

$$4kbf^3 f_x^2 - 12bff_y f_x^3 + \cdots + 12bmf^4 f_{xxxx} + 4b^2 f^3 f_x f_{xxxx} = 0. \tag{4}$$

Now the function f in equation (5) can be assumed as [36],

$$f = a_7 + g^2 + h^2, \tag{5}$$

where $g^2 = a_1x + a_2t + a_3$, $h^2 = a_4x + a_5t + a_6$. However, a_i $(1 \le i \le 7)$ are all real parameters to be measured. Now, substituting f into equation (5) and associating the coefficients of x and t imply us the subsequent result on parameters:

Set I.

$$a_1 = \sqrt{\frac{-3m}{50b}}, \quad a_2 = \frac{-1}{5}\sqrt{\frac{-3m}{50b}}\,m, \quad a_3 = a_3, \quad a_4 = 0, \quad a_5 = 0. \tag{6}$$

The parameters in equation (6) prevent the lump solutions to equation (1)

$$u(x,t) = m + \frac{2\sqrt{6}\,b\sqrt{\frac{-m}{b}}\left(a_3 - \frac{1}{25}\sqrt{\frac{-3m}{2b}}\,mt + \frac{1}{5}\sqrt{\frac{-3m}{2b}}\,x\right)}{5\left(a_6^2 + a_7\left(a_3 - \frac{1}{25}\sqrt{\frac{-3m}{2b}}\,mt + \frac{1}{5}\sqrt{\frac{-3m}{2b}}\,x\right)^2\right)}. \tag{7}$$

Set II.

$$a_1 = a_1, \quad a_2 = a_2, \quad a_3 = a_3, \quad a_4 = \sqrt{-1}\,a_1, \quad a_5 = \sqrt{-1}\,a_2. \tag{8}$$

The parameters in equation (8) imply the lump solutions to equation (1)

$$u(x,t) = m + \frac{2b\{2ia_1(a_6 + ia_2t + ia_1x) + 2a_1(a_3 + a_2t + a_1x)\}}{a_7 + (a_6 + ia_2t + ia_1x)^2 + (a_3 + a_2t + a_1x)^2}. \tag{9}$$

3 Lump-one stripe soliton interaction solution

To this aim, f in the bilinear equation can be assumed as [43],

$$f = g^2 + h^2 + b_1\exp(f_1) + a_7, \tag{10}$$

where $g^2 = a_1x + a_2t + a_3$, $h^2 = a_4x + a_5t + a_6$, $f_1 = k_1x + k_2t$. However, a_i $(1 \le i \le 7)$, k_1 and k_2 are all real parameters to be found. Now, inserting f in equation (3) and relating the coefficients of x and t give us:

Set I.

$$a_1 = \sqrt{-1}\,a_4, \quad a_2 = \frac{\frac{\sqrt{-1}}{5}(-mk_1^2 + 2k)}{k_1a_4}, \quad a_5 = \frac{2k - mk_1^2}{2k_1a_4}, \quad k_2 = \frac{2k - mk_1^2}{k_1}a_1. \tag{11}$$

The parameters in equation (11) create the required solutions to equation (1)

$$u(x,t) = m + \frac{2b(b_1\Delta_1k_1 + 2ia_4(a_3 + i\mu + ia_4x) + 2a_4(a_6 + \mu + a_4x))}{a_7 + b_1\Delta_1 + (a_3 + i\mu + ia_4x)^2 + (a_6 + \mu + a_4x)^2}, \tag{12}$$

where $\Delta_1 = e^{\frac{ia_4(2k - mk_1^2)t}{k_1} + k_1x}$ and $\mu = \frac{i(2k - mk_1^2)}{2a_4k_1}$.

Set II.

$$a_1 = \sqrt{-1}\,a_4, \quad a_2 = \frac{\sqrt{-1}\,(a_4a_5k_1 + mk_1^2 - 2k)}{k_1a_4}, \quad a_5 = \frac{2k - mk_1^2}{2k_1a_4}, \quad k_2 = \frac{2k - mk_1^2}{k_1}a_1. \tag{13}$$

The parameters in equation (13) reveal the required solutions to equation (1)

$$u(x,t) = m + \frac{2b(b_1\Delta_1k_1 + 2ia_4(a_3 + i\nu + ia_4x) + 2a_4(a_6 + a_5t + a_4x))}{a_7 + b_1\Delta_1 + (a_3 + i\nu + ia_4x)^2 + (a_6 + a_5t + a_4x)^2}, \tag{14}$$

where $\Delta_1 = e^{\frac{ia_4(2k - mk_1^2)t}{k_1} + k_1x}$ and $\nu = \frac{i(-2k + a_5a_4k_1 - mk_1^2)t}{a_4k_1}$.

