Research Article

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# On the relations between some well-known methods and the projective Riccati equations

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**Abstract:** Solving nonlinear evolution equations is an important issue in the mathematical and physical sciences. Therefore, traditional methods, such as the method of characteristics, are used to solve nonlinear partial differential equations. A general method for determining analytical solutions for partial differential equations has not been found among traditional methods. Due to the development of symbolic computational techniques many alternative methods, such as hyperbolic tangent function methods, have been introduced in the last 50 years. Although all of them were introduced as a new method, some of them are similar to each other. In this study, we examine the following four important methods intensively used in the literature: the tanh–coth method, the modified Kudryashov method, the F-expansion method and the generalized Riccati equation mapping method. The similarities of these methods attracted our attention, and we give a link between the methods and a system of projective Riccati equations. It is possible to derive new solution methods for nonlinear evolution equations by using this connection.

**Keywords:** the tanh–coth method, the modified Kudryashov method, the F-expansion method, the generalized Riccati equation mapping method, the projective Riccati equations

## 1 Introduction

Many scientific phenomena such as heat flow, wave propagation, population models and dispersion of chemically reactive materials are characterized by partial differential equations. Therefore, solutions of the partial differential equations have attracted the attention of many researchers in many scientific fields. As a result, many solution methods have been introduced and the applications of these solution methods have been published. For instance, one of the important research areas in modern applied mathematics is the theory of non-integer derivative. Particularly in engineering sciences, fractional derivatives are a very powerful tool for modeling many problems [1–16].

It is well known that a general method for determining analytical solutions for partial differential equations has not been found, unfortunately. For this purpose, certain methods have been introduced that solve certain groups of partial differential equations. Although each method is introduced as a new solution technique, several methods give the same solutions to the differential equations. Thus, the literature is filled with a lot of methods similar to each other, such as the tanh–coth method, the modified Kudryashov method, the F-expansion method and the generalized Riccati equation mapping method.

Huibin and Kelin were pioneer researchers to introduce the first member of the hyperbolic function methods family, and they called their method as the tanh function method [17]. Following the ideas of Huibin and Kelin, Malfliet and Hereman introduced another version of the tanh function method [18]. Wazwaz improved and applied the method to a wider class of equations [19]. After these valuable works, several modifications of the tanh function method were used to solve nonlinear partial differential equations (NPDEs). In our previous work, we compared several hyperbolic tangent function methods and combined all of them into one called the unified method [20]. Kudryashov considered the Backlund transformation and discussed the modifications of the method and introduced a new method named the Kudryashov method [21]. A new and effective modified version of Kudryashov method was used to obtain new exact solutions of some equations with quadratic and cubic nonlinearities [22]. Zhou et al. generalized all the Jacobi elliptic function expansion methods via the F-expansion method in order to obtain periodic wave solutions of some equations [23]. Jin-Liang

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Zhang et al. considered coupled nonlinear evolution equations and obtained solitary wave solutions by applying the F-expansion method in the limit case [24]. Zhu reconsidered the tanh function method and improved the method by introducing the generalized Riccati equation mapping method [25].

In this study, we handle four basic methods that are used extensively to solve nonlinear evolution equations in the field of mathematical physics: the tanh–coth method, the modified Kudryashov method, the F-expansion method and the generalized Riccati equation mapping method. When these methods are investigated, the following similarities can be seen:

- All of them use similar ansatzs.
- All of them use the solutions of Riccati differential equations.
- All of them use similar solution steps.
- All of them use the same balance procedure.
- Sometimes they give the same solutions.

We discuss these similarities and give the links between the methods and a system of projective Riccati equations. This connection clarifies why the methods are so similar to each other. So, new solution methods can be obtained by using this idea.

This paper is organized as follows: in Section 2, we summarize the methods. In Section 3, using a kind of system of the projective Riccati equations we deduce a link for the methods. In Section 4, we summarize our conclusions and explain how to create an alternative solution method via an example.

