Research Article

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Some probability effects in the classical context

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Abstract: This article gives an account of a theoretical research subdivided into three parts. The first stage recalls the theorem of large numbers and the theorem a single number which demonstrate how the frequentist and the subjective models of probability can be used according to distinct assumptions. Based on these achievements, the second stage describes the temporal evolution of the random outcome (this article distinguishes the outcome from the random event or phenomenon that produces it). It is proved that the outcome of a single unpredictable event switches from the indeterministic to the deterministic state when the event winds up. The third stage means to show how the theoretical results apply to physical situations.

Keywords: mathematical physics, frequentism, subjectivism, aleatory and deterministic states, superposition

1 Introduction

In the nineteenth century, the probability calculus began to infiltrate the scientific and engineering sectors and experts felt the need for an accurate definition of probability. Various models have been put forward [1]. In particular, the frequentist school sees probability as the limit value of the frequency in a sequence of trials and underpins the classical statistic in a way [2]. The subjective school holds that probability expresses the degree of personal credence and refers to the Bayesian statistics [3]. The two perspective and more recent solutions [4] appear to be irreconcilable in the point of logic, and the scientific community still wrestles with the probability interpretation issue [5].

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Why have authors not found a conclusive answer so far? A recent research [6] brings evidence how eminent authors – despite the diverging conclusion – share a similar approach, which is strongly influenced by personal opinions and philosophical arguments. We believe that the philosophers' methodology is not so appropriate to science, and we have made an attempt to develop a purely mathematical approach to the probability interpretation which rejects any philosophical consideration. Section 2 recalls concisely some results already published, and Section 3 develops three theorems on the basis of the previous achievements. Section 4 discusses how the theoretical results apply to physical contexts.

2 Approaching the interpretation of probability via mathematics

Let us recollect a few notions about the abstract theory of probability [7,8]. For the sake of simplicity, we reason inside the Bernoulli scheme where e is the successful result and $\bar{e}$ the failure.

Definition. The outcome or result e is a subset of the space Ω and the probability P is a real number that expresses how likely e is to occur

$$P = P(e), \quad E \subset \Omega. \tag{1}$$

Definition. The outcome whose resulting behavior is not entirely determined by its initial state and mechanism is called random or indeterministic.

$$0 < P(e) < 1, \quad P \in \mathbb{R}. \tag{2}$$

If the initial conditions establish that e will certainly occur or will not occur, then e is deterministic. The following extremes qualify the certain and the impossible outcomes, respectively

$$\begin{aligned} P(e) &= 1, \\ P(e) &= 0, \quad P \in \mathbb{R}. \end{aligned} \tag{3}$$

The abstract definition of probability needs to be approved or rejected by means experiments. Speaking in general, an experimentalist cannot verify the entire extension of the intended variable, but he or she can test a special range or a special typology of that variable. For example, nobody can check the concept of force; instead one tests the gravitational force or the attrition force or another special type of force. Therefore, an operator cannot control the concept of probability in general but drives a test in a special situation. In accordance with the current literature we investigate:

- $P(e_\infty)$ is the probability of e in n trials with $n \to \infty$; e.g. $P(H_\infty)$ is the probability of heads when a coin is tossed a large number of times.
- $P(e_1)$ is the probability of e in a single trial; e.g. $P(H_1)$ is the probability of heads when a coin is tossed once.

Let us briefly recall two theorems proved and commented in [9].

Theorem of large numbers (TLN): *Let the variables be independent and identically distributed (i.i.d.). When the number of experiments tends to infinity, the relative frequency of success $F(e_n)$ approaches the probability*

$$F(e_n) \xrightarrow{\text{a.s.}} P(e_\infty), \quad \text{as } n \to \infty. \tag{4}$$

Theorem of a single number (TSN): *In a single trial, the relative frequency of success $F(e_n)$ is not equal to the probability*

$$F(e_n) \neq P(e_1) \quad n = 1. \tag{5}$$

Let us make concise remarks on the theorems.

TLN demonstrates that $P(e_\infty)$ can be tested, at least in principle and therefore the frequentist probability is a physical quantity.

It is necessary to underline how TLN deals with the testability issue and for this reason Equation (4) specifies the involved result (e_∞). The strong law of large numbers (LLN) established by Borel describes the convergence of empirical frequency to the expected value of probability

$$F(e_n) \xrightarrow{\text{a.s.}} P, \quad \text{as } n \to \infty. \tag{6}$$

LLN deals with the probability P in general; instead TLN deals with $P(e_\infty)$, a specific form of probability, since TLN investigates the testability problem. LLN is a law, but TLN is an applied theorem.