4 Multiwave solutions

For finding multiwave solutions, we use the succeeding transformation in equation (1) [44],

$$u(x,t) = \psi(\xi), \quad \xi = k_1 x - c_1 t. \tag{15}$$

With the help of the above transformation, we obtain:

$$-2kk_1\psi' + 2k_1^3 k_2 \psi'\psi'' - c_1 k_1^2 \psi^3 + k_1^3 \psi\psi^3 = 0. \tag{16}$$

Now with the aid of the following assumption in equation (16)

$$\psi = 2(\ln f)_\xi, \tag{17}$$

we get,

$$\begin{aligned} &2k_1(-20k_1^2 f_\xi^5 + 2k_1 f f_\xi^3 (3c_1 f_\xi + 22k_1 f_{\xi\xi} - 6k_1 f^2 f_\xi)) \\ &\quad - \cdots + 2k_1 f_\xi (2c_1 f_{\xi\xi\xi} + k_1 f_{\xi\xi\xi\xi}) = 0. \end{aligned} \tag{18}$$

To find the multiwave solutions of equation (18), we apply the subsequent hypothesis [40]:

$$\begin{aligned} f &= b_0 \cosh(a_1\xi + a_2) + b_1 \cos(a_3\xi + a_4) \\ &\quad + b_2 \cosh(a_5\xi + a_6), \end{aligned} \tag{19}$$

where a_1, a_2, a_3, a_4, a_5 and a_6 are any constants to be examined. Substituting equation (19) into equation (18) via symbolic computation and collecting the coefficients of all powers of $\sinh(a_1\xi + a_2)$ and $\sinh(a_5\xi + a_6)$, $\cos(a_3\xi + a_4)$, $\cosh(a_1\xi + a_2)$, $\cosh(a_5\xi + a_6)$, $\sin(a_3\xi + a_4)$, functions to be zero, we get a system of equations. After solving this algebraic system, we obtain some different parametric values:

Set I.

$$\begin{aligned} &a_1 = a_1, \quad b_0 = 0, \quad a_2 = a_2, \quad c_1 = \frac{24k_1^2(a_3^2 - a_5^2)}{(a_3^2 - 3a_5^2)}, \\ &a_3 = a_3, \quad a_4 = a_4. \end{aligned} \tag{20}$$

By substituting equation (20) into equation (19), we get

$$\begin{aligned} f &= b_0 \cosh(a_1\xi + a_2) + b_1 \cos(a_3\xi + a_4) \\ &\quad + b_2 \cosh(a_5\xi + a_6). \end{aligned} \tag{21}$$

As f solves equation (18), then ψ solves equation (16) via $\psi = 2(\ln f)_\xi$ we obtain

$$u(x,t) = \frac{2[-a_3 b_1 \cos(a_4 + a_3\Omega_1)]\sin(a_4 + a_3\Omega_1) + a_5 b_2 \cos(a_6 + a_5\Omega_1)\sin(a_6 + a_5\Omega_1)}{b_1\cos(a_4 + a_3\Omega_1) + b_2\cos(a_6 + a_5\Omega_1)}, \tag{22}$$

where $\Omega_1 = \frac{-24(a_3^2 - a_5^2)k_1^2 t}{a_3^2 - 3a_5^2} + k_1 x$.

5 Breather solutions

For finding the breathers of equation (18), we assume the successive transformation [45]:

$$\begin{aligned} f \;&=\; e^{-p(a_2 + a_1\xi)} + b_1 e^{p(a_4 + a_3\xi)} \\ &\quad + b_0 \cos(p_1(a_6 + a_5\xi)), \end{aligned} \tag{23}$$

where $a's$ are any real constants to be found. Inserting equation (23) into (18) via computational Mathematica and collecting all the coefficients of trigonometric and exponential functions to be zero we get a system of equations. After solving this system, we found:

Set I.

$$\begin{aligned} &a_2 = a_2, \quad a_3 = \frac{(4)^{\frac{1}{3}}\left(\frac{k}{k_1^2}\right)^{\frac{1}{3}}}{p}, \quad a_4 = a_4, \quad a_5 = 0, \\ &c_1 = \frac{-(4)^{\frac{1}{3}}\left(\frac{k}{k_1^2}\right)^{\frac{1}{3}}}{2} k_1, \quad b_0 = b_0. \end{aligned} \tag{24}$$

As f solves equation (18), after that ψ solves equation (16) through applying $\psi = 2(\ln f)_\xi$, we get

$$u(x,t) = \frac{2(2)^{\frac{2}{3}} b_1 e^{p\omega} \left(\frac{k}{k_1^2}\right)^{\frac{1}{3}}}{b_0 + e^{-a_2 p} + b_1 e^{p\omega}\left(\frac{k}{k_1^2}\right)^{\frac{1}{3}}}, \tag{25}$$

where $\omega = a_4 + \frac{2^{\frac{2}{3}}\left(\frac{k}{k_1^2}\right)^{\frac{1}{3}}\left(\frac{\left(\frac{k}{k_1^2}\right)^{\frac{1}{3}} k_1 t}{2^{\frac{1}{3}}} + k_1 x\right)}{p}$.