## 2 The outlines of the methods

The methods developed by the solutions of some Riccati equations are quite similar to each other. All of these methods transform an NPDE of the form:

$$P(u, u_t, u_x, u_{xx}, \ldots) = 0, \tag{1}$$

to an ordinary differential equation (ODE) of the form:

$$Q(U, U', U'', \ldots) = 0, \tag{2}$$

by using the wave variable $\xi = x - ct$ ($c$ is a constant). If all terms of the resulting ODE contain derivatives in $\xi$, then equation (2) is integrated and adhering to the boundary conditions

$$U(\xi) \to 0, \frac{\mathrm{d}^n(\xi)}{\mathrm{d}\xi} \to 0, (n = 1, 2, 3, \ldots) \text{ for } \xi \to \pm\infty, \tag{3}$$

the constant of integration is considered being zero. So, a lower order ODE is obtained. All methods up to this step are similar to each other. The differences between the methods start after this step. Now we focus on the differences.

1. The tanh–coth method

The solution of (2) can be expressed as a polynomial,

$$U(\xi) = S(\sigma) = \sum_{i=0}^{M} a_i \sigma^i(\xi) + \sum_{i=1}^{M} b_i \sigma^{-i}(\xi), \tag{4}$$

where $\sigma = \sigma(\xi)$ satisfies the equation:

$$\sigma'(\xi) = \alpha - \sigma^2(\xi), \tag{5}$$

where $\sigma'(\xi) = \frac{\mathrm{d}\sigma(\xi)}{\mathrm{d}\xi}$ and $a_i$, $b_i$, $\alpha$ and $\xi_0$ are constants. Equation (5) has the following solutions:

(i) when $\alpha > 0$

$$\begin{aligned} \sigma(\xi) &= \sqrt{\alpha}\ \tanh(\sqrt{\alpha}(\xi + \xi_0)) \quad \text{or} \\ \sigma(\xi) &= \sqrt{\alpha}\ \coth(\sqrt{\alpha}(\xi + \xi_0)), \end{aligned} \tag{6}$$

(ii) when $\alpha < 0$

$$\begin{aligned} \sigma(\xi) &= -\sqrt{\alpha}\ \tan(\sqrt{\alpha}(\xi + \xi_0)) \quad \text{or} \\ \sigma(\xi) &= \sqrt{\alpha}\ \cot(\sqrt{\alpha}(\xi + \xi_0)), \end{aligned} \tag{7}$$

(iii) when $\alpha = 0$

$$\sigma(\xi) = \frac{1}{\xi + \xi_0}. \tag{8}$$

Since the derivatives of the hyperbolic tangent function are a polynomial in hyperbolic tangent function, in the tanh–coth method, the value of $\alpha$ is accepted as 1 and only the solution $\sigma(\xi) = \tanh(\xi)$ is taken into account. This choice makes it easier to calculate the higher order derivatives of $\sigma = \sigma(\xi)$.

2. The modified Kudryashov method

The solutions of equation (2) are considered as a polynomial,

$$U(\xi) = \sum_{i=0}^{M} a_i \sigma^i(\xi), \tag{9}$$

where $a_i$ and $i = 0, 1, 2, \ldots, M$ are constants but $a_m \neq 0$. $\sigma = \sigma(\xi)$ is the following function:

$$\sigma(\xi) = \frac{1}{1 + \mathrm{d}a^{\xi}}, \tag{10}$$

where $d$ is an arbitrary constant and $a \neq 0, 1$. Also, $\sigma = \sigma(\xi)$ satisfies the Riccati differential equation:

$$\sigma'(\xi) = -\sigma(\xi)\ln a + \sigma^2(\xi)\ln a. \tag{11}$$

The coefficients in (11) are constants and so this equation can be reduced to a separable differential equation. Therefore,

$$\frac{d\sigma(\xi)}{d\xi} = (\sigma^2(\xi) - \sigma(\xi))\ln a \Rightarrow \int \frac{d\sigma(\xi)}{\sigma^2(\xi) - \sigma(\xi)} = \int \ln a d\xi. \tag{12}$$

This integral can be easily calculated and we have

$$\ln\left(\frac{\sigma(\xi) - 1}{\sigma(\xi)}\right) = \ln(a^{\xi}c), \quad \sigma(\xi) = \frac{1}{1 - ca^{\xi}}, \tag{13}$$

where $c$ is an arbitrary constant. Thus, solution (10) is obtained.

3. The F-expansion method

The F-expansion method uses the similar steps to the tanh–coth method and the modified Kudryashov method. In this method, $U(\xi)$ can be expressed as a finite series in the form:

$$U(\xi) = \sum_{n=0}^{M} a_n\sigma(\xi),\ a_n \neq 0, \tag{14}$$

where $a_0,\ldots,a_n$ are constants and $\sigma = \sigma(\xi)$ is a solution for the nonlinear ODE

$$(\sigma'(\xi))^2 = P\sigma^4 + Q\sigma^2 + R, \tag{15}$$

where $P$, $Q$ and $R$ are constants. $M$ can be determined by the same procedure used in the tanh–coth method and the modified Kudryashov method.

