Research Article

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Beta Generalized Exponentiated Frechet Distribution with Applications

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Abstract: In this article we introduce a new six - parameters model called the Beta Generalized Exponentiated-Frechet (BGEF) distribution which exhibits decreasing hazard rate. Many models such as Beta Frechet (BF), Beta ExponentiatedFrechet (BEF), Generalized Exponentiated-Frechet (GEF), ExponentiatedFrechet (EF), Frechet (F) are sub models. Some of its properties including r^{th} moment, reliability and hazard rate are investigated. The method of maximum likelihood isproposed to estimate the model parameters. The observed Fisher's information matrix is given. Moreover, we give the advantage of the (BGEF) distribution by an application using two real datasets

Keywords: ExponentiatedFrechet distribution; Beta generalized exponentiatedFrechetdistribution; Maximum likelihood estimation; Fisherinformation matrix; Monte Carlo simulation

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1 Introduction

Statistical distributions are very useful in describing and predicting real data analysis. Although many distributions have been developed, there are always techniques for developing distributions which are either more flexibly or for fitting specific real data analysis. The Frechet distribution has found wide applications in extreme value theory. Some extensions of the Frechet distribution are suggested to attract representing various types of data. In this article, we introduce and study mathematical properties of a new model referred to as the BetaGeneralized Exponentiated-Frechet(BGEF) distribution. The Frechet distribution represents a special case of the new model. We hope that it will attract wider applications in many other areas of scientific research.Some extensions of the Frechet distribution are available in the literature, see for example [1–7].

Consider the cumulative distribution function (cdf) and probability density function (pdf) for the ExponentiatedFrechet (EF) distribution respectively as following:

$$G_{EF}(x) = 1 - \left(1 - e^{-x^{-\alpha}}\right)^{\theta}, \quad \alpha, \theta > 0, \quad x > 0. \tag{1}$$

$$g_{EF}(x) = \alpha\theta x^{-(\alpha+1)} e^{-x^{-\alpha}} \left(1 - e^{-x^{-\alpha}}\right)^{\theta-1}, \quad \alpha, \theta > 0, x > 0. \tag{2}$$

The generalized or exponentiated form for (1) is given by

$$F_{GEF}(x) = [1 - \left(1 - e^{-x^{-\alpha}}\right)^{\theta\lambda}]^{\beta}, \quad \theta, \alpha, \lambda \text{ and } \beta > 0, x > 0. \tag{3}$$

And its probability density function (pdf) of the Generalized Exponentiated Frechet (GEF) distribution is

$$f_{GEF}(x) = \alpha\theta\beta\lambda x^{-(\alpha+1)} e^{-x^{-\alpha}} \left(1 - e^{-x^{-\alpha}}\right)^{\theta\lambda-1} \left[1 - \left(1 - e^{-x^{-\alpha}}\right)^{\theta\lambda}\right]^{\beta-1}, \tag{4}$$

$\theta, \alpha, \lambda and \beta > 0, x > 0,$

where θ, α, λ and β are shape parameters.

[8] defined the beta G distribution from a quite arbitrary cumulative distribution function (cdf) by $G(x)$,

$$G(x) = \frac{1}{B(a,b)} \int_0^{F(x)} t^{a-1}(1-t)^{b-1} dt, 0 < a, b < \infty.$$

The generalized class of distribution can be generated by applying the (cdf) to a beta distribution random variable to obtain:

$$G(x) = \frac{1}{B(a,b)} \int_0^{F_{GEF}(x)} t^{a-1}(1-t)^{b-1} dt, \quad 0 < a, b < \infty, \tag{5}$$

where

$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$ is the beta function.

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The corresponding density function for $G(x)$ is given by:

$$g(x) = \frac{1}{B(a,b)}[F_{GEF}(x)]^{a-1}[1 - F_{GEF}(x)]^{b-1} f_{GEF}(x), \quad (6)$$

$0 < a, b < \infty$.

In this paper, a new distribution is introduced by taking $G(.)$ to be the GEF, to generate the new family called the generalized exponentiated distribution Frechet distribution. The rest of the paper is organized as follows: In Section 2, we define the BGEF distribution, provide its special models, and give some plots for its probability density function(pdf), reliability function (rf) and hazard rate function (hrf). We derived the r^{th} moment in Section 3. The maximum likelihood estimates of the BGEF distribution parameters are derived in Section 4. In Section 5, the Fisher information matrix is discussed. In Section 6, a simulation study is carried out to assess the performance of the maximum likelihood estimates. In Section 7, the usefulness of the BGEF distribution is illustrated by means of two real data sets. Finally, Section 8 is devoted to some concluding remarks.