6 Results and discussion

In this section, we have made a detailed comparison of our accomplished results with the earlier literature. Many researchers used various methods for calculating solitary wave solutions of the Hunter–Saxton equation. Particularly, Beals et al. applied inverse scattering technique [46], Alberto et al. used distance functional [47], Lenells applied properties of the Riemannian [48], Lenells et al. applied a geometric approach [49], Bressan et al. utilized Lipschitz metric [50], Zhao et al. applied conservation laws [51], Korpinar used symmetry analysis [51] and Zhao applied conservation laws to obtain the exact solutions for the presented model [52]. But here, in this work we have found the lump, lump-one stripe, multiwave and breather solutions for the Hunter–Saxton equation with the aid of Hirota bilinear approach. These types of solutions have been utilized in many fields of science, for example, physics, chemistry, biology, finance, oceanographic engineering, capillary flow and nonlinear optics [42–45]. Now we will notice how our obtained results

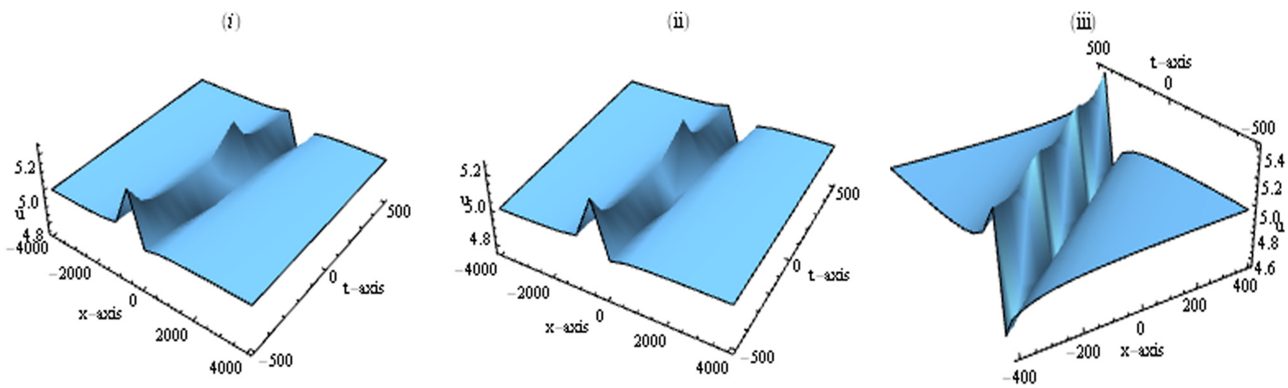

Figure 1: The graphs of the solution $u(x, t)$ in equation (7) are shown via suitable parameters $m = 5$, $b = -5$, $a_7 = 7$, $a_6 = 5$. 3D graphs at (i) $a_3 = 0.2$, (ii) $a_3 = 0.8$ and (iii) $a_3 = 2$, respectively.

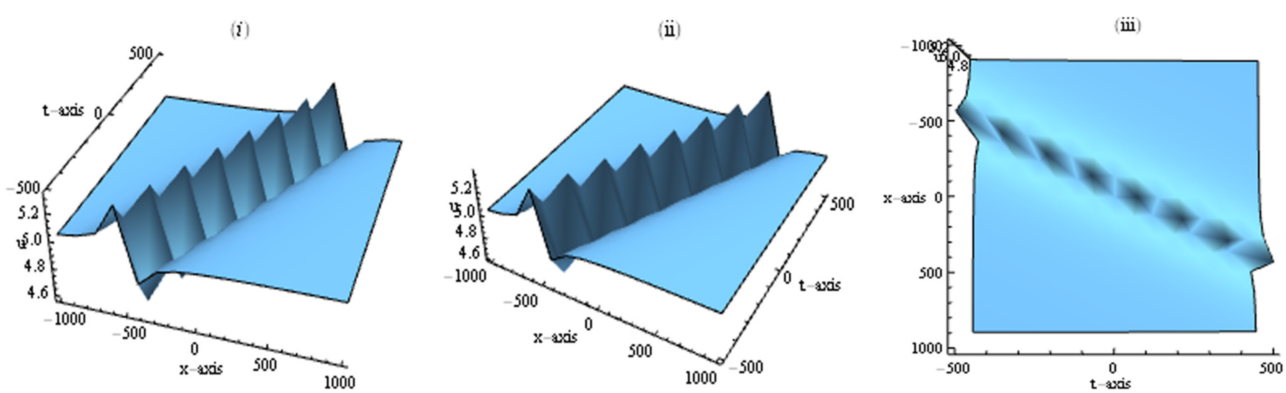

Figure 2: The profiles of the solution $u(x, t)$ in equation (7) are shown by different choices of parameters $m = 5$, $b = -5$, $a_7 = 7$, $a_6 = 5$. 3D graphs at (i) $a_3 = 5$, (ii) $a_3 = 8$ and (iii) $a_3 = 15$, respectively.