The F-expansion method uses some solutions of (15) to obtain the solutions of (2) by choosing properly $P, Q$ and $R$. Some values of $P, Q, R$ and the corresponding Jacobi elliptic function solutions of equation (15) are given as follows:

| Equation | Solution |
|---|---|
| $(\sigma'(\xi))^2 = m^2\sigma^4 - (1+m^2)\sigma^2 + 1$ | $\sigma = \mathrm{sn}\xi$, |
| | $\sigma = \mathrm{cd}\xi$ |
| $(\sigma'(\xi))^2 = -m^2\sigma^4 + (-1+2m^2)\sigma^2 + 1 - m^2$ | $\sigma = \mathrm{cn}\xi$ |
| $(\sigma'(\xi))^2 = -\sigma^4 + (2-m^2)\sigma^2 - 1 + m^2$ | $\sigma = \mathrm{dn}\xi$ |
| $(\sigma'(\xi))^2 = \sigma^4 - (m^2+1)\sigma^2 + m^2$ | $\sigma = \mathrm{ns}\xi$ (16) |
| $(\sigma'(\xi))^2 = (1-m^2)\sigma^4 + (2m^2-1)\sigma^2 - m^2$ | $\sigma = \mathrm{nc}\xi$ |
| $(\sigma'(\xi))^2 = (m^2-1)\sigma^4 + (2-m^2)\sigma^2 - 1$ | $\sigma = \mathrm{nd}\xi$ |
| $(\sigma'(\xi))^2 = (1-m^2)\sigma^4 + (2-m^2)\sigma^2 + 1$ | $\sigma = \mathrm{sc}\xi$ |
| $(\sigma'(\xi))^2 = (m^4-m^2)\sigma^4 + (2m^2-1)\sigma^2 + 1$ | $\sigma = \mathrm{sd}\xi$ |
| $(\sigma'(\xi))^2 = \sigma^4 + (2-m^2)\sigma^2 + 1 - m^2$ | $\sigma = \mathrm{cs}\xi$ |
| $(\sigma'(\xi))^2 = \sigma^4 + (2m^2-1)\sigma^2 + m^4 - m^2$ | $\sigma = \mathrm{ds}\xi$. |

Taking $m \to 1$ the following equalities hold:

$$\mathrm{sn}\xi = \tanh\xi, \quad \mathrm{ns}\xi = \coth\xi,$$

so some solutions in (16) degenerate into the solutions obtained by the tanh–coth method.

4. The generalized Riccati equation mapping method

The fundamental steps are the same as the aforementioned methods. In this solution method, the solution(s) of (2) is of the form:

$$U(\xi) = \sum_{j=0}^{M} a_j^j\sigma(\xi),\ a_j \neq 0, \tag{17}$$

where $a_j$ are functions and $M$ is fixed by the balancing procedure. Also, $\sigma(\xi)$ is a solution of the equation:

$$\sigma'(\xi) = r + p\sigma(\xi) + q\sigma^2(\xi), \tag{18}$$

where $r$, $p$ and $q$ are all real constants. The generalized Riccati equation mapping method uses some solutions of (18).

More general form of equation (18) is the following form:

$$(\phi'(\xi))^2 = h_0 + h_1\phi(\xi) + h_2\phi^2(\xi) + h_3\phi^3(\xi) + h_4\phi^4(\xi) + h_5\phi^5(\xi) + h_6\phi^6(\xi), \tag{19}$$

and it involves a sixth-degree nonlinear term. Sirendaoreji has classified the solutions of (19) by means of the Backlund transformations and the superposition formulas [26]. He has used the relations

$$\phi^2(\xi) = \sigma(\xi),\ h_2 = \frac{c_2}{4},\ h_4 = \frac{c_3}{4},\ h_6 = \frac{c_4}{4},$$

and showed that the solutions of (19) obtained before can be reduced to the solutions of the equation:

$$(\sigma'(\xi))^2 = c_2\sigma^2(\xi) + c_3\sigma^3(\xi) + c_4\sigma^4(\xi), \tag{20}$$