2 The BGEF Distribution

In this section, based on (3), (4) and (6), we define the BGEF distribution and provide some plots for its pdf and hrf with different parameter values.

$$g_{BGEF}(x) = \frac{\alpha\theta\lambda\beta}{B(a,b)} x^{-(\alpha+1)} e^{-x^{-\alpha}} \left(1 - e^{-x^{-\alpha}}\right)^{\theta\lambda-1} \left[1 - \left(1 - e^{-x^{-\alpha}}\right)^{\theta\lambda}\right]^{\beta a-1} \left[1 - \left[1 - \left(1 - e^{-x^{-\alpha}}\right)^{\theta\lambda}\right]^{\beta}\right]^{b-1}, \quad (7)$$

$a, b, \theta, \alpha, \lambda$ and $\beta > 0, x > 0$.

For a positive real value >0 , (7), can be written as an infinite power series in the form:

$$g_{BGEF}(x) = \frac{\alpha\theta\lambda\beta}{B(a,b)} x^{-(\alpha+1)} e^{-x^{-\alpha}} \left(1 - e^{-x^{-\alpha}}\right)^{\theta\lambda-1} \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b)}{i!\,\Gamma(b-i)} \left(1 - \left(1 - e^{-x^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta(a+i)-1}, \quad (8)$$

$a, b, \theta, \alpha, \lambda$ and $\beta > 0, x > 0$.

From (8), the corresponding (cdf) can be written as follows:

$$G_{BGEF}(x) = \frac{1}{B(a,b)} \sum_{i=0}^{\infty} \frac{(-1)^i}{b(a+i)B(b-i,i+1)} \left(1 - \left(1 - e^{-x^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta(a+i)}, \quad (9)$$

$a, b, \theta, \alpha, \lambda$ and $\beta > 0, x > 0$.

The reliability function of the BGEF distribution can be written as:

$$R(x) = 1 - \frac{1}{B(a,b)} \sum_{i=0}^{\infty} \frac{(-1)^i}{b(a+i)B(b-i,i+1)} \left[1 - \left(1 - e^{-x-\alpha}\right)^{\theta\lambda}\right]^{\beta(a+i)}, \quad (10)$$

$a, b, \theta, \alpha, \lambda$ and $\beta > 0, x > 0$.

The hazard function h(x)of the BGEF distribution is given by:

$$h(x) = \frac{A^{\star}}{1 - \frac{1}{B(a,b)} \sum_{i=0}^{\infty} \frac{(-1)^i}{b(a+i)B(b-i,i+1)} \left[1 - \left(1 - e^{-x^{-\alpha}}\right)^{\theta\lambda}\right]^{\beta(a+i)}}, \quad (11)$$

$a, b, \theta, \alpha, \lambda$ and $\beta > 0, x > 0$.

where

$$A^{\star} = \frac{\alpha\theta\lambda\beta}{B(a,b)} x^{-(\alpha+1)} e^{-x^{-\alpha}} \left(1 - e^{-x^{-\alpha}}\right)^{\theta\lambda-1} \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b)}{i!\,\Gamma(b-i)} \left[1 - \left(1 - e^{-x^{-\alpha}}\right)^{\theta\lambda}\right]^{\beta(a+i)-1}$$

Figure 1: Shows the density function of the BGEF for various values of parameters, α, θ, λ, and β. Note that the BGEF reduces to the generalized exponentiated Frechet distribution when: $a = 1$, $b = 1$ and the exponentiated Frechet distribution when: $a = 1$, $b = 1$, $\lambda = 1$, $\beta = 1$.

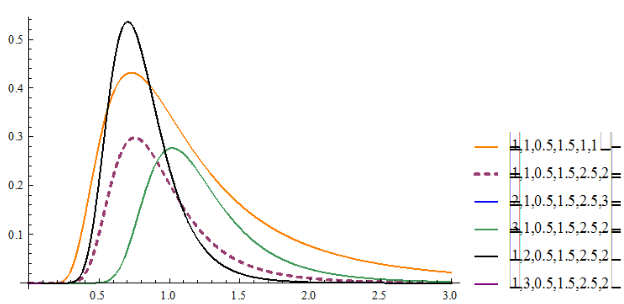

Figure 1: The pdf curves of the BGEF with (a,b,α,θ,λ,β)

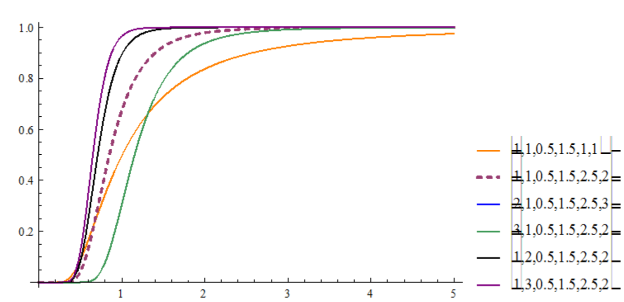

Figure 2: The reliability curves of the BGEF distribution with (a,b,α,θ, λ,β).

Figure 3: showsthe decreasing the curves of the hazard function of different parameter values for the BGEF distribution.