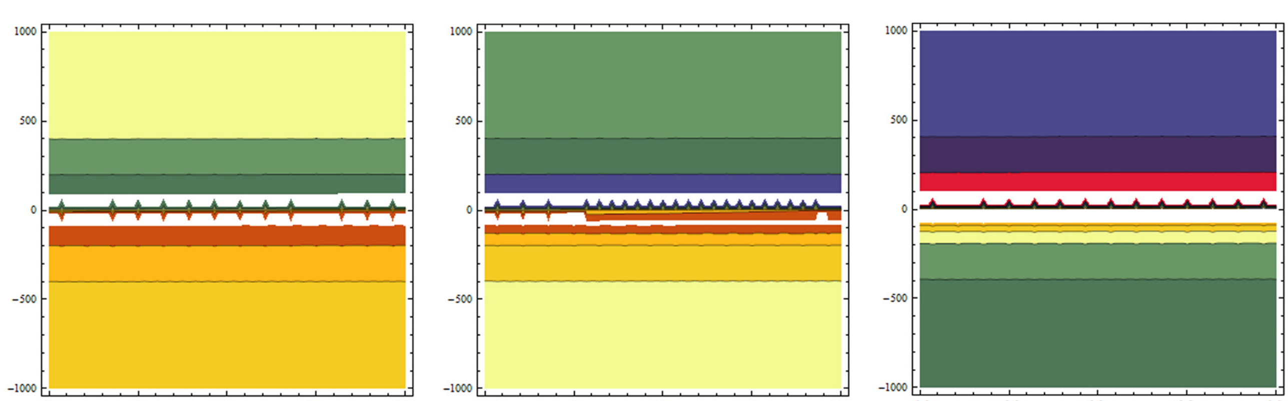

Figure 3: The corresponding contour profiles for Figure 1.

alter their shapes via different values of a_3 from Figure 1(i), and we can see at $a_3 = 0.2$ that u has a maximum value at some points, expressing 3D shape of u making two bright lump solutions. Similarly, in Figure 1(ii) at $a_3 = 0.8$ the same features are observed. But at $a_3 = 2$ in Figure 1(iii), we obtain three bright lump solutions. In the same way, for $a_3 = 5$ in Figure 2(i) we have achieved seven bright lump solutions and the process was repeated for gradually increasing values of a_3, for instance, $a_3 = 8$, $a_3 = 15$ (Figure 2(ii) and (iii)) respectively. Figures 3 and 4 express

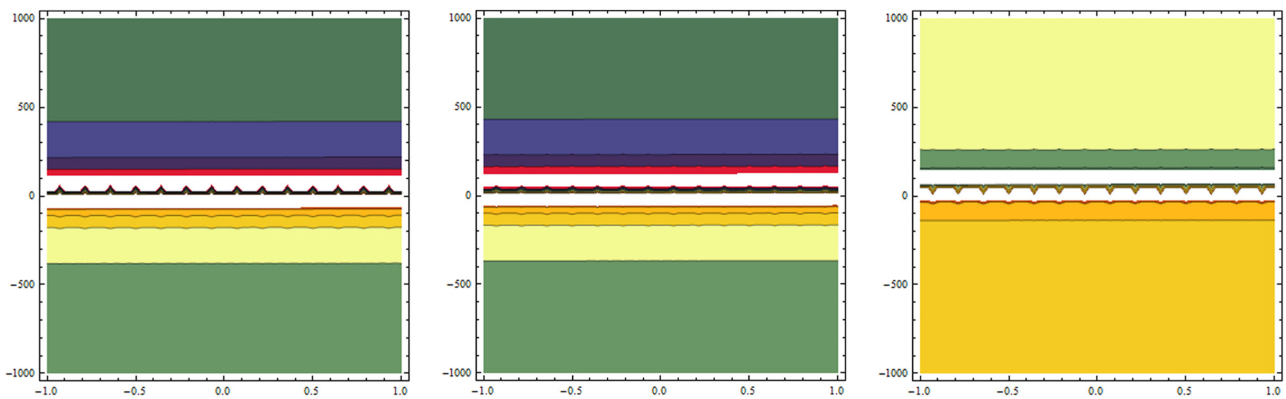

Figure 4: The associating contour graphs for Figure 2.