TSN proves that $P(e_1)$ is out of control; it cannot be tested and is usually used as a subjective probability; notably $P(e_1)$ qualifies the credence of an individual on the occurrence of the single result (e_1).

TLN and TSN entail the following consequences:

(1) Philosophers claim that only one type of probability is true; however, TLN and TSN disprove this opinion. Two types of probability are the frequentist and the subjective, and they have different properties from the mathematical and physical perspectives.
(2) TLN and TSN assume the following values: $n \to \infty$ and $n = 1$ in the order; they refer to disjoint situations and therefore $P(e_\infty)$ and $P(e_1)$ do not contradict as many believe.
(3) The classical and Bayesian statistics underpinned by the frequentist and subjective models are to be selected depending on $n \to \infty$ and $n = 1$, respectively. In this manner, experts have a precise rule to pick the suitable statistical methods in a project, whereas so far experts adopt empirical and sometimes occasional criteria.

On one side, TLN and TSN seem somewhat trivial from the mathematical viewpoint; on the other side, they provide nontrivial answers to issues debated since long.

3 Analysis of random results

Probabilists use the terms *event* and *outcome* (or *result*) as synonyms; but we use these terms to denote different entities in accordance with the cognition shared in technology, engineering, etc. The noun event stands for the phenomenon, the trial or the system E which brings into being the outcome e. The event and the outcome are distinct elements even if they have a close relation.

Definition. The random event $E = (\iota, r; e)$ is a *structure* or *triad* where e is the successful result, and the antecedent or input i enables the process r to create e. In consequence of TLN and TSN, we have the long-term event and the single event in the order

$$\begin{aligned} E_n &= [i, r; (e_\infty)], \\ E_1 &= [i, r; (e_1)]. \end{aligned} \tag{7}$$

The structural model of E has been introduced in [10], and the book [11] goes into the mathematical properties of that model.

From the material viewpoint, the generic random event is a dynamical process located in the time scale. It begins in the instant $t_o = 0$ and finishes in t_e, which is the delivery time of e. In particular, the single event E_1 begins at t_o and delivers the outcome e_1 at time t_e. The long-term event E_n begins with the first trial at t_o; it emits the sequence of n outcomes and finishes with the last trial. No result is extant in advance of t_o. During the interval (t_o, t_e), the outcome is under preparation, and finally it is available in t_e. For example, when one flips a coin at time t_o, the preliminary element i consists of the actions that throw the coin in the air. The process r occurs in the interval (t_o, t_e) and presents tails e or otherwise heads $\bar{e}$. The random event ends when the coin falls on the table. When one flips a coin several times, the overall event starts with the first toss at t_o and finishes with the last toss at t_e.

Definition. Two main time intervals are the following:
- First time-interval (T1) with $0 < t < t_e$;
- Second time-interval (T2) with $t \geq t_e$.

The result e is qualified by the probability in T1 and is controlled by the relative frequency in T2.

Definition. In agreement with Definitions (2) and (3) we have that:
- The result e has the indeterministic state $e^{(i)}$ when one or both of the following is/are true:

$$0 < P(e) < 1; \quad 0 < F(e) < 1. \tag{8a}$$

- The result e has the deterministic state $e^{(d)}$ when it is qualified by the extremes:

$$\begin{aligned} P(e) &= 0; \quad F(e) = 0, \\ P(e) &= 1; \quad F(e) = 1. \end{aligned} \tag{8b}$$

The present work means to answer questions of these kinds: What is probability when the event is in progress? What is probability when E finishes?

Let us analyze the properties of e_∞ and e_1 that are the outcomes of the long-term and the single random events, respectively.

Theorem of continuity: *The outcome e_∞ keeps the indeterministic state in T1 and T2*

$$e^{(i)}{}_\infty, \quad 0 \leq t. \tag{9}$$

Proof. The result e_∞ is indeterministic in T1 because of the assumptions of TLN

$$0 < P(e_\infty) < 1, \quad 0 \leq t < t_e. \tag{10}$$

TLN demonstrates that the relative frequency – measured in T2 – converges toward the probability, hence e_∞ remains indeterministic from $t = 0$ onward. □

Let us examine the evolution of the outcomes in the following situations.