where $c_2, c_3$ and $c_4$ are constants. So, all the solution methods in the literature, such as the F-expansion method and the generalized Riccati equation mapping method, use the following exact solutions of equation (20) directly or the solutions derivable from these solutions:

$$\sigma(\xi)=\begin{cases}\dfrac{-c_2c_3\operatorname{sech}^2\left(\frac{\sqrt{c_2}}{2}\xi\right)}{c_3^2-c_2c_4\left(1+\varepsilon\tanh\left(\frac{\sqrt{c_2}}{2}\xi\right)\right)^2}, & c_2>0\\ \dfrac{c_2c_3\operatorname{csch}^2\left(\frac{\sqrt{c_2}}{2}\xi\right)}{c_3^2-c_2c_4\left(1+\varepsilon\coth\left(\frac{\sqrt{c_2}}{2}\xi\right)\right)^2}, & c_2>0\\ \dfrac{-c_2\operatorname{sech}^2\left(\frac{\sqrt{c_2}}{2}\xi\right)}{c_3+2\epsilon\sqrt{c_2c_4}\tanh\left(\frac{\sqrt{c_2}}{2}\xi\right)}, & c_2>0 \text{ and } c_4>0\\ \dfrac{c_2\operatorname{csch}^2\left(\frac{\sqrt{c_2}}{2}\xi\right)}{c_3+2\epsilon\sqrt{c_2c_4}\coth\left(\frac{\sqrt{c_2}}{2}\xi\right)}, & c_2>0 \text{ and } c_4>0\\ \dfrac{-c_2\sec^2\left(\frac{\sqrt{-c_2}}{2}\xi\right)}{c_3+2\epsilon\sqrt{-c_2c_4}\tan\left(\frac{\sqrt{-c_2}}{2}\xi\right)}, & c_2<0 \text{ and } c_4>0\\ \dfrac{-c_2\csc^2\left(\frac{\sqrt{-c_2}}{2}\xi\right)}{c_3+2\epsilon\sqrt{-c_2c_4}\cot\left(\frac{\sqrt{-c_2}}{2}\xi\right)}, & c_2<0 \text{ and } c_4>0\\ \dfrac{4c_2e^{\epsilon\sqrt{c_2}\xi}}{(e^{\epsilon\sqrt{c_2}\xi}-c_3)^2-4c_2c_4}, & c_2>0\\ \dfrac{4\epsilon e^{\epsilon\sqrt{c_2}\xi}}{1-4c_2c_4e^{2\epsilon\sqrt{c_2}\xi}}, & c_2>0 \text{ and } c_3=0.\end{cases} \tag{21}$$

Many authors have considered and investigated the solutions of (19). We refer the reader to ref. [27–32] for further references.

# 3 The main idea: the projective Riccati equations and the methods

Let us consider an NPDE of the form:

$$P(u, u_t, u_x, u_{xx},\ldots)=0. \tag{22}$$

The wave transformation

$$u(x,t)=U(\xi),\ \xi=x-ct,$$

where $c$ is a constant, converts (22) to an ODE:

$$Q(U, U', U'',\ldots)=0. \tag{23}$$

If we can solve (23), then we can obtain travelling wave solution(s) of (22). To find solutions to (23), we suppose that $U(\xi)$ can be expressed as one of the following ansatzs:

$$\begin{aligned}U(\xi)&=a_0+\sum_{j=1}^{n}a_j\sigma^j(\xi)+\sum_{j=1}^{n}b_j\sigma^{j-1}(\xi)\tau(\xi),\\ U(\xi)&=a_0+\sum_{j=1}^{n}a_j\sigma^j(\xi),\\ U(\xi)&=a_0+\sum_{j=1}^{n}a_j\sigma^j(\xi)+\sum_{j=1}^{n}b_j(\sigma(\xi))^{-j},\end{aligned} \tag{24}$$

where $\sigma=\sigma(\xi)$ and $\tau=\tau(\xi)$ satisfy the system

$$\begin{aligned}\sigma'(\xi)&=\varepsilon\sigma(\xi)\tau(\xi),\\ \tau'(\xi)&=\varepsilon\omega(\sigma(\xi),\tau(\xi)),\end{aligned} \tag{25}$$

where $\varepsilon=\pm1$ and $\omega$ is a rational function in the variables $\sigma=\sigma(\xi)$ and $\tau=\tau(\xi)$.