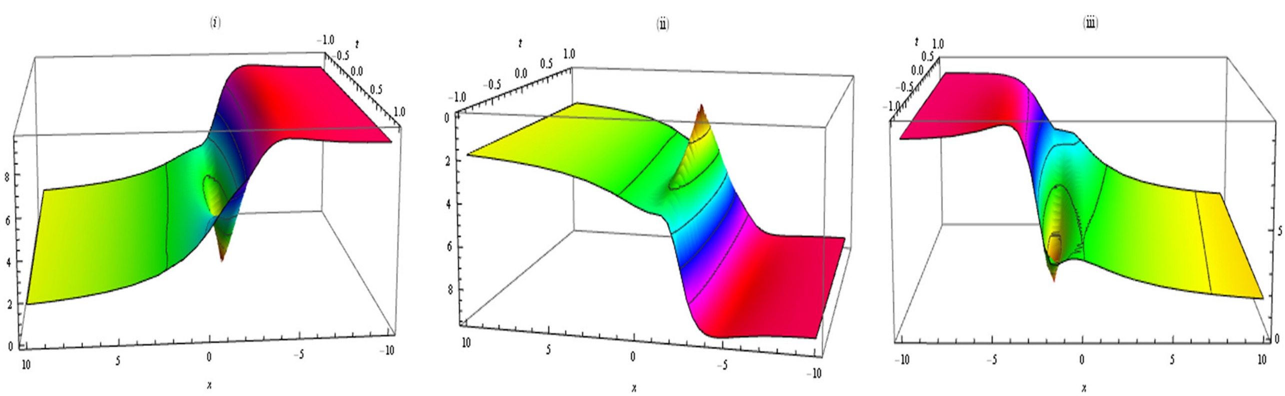

Figure 5: The profiles of the solution $u(x, t)$ in equation (14) via various choices of parameters $m = 1$, $b = 5$, $a_3 = 0.2$, $a_4 = 1$, $a_5 = 2$, $a_6 = 5$, $b_1 = 2$, $k_1 = -1$, $k = 1$. Contour profiles at (i) $a_3 = 0.2$, (ii) $a_3 = 0.8$ and (iii) $a_3 = 2$, respectively.

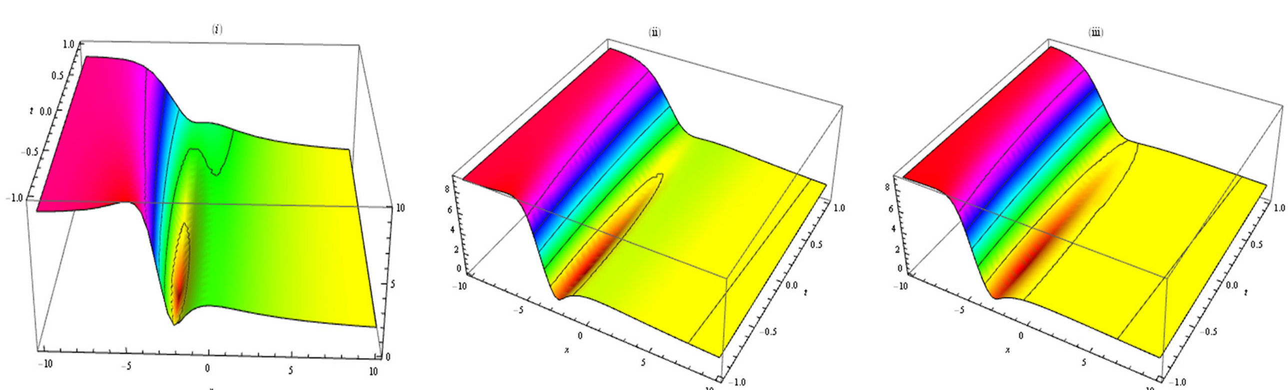

Figure 6: The shapes of the solution $u(x, t)$ in equation (14) are shown by various choices of parameters $m = 1$, $b = 5$, $a_3 = 0.2$, $a_4 = 1$, $a_5 = 2$, $a_6 = 5$, $b_1 = 2$, $k_1 = -1$, $k = 1$. 3D graphs at (i) $a_3 = 5$, (ii) $a_3 = 10$ and (iii) $a_3 = 15$, respectively.

the relating contour profiles for Figures 1 and 2, respectively. Now we have observed how our obtained solutions change their wave structure via appropriate choices of a_3 from Figure 5(i), and we can notice a lump-one stripe soliton at $a_3 = 0.2$. Similarly, Figures 5(ii), 5(iii) and 6(i)–(iii) show how a lump-one stripe soliton rises or descends for different values of a_3. Figures 7 and 8 present the associating contour graphs for Figures 5 and 6, respectively.

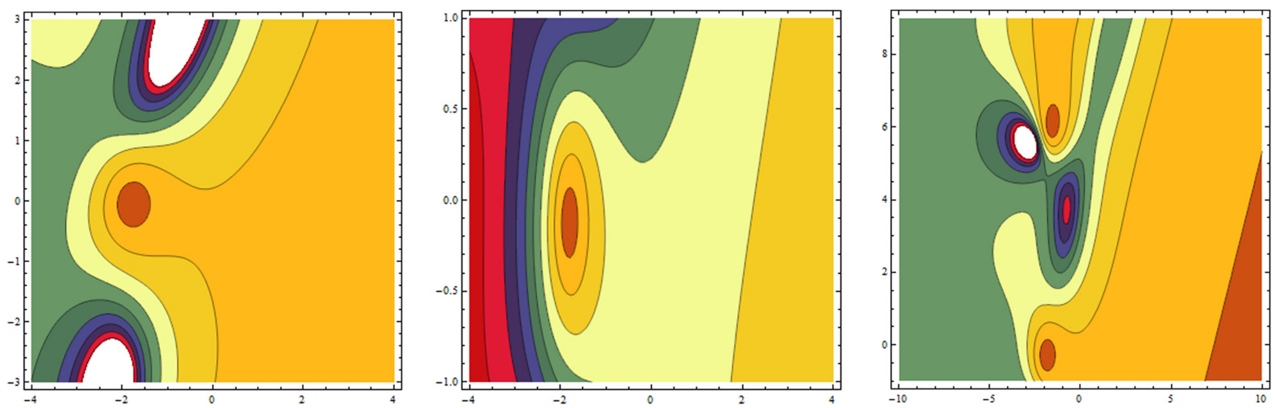

Figure 7: The relating contour graphs for Figure 5.