3.1 A single potential outcome

Theorem of discontinuity (TD): *The outcome e_1 of single trial switches from the indeterministic state to the deterministic state at the end of T1*

$$[e_1^{(i)}(i)] \rightarrow [e_1^{(d)}(t)], \quad t = t_e. \tag{11}$$

Proof. The result e_1 is indeterministic during T1 in consequence of the assumptions of TSN. The TSN proves that the relative frequency – measured in T2 – has an extreme value. Hence, (e_1) switches from the random to the deterministic state in t_e and Equation (11) is proved. □

Example. The result heads is uncertain as long as the coin flies in the air. When the coin stays at rest with heads upward, the result is certain and no longer aleatory.

3.2 Multiple potential outcomes

Speaking in general, the event E can create the outcome in various manners. For example, E prepares the final product ej step by step or using parallel processes. In the probability theory, an interesting event handles the outcomes ea_1, eb_1, ec_1,...,em_1 altogether during interval T1. All the potential outcomes are physically ready and only one of them will be delivered at the end of T1. This situation is typical of several games of chance.

Definition. When the random potential results ea_1, eb_1, ec_1,...,em_1 are ready, they superpose

$$e = (ea_1 \text{ AND } eb_1 \text{ AND} \dots \text{AND } em_1), \quad t < t_e. \tag{12}$$

The results coexist because all of them are under preparation during T1. When the event winds up, only one result comes to light out of m potential results

$$e = (ea_1 \text{ OR } eb_1 \text{ OR} \dots \text{OR } em_1), \quad t_e \leq t. \tag{13}$$

The following theorem describes the passage from equations (12) to (13).

Theorem of reduction (TR): *If equation (12) is true, the emission of the outcome ex_1 in t_e causes the reduction in m potential outcomes to just one effective outcome*

$$\begin{gathered} [ea_1^{(i)} + eb_1^{(i)} + ec_1^{(i)} + \dots + em_1^{(i)}] \rightarrow [ex_1^{(d)}], \\ ex = \text{any of } ea, eb, \dots, em; \quad t = t_e. \end{gathered} \tag{14}$$

Proof. TD proves that the generic outcome ex_1 released in t_e switches from the indeterministic to the deterministic state; hence, the following expression is true for ex_1 that becomes certain

$$\begin{gathered} [0 < P(ex_1) < 1] \rightarrow [P(ex_1) = 1], \\ ex = \text{any of } ea, eb, \dots em; \ t = t_e. \end{gathered} \tag{15}$$

□

TD also applies to the remaining potential outcomes that become impossible in agreement with the normalization condition, and TR is proven

$$[0 < P(ey_1) < 1] \rightarrow [P(ey_1) = 0],$$
$$ey = ea, eb, (ex - 1), (ex + 1), \ldots em; \quad t = t_e. \tag{16}$$

TD and TR do not mention any observer. The transition Equation (14) is strictly inherent to the physical phenomenon and do not depend on human action or knowledge.

4 Discussing the results in practical contexts

Let us look into three physical cases regarding TR and TD.

Example – Machines and devices are subject to aging due to the various degradation processes [12]. The probability of good functioning of systems approximates the exponential function

$$P(t) = e^{-\int_0^t \lambda(t)\mathrm{d}t} \quad \lambda > 0. \tag{17}$$

where $\lambda(t)$ is the hazard rate. The probability (17) for the device X has decimal values as long as X runs

$$0 < P_X(t) < 1, \quad t < t_x. \tag{18}$$

As soon as X breaks down in t_x, the probability of working drops to zero

$$P_X(t) = 0, \quad t \geq t_x. \tag{19}$$

The state of X changes from equations (18) to (19) in accordance with the TD equation (11)

$$[X^{(i)}] \rightarrow [X^{(d)}], \quad t = t_x.$$

The deterministic state of the device does not depend on any observer; X is out of service if nobody sees it (Figure 1).

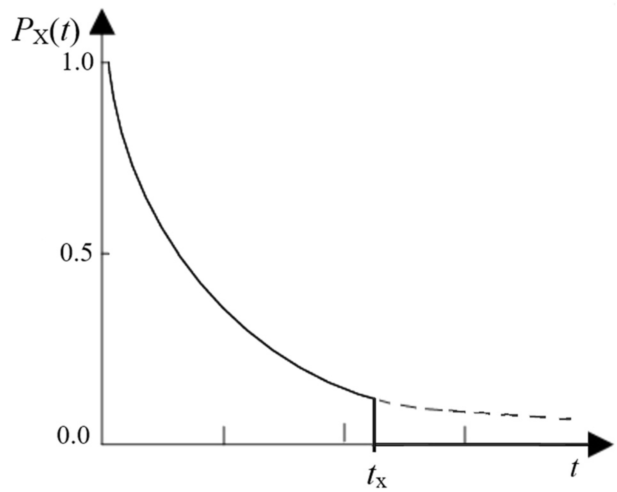

Figure 1: Probabilistic state of the device X with time.