From (25), we get a differential equation of the form:

$$\begin{aligned}\left(\frac{\sigma'(\xi)}{\sigma(\xi)}\right)'&=\omega(\sigma(\xi),\tau(\xi)),\\ \sigma'(\xi)&=\sigma(\xi)\int\omega(\sigma(\xi),\tau(\xi))\mathrm{d}\xi.\end{aligned} \tag{26}$$

If the function $\omega$ is selected properly, then system (25) can be solved exactly. For instance, taking $\varepsilon=-1$ and

$$\omega(\sigma(\xi),\tau(\xi))=\tau^2(\xi)+\frac{\mu}{K}\sigma(\xi)-1, \tag{27}$$

where $\mu=\pm1$ and $K\neq0$ we get

$$\begin{aligned}\sigma'(\xi)&=-\sigma(\xi)\tau(\xi),\\ \tau'(\xi)&=-\tau^2(\xi)-\frac{\mu}{K}\sigma(\xi)+1,\end{aligned} \tag{28}$$

which admits the first integral

$$\left(\frac{1}{\sigma(\xi)}-\frac{\mu}{K}\right)^2-\frac{\tau^2(\xi)}{\sigma^2(\xi)}=K^{-2}. \tag{29}$$

The general two-parameter solution of (29) is

$$\sigma(\xi)=\frac{K}{\cosh\xi+\mu},\ \tau(\xi)=\frac{\sinh\xi}{\cosh\xi+\mu}. \tag{30}$$

For more details about (29), we refer the reader to ref. [33].

Now we can give the link between the methods and the projective Riccati equations (25) and (26). As a result, we will see why the methods are so similar to each other. Also, this relationship will also clarify how to derive alternative solution methods.

1. The tanh–coth method

Setting

$$\omega(\sigma(\xi),\tau(\xi))=\varepsilon\left(-\sigma(\xi)\tau(\xi)-\alpha\frac{\tau(\xi)}{\sigma(\xi)}\right), \tag{31}$$

in (26), then we get the following equation:

$$\sigma'(\xi)=\sigma(\xi)\int\left(-\epsilon\sigma(\xi)\tau(\xi)-\epsilon\alpha\frac{\tau(\xi)}{\sigma(\xi)}\right)\mathrm{d}\xi. \tag{32}$$

From (23), we get

$$\begin{aligned}\sigma'(\xi)&=\sigma(\xi)\int\left(-\sigma'(\xi)-\alpha\frac{\sigma'(\xi)}{\sigma^2(\xi)}\right)\mathrm{d}\xi\\ &=\sigma(\xi)\left(-\sigma(\xi)+\alpha\frac{1}{\sigma(\xi)}+c\right).\end{aligned} \tag{33}$$

Choosing the integration constant $c$ as zero, then the last equation gives (5). Thus, we obtain the tanh–coth method.

2. The modified Kudryashov method
   Choosing

$$\omega(\sigma(\xi),\tau(\xi)) = \varepsilon\sigma(\xi)\tau(\xi)\ln a \tag{34}$$

in (26), we get

$$\begin{aligned}\sigma'(\xi) &= \sigma(\xi)\int \varepsilon\sigma(\xi)\tau(\xi)\ln a\mathrm{d}\xi\\ &= \sigma(\xi)\int \sigma'(\xi)\ln a\mathrm{d}\xi\\ &= \sigma(\xi)(\sigma(\xi)\ln a + c).\end{aligned} \tag{35}$$

If the integration constant $c$ equals $-\ln a$, then this choice gives us the modified Kudryashov method.

3. The F-expansion method
   Setting

$$\begin{aligned}&\omega(\sigma(\xi),\tau(\xi))\\ &= \frac{2P\varepsilon\sigma^2(\xi)\tau(\xi) + 1 - 2R\varepsilon\sigma^{-2}(\xi)\tau(\xi)}{2\sqrt{P\sigma^2(\xi) + Q + R\sigma^{-2}(\xi)}},\end{aligned} \tag{36}$$

in (26), where $P$, $Q$ and $R$ are real constants. After using these choices, equation (36) takes the following form:

$$\begin{aligned}&\sigma'(\xi)\\ &= \sigma(\xi)\int \frac{2P\sigma(\xi)\sigma'(\xi) + 1 - 2R\sigma^{-3}(\xi)\sigma'(\xi)}{2\sqrt{P\sigma^2(\xi) + Q + R\sigma^{-2}(\xi)}}\mathrm{d}\xi\\ &= \sigma(\xi)\left(\sqrt{P\sigma^2(\xi) + Q + R\sigma^{-2}(\xi)} + c\right).\end{aligned} \tag{37}$$