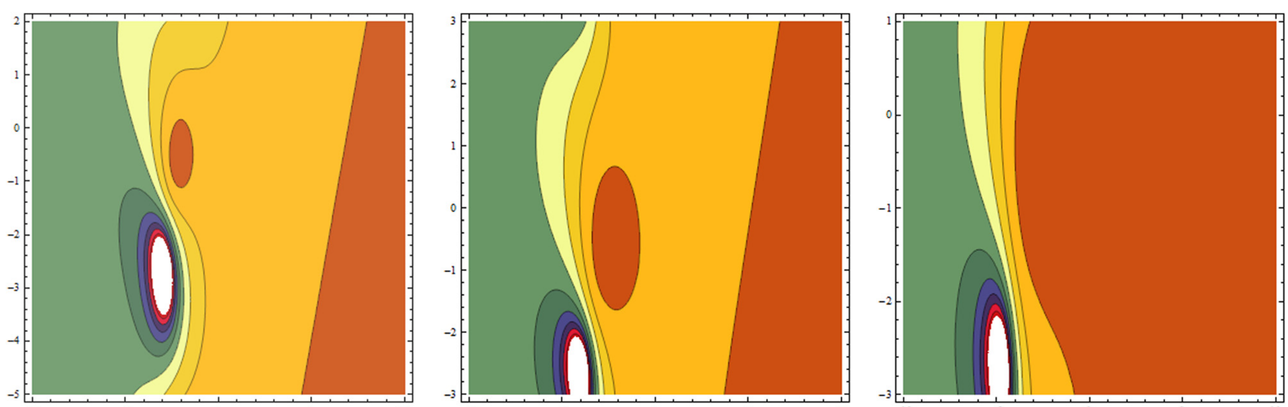

Figure 8: The associating contour profiles for Figure 6.

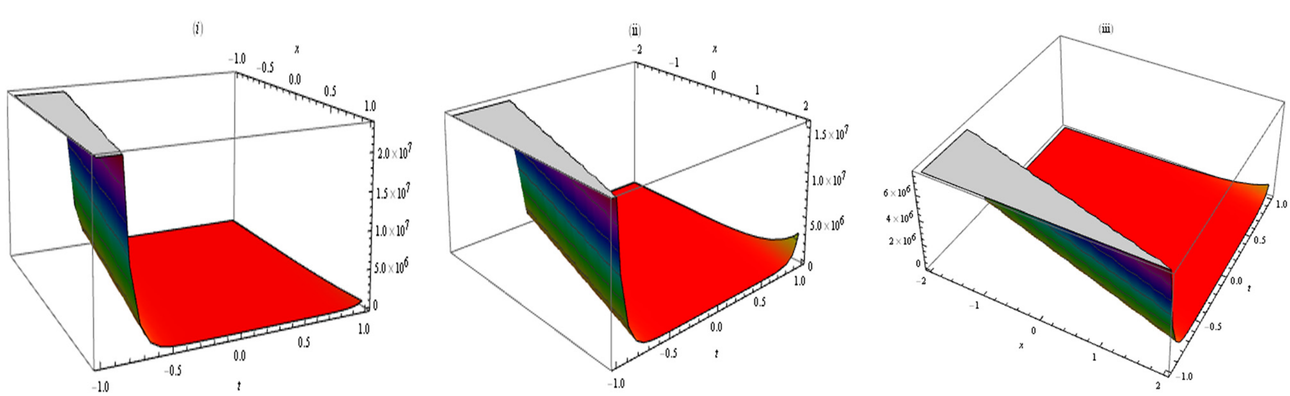

Figure 9: The profiles of the solution $u(x, t)$ in equation (22) via various choices of parameters $m = 1$, $b = 5$, $a_4 = 1$, $a_5 = 2$, $a_6 = 5$, $a_7 = 1$, $b_1 = 2$, $b_2 = 1$, $k_1 = -1$, $k = 1$. 3D graphs at (i) $a_3 = 0.2$, (ii) $a_3 = 0.8$ and (iii) $a_3 = 2$, respectively.

Also, Figures 9(i)–(iii) and 10(i)–(iii) show kink wave and their changes in wave shape via $a_3 = -1$, $a_3 = 0.2$, $a_3 = 0.5$, $a_3 = 0.8$, $a_3 = 2$ and $a_3 = 5$, respectively. Similarly, Figures 11(i)–(iii) and 12(i)–(iii) show solitary wave and the changes in their structure through $a_2 = -1$, $a_2 = 0.2$, $a_2 = 0.5$, $a_2 = 0.8$, $a_2 = 2$ and $a_2 = 5$, respectively. Finally, Figures 13