Example. Suppose one asks Mr. Y: How old are you?

Mr. Y provides the following answer: I am 32 years old.

As long as Mr. Y keeps on living, the probability of living $P_Y(t)$ complies with the exponential function (17) [13]

$$0 < P_Y(t) < 1, \quad t < t_Y. \tag{20}$$

At the time of the query t_Y, the value "32 years of age" virtually places an end point to the age of Mr. Y. The measurement establishes a provisional close of Y's life, and in effect though not in fact, the life span of Mr. Y keeps the deterministic state

$$P_Y(t) = 1$$
$$t = t_Y = 32 \text{ years}. \tag{21}$$

Equations (20) and (21) yield the following expression in accordance to TD

$$[Y^{(i)}] \rightarrow [Y^{(d)}], \quad t = t_Y. \tag{22}$$

Transition (11) pertains to the single random event by its very nature; hence also the virtual or simulated halt of the random event causes the outcome state to change (Figure 2).

Example. Suppose rolling a die which at rest exhibits the number 5 upward. As matter of facts, the die has six potential states until it is rolling. At rest only one state becomes certain, while the remaining states vanish in accordance with TR

$$[1^{(i)} + 2^{(i)} + 3^{(i)} + 4^{(i)} + 5^{(i)} + 6^{(i)}] \rightarrow [5^{(d)}]. \tag{23}$$

In particular at the end of T1, one registers the following analytical situation that is consistent with Equations (15) and (16)

$$[0 < P_5(t) < 1] \rightarrow [P_5(t) = 1],$$
$$[0 < P_k(t) < 1] \rightarrow [P_k(t) = 0],$$
$$k = 1, 2, 3, 4, 6.$$

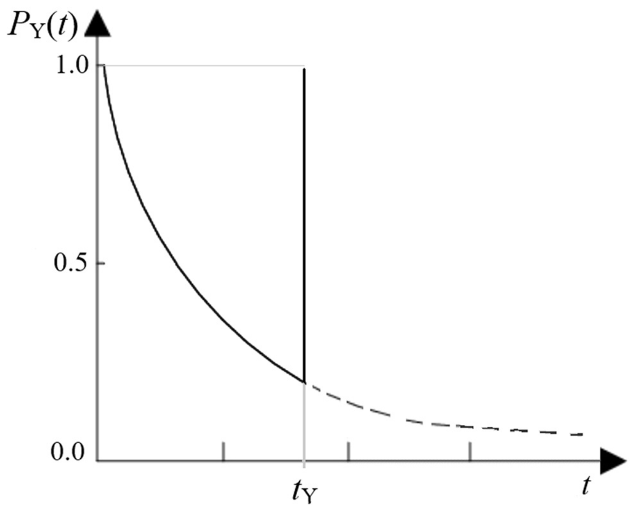

Figure 2: Probabilistic state of Mr. Y with time.

The die has the numbers 5 upward even if no observer examines it. The transition of 5 is a material process independent of human measurement and cognition.

5 Conclusion

This article recalls the TLN and the TSN that establish the properties of two distinct probability models. The theorems underpin the frequentist and subjective views, respectively, and at the same time deny the existence of a sole kind of probability.

The second part of the article analyses the behavior of aleatory outcomes. According to the usual physical cognition, the event is a process that prepares the result and lastly outputs it. The triad (7) is the specific model of the events. The TD proves that the outcome of the single event moves from the indeterministic to the deterministic state as soon as the event winds up even if nobody interferes with it. The TR shows how m potential results become reduced to a single effective result.

The third part applies the theorems to classical physics. The various concepts are discussed and verified in relation to the current literature and material situations.

Speaking in general, thinkers develop abundant annotations, whereas the mathematical language offers concise accounts.

Dirac, Einstein and other eminent physicists deem simplicity a sign of elegance and effectiveness. The theorems presented in this article do not seem complex from the mathematical perspective, yet they are able to provide answers to controversies debated since long.

In the future, we mean to export the present theoretical result into the quantum context.

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