Setting $c = 0$, we obtain

$$\begin{aligned}(\sigma'(\xi))^2 &= \sigma^2(\xi)(P\sigma^2(\xi) + Q + R\sigma^{-2}(\xi))\\ &= P\sigma^4(\xi) + Q\sigma^2(\xi) + R.\end{aligned}$$

So, we get (15) and the F-expansion method.

4. The generalized Riccati equation mapping method
   Now, choosing

$$\omega(\sigma(\xi),\tau(\xi)) = -r\varepsilon\frac{\tau(\xi)}{\sigma(\xi)} + 1 + q\varepsilon\sigma(\xi)\tau(\xi), \tag{38}$$

in (26), where $r$, $p$ and $q$ are real constants, turns (38) into the following form:

$$\begin{aligned}\sigma'(\xi) &= \sigma(\xi)\int\left(-r\frac{\sigma'(\xi)}{\sigma^2(\xi)} + 1 + q\sigma'(\xi)\right)\mathrm{d}\xi\\ &= \sigma(\xi)\left(\frac{r}{\sigma(\xi)} + \xi + q\sigma(\xi)\right).\end{aligned} \tag{39}$$

We can write $p$ instead of $\xi$. So the last equation takes the following form:

$$\sigma'(\xi) = r + p\sigma(\xi) + q\sigma^2(\xi). \tag{40}$$

Equation (40) is the same as equation (18). As a result, we obtain the extended generalized Riccati equation mapping method easily.

## 4 Conclusions

We have considered and investigated the similarities and the differences of four important solution methods: the tanh–coth method, the modified Kudryashov method, the F-expansion method and the generalized Riccati equation mapping method. In fact, when we put aside their differences, the similarities of these methods attract much more attention. We have revealed that the origin of these methods is based on the same system of projective Riccati equation. So the methods are similar to each other. This study also gives the clues about alternative solution methods. We can derive new solution methods to find solutions to NPDEs by using (25) and (26). The main point we need to focus on is: if we can solve (26) exactly, then we get $\sigma = \sigma(\xi)$ and $\tau = \tau(\xi)$. We have to choose $\omega(\sigma(\xi),\tau(\xi))$ appropriately. Using $\sigma(\xi)$ and $\tau(\xi)$, we can determine one of the ansatzs (24). So, we solve (23) and then we can obtain travelling wave solution(s) of (22). Every alternative choice for the function $\omega = \omega(\sigma(\xi),\tau(\xi))$ gives a new solution method.

Let us explain via an example. Choosing the function $\omega = \omega(\sigma(\xi),\tau(\xi))$ as

$$\omega(\sigma(\xi),\tau(\xi)) = \varepsilon\left(\sigma(\xi)\tau(\xi) - \frac{\tau(\xi)}{\sigma(\xi)}\right), \tag{41}$$

we get

$$\sigma'(\xi) = \sigma(\xi)\int\left(\varepsilon\sigma(\xi)\tau(\xi) - \varepsilon\frac{\tau(\xi)}{\sigma(\xi)}\right)\mathrm{d}\xi. \tag{42}$$

Considering $\sigma'(\xi) = \varepsilon\sigma(\xi)\tau(\xi)$, $\varepsilon = \pm 1$ and the integration constant as 1 yield a nonlinear differential equation:

$$\sigma'(\xi) = 1 + \sigma(\xi) + \sigma^2(\xi). \tag{43}$$

The exact solution of (43) is

$$\sigma(\xi) = \frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}}{2}\xi + c\right) - \frac{1}{2}, \tag{44}$$

where $c$ is a constant. Using (25) we get $\tau = \tau(\xi)$ and substituting any of the ansatzs in (24) into (2), we obtain a polynomial equation in the variables $\sigma(\xi)$ and $\tau(\xi)$. Equating the coefficients of $\sigma^i(\xi)\tau^i(\xi)$, $i = 0, 1, 2, \ldots, n$, to zero, we get a system of polynomial equations in the variables $a_i$, $b_i$. Solving this system with the aid of Mathematica or Maple, we obtain the desired solutions.

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