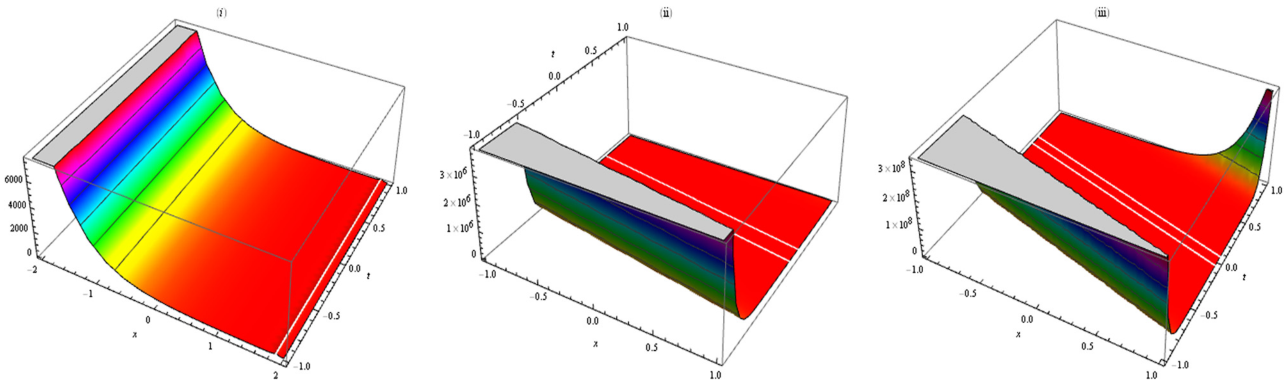

Figure 10: The profiles of the solution $u(x, t)$ in equation (22) via various choices of parameters $m = 1$, $b = 5$, $a_4 = 1$, $a_5 = 2$, $a_6 = 5$, $a_7 = 1$, $b_1 = 2$, $b_2 = 1$, $k_1 = -1$, $k = 1$. 3D graphs at (i) $a_3 = 5$, (ii) $a_3 = 10$ and (iii) $a_3 = 15$, respectively.

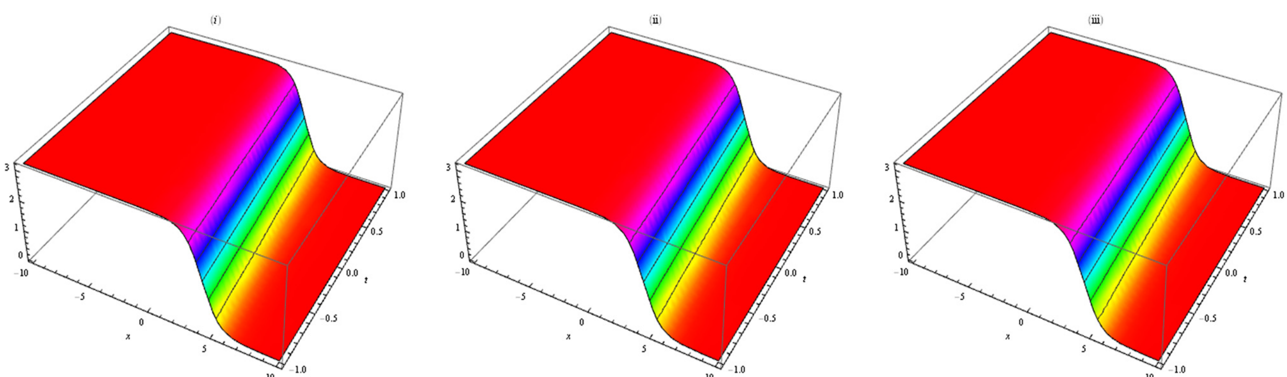

Figure 11: The graphs of the solution $u(x, t)$ in equation (25) through different choices of parameters $m = 1$, $p = 5$, $a_4 = 1$, $b_0 = 1$, $b_1 = 2$, $k_1 = -1$, $k = 1$. 3D graphs at (i) $a_2 = -1$, (ii) $a_2 = 0.2$ and (iii) $a_2 = 0.4$, respectively.

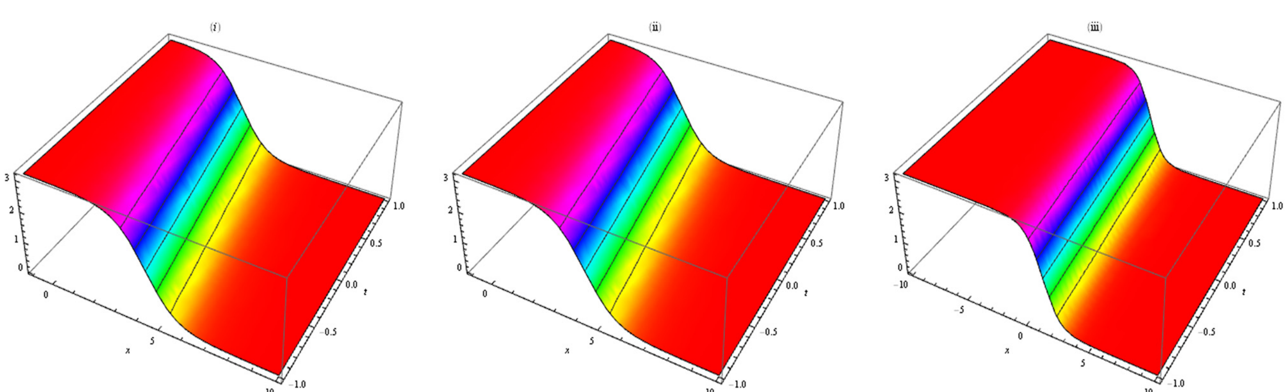

Figure 12: The shapes of the solution $u(x, t)$ in equation (25) via appropriate choices of parameters $m = 1$, $p = 5$, $a_4 = 1$, $b_0 = 1$, $b_1 = 2$, $k_1 = -1$, $k = 1$. 3D graphs at (i) $a_2 = 0.8$, (ii) $a_2 = 2$ and (iii) $a_2 = 15$, respectively.

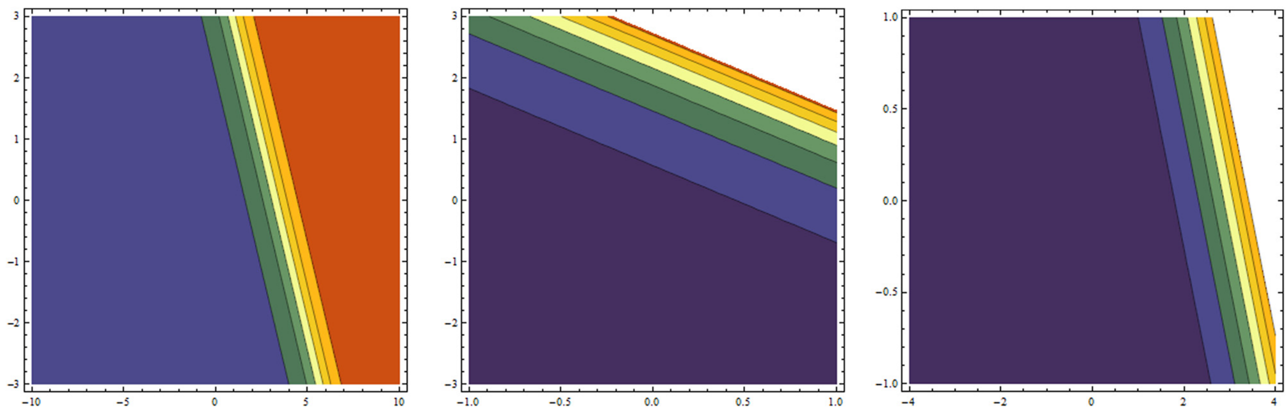

Figure 13: The corresponding contour shapes of Figure 11.

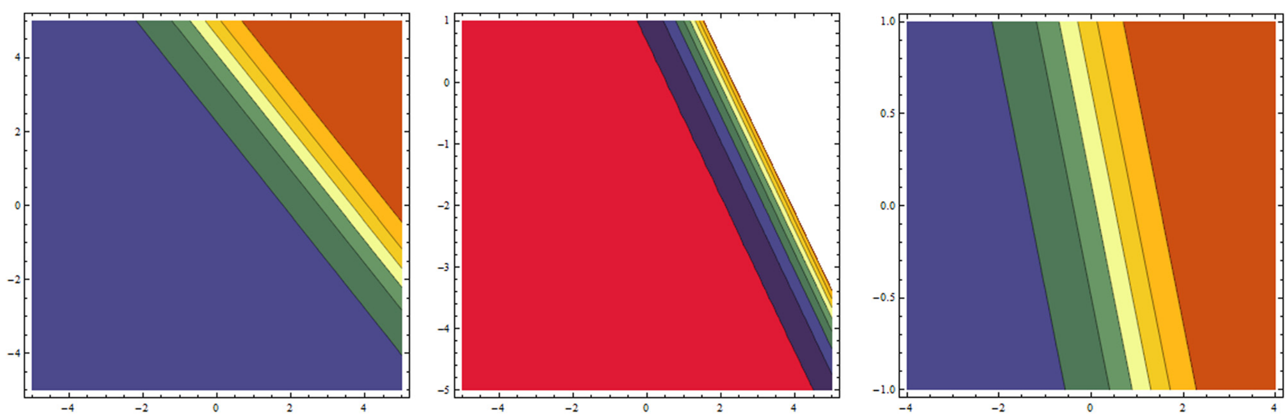

Figure 14: The relating contour profiles of Figure 12.

and 14 present the concerning contour graphs for Figures 11 and 12, respectively.

7 Concluding remarks

The purpose of this article is to accumulate the lump, lump-one stripe, multi wave and breather solutions for the Hunter–Saxton equation by way of Hirota bilinear scheme and through defining appropriate transformations. We have successfully generated some new exact solutions to the concerning model. The 3D and contour graphs mapped different numeric values, to observe the physical behavior of the system. For better understanding and more effectiveness, we have also explained the geometry of the graphs. The attained solutions show that the proposed method is very reliable, aggressive and simple, and so, the recommended idea could be extended for further nonlinear models in mathematical physics.

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