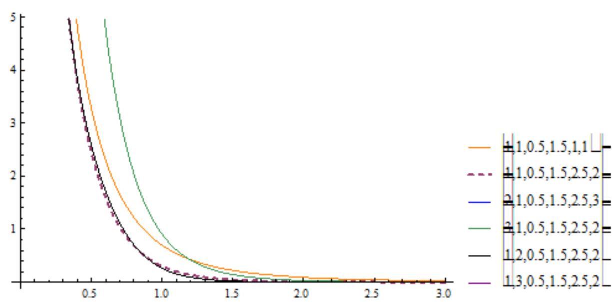

Figure 3: The hazard curve of the BGEF distribution with (a,b,α,θ, λ,β).

3 Statistical properties of the BGEF

In this section, we discuss some properties of the BGEF distribution. The non central moment, mean and variance of BGEF distribution.

The r^{th} moment around zero of a BGEF distribution is given by

$$\acute{\mu}_r = E\left(X^r\right) = \int_0^\infty x^r \frac{\alpha\theta\lambda\beta}{B(a,b)} x^{-(\alpha+1)} e^{-x^{-\alpha}} \left(1-e^{-x^{-\alpha}}\right)^{\theta\lambda-1} \left[1 - \left(1-e^{-x^{-\alpha}}\right)^{\theta\lambda}\right]^{\beta a-1} \times [1-[1-\left(1-e^{-x^{-\alpha}}\right)^{\theta\lambda}]^{\beta}]^{b-1} dx,$$

$r = 1, 2, \ldots.$

Using the generalized binomial series, we get:

$$\acute{\mu}_r = \frac{\theta\lambda\beta}{B(a,b)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i+j+k} \binom{b-1}{i} \binom{\beta(a+i)-1}{j} \binom{\theta\lambda(j+1)-1}{k} \frac{\Gamma[1-\frac{r}{\alpha}]}{(k+1)^{1-\frac{r}{\alpha}}}, \quad r < \alpha. \tag{12}$$

Substituting $r = 1$ in (12), we obtain the mean of BGEF distribution as follows:

$$\acute{\mu} = \frac{\theta\lambda\beta}{B(a,b)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i+j+k} \binom{b-1}{i} \binom{\beta(a+i)-1}{j} \binom{\theta\lambda(j+1)-1}{k} \frac{\Gamma[1-\frac{1}{\alpha}]}{(k+1)^{1-\frac{r}{\alpha}}}; \quad \alpha > 1. \tag{13}$$

At $r = 2$ in (12), we obtain the second moment foe BGEF distribution as to:

$$\acute{\mu}_2 = \frac{\theta\lambda\beta}{B(a,b)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i+j+k} \binom{b-1}{i} \binom{\beta(a+i)-1}{j} \binom{\theta\lambda(j+1)-1}{k} \frac{\Gamma[1-\frac{2}{\alpha}]}{(k+1)^{1-\frac{r}{\alpha}}}; \quad \alpha > 2. \tag{14}$$

By using (13) and (14), we get the variance

$$Var(x) = \frac{\theta\lambda\beta}{B(a,b)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i+j+k} \binom{b-1}{i} \binom{\beta(a+i)-1}{j} \binom{\theta\lambda(j+1)-1}{k} \left\{ \frac{\Gamma[1-\frac{2}{\alpha}]}{(k+1)^{1-\frac{r}{\alpha}}} - \left(\frac{\Gamma[1-\frac{1}{\alpha}]}{(k+1)^{1-\frac{r}{\alpha}}}\right)^2 \right\}. \tag{15}$$

4 Maximum likelihood estimates of the parameters

Let $X_1, X_2, \ldots, X_n$ be a random sample from BGEF distribution with parameters (a, b, α,θ,λ,β). The likelihood function is

$$L(\underline{\theta}|\, x) = \prod_{i=1}^{n} f(x_i). \tag{16}$$

Substitute by $f(.)$ in (7) and $\underline{\theta}$ = (a,b,α,θ,λ,β). Taking the logarithm of the likelihood function (16) we have:

$$l = \ln L = n\ln[\alpha\times\theta\times\lambda\times\beta] - n\ln[B(a,b)] - (\alpha+1) \sum_{i=1}^{n} \ln[x_i] - \sum_{i=1}^{n} x_i^{-\alpha} + (\theta\lambda-1)\sum_{i=1}^{n} \ln\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda} + (a\beta-1)\sum_{i=1}^{n} \ln\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right] + (b-1) \sum_{i=1}^{n} \ln\left[1-\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right]^{\beta}\right]. \tag{17}$$

Differentiae (17) with respect to a, b, $alpha$, θ ,λ and β respectively, to have:

$$l_j = \frac{\partial l}{\partial \theta_j} = \frac{1}{L}\frac{\partial L}{\partial \theta_j}, j = 1,2,3,4,5,6.$$

From (17), we have:

$$l_1 = \frac{\partial l}{\partial a} = -\frac{n}{B(a,b)}\tau_1 + \beta \sum_{i=1}^{n} \ln\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right],$$

where:

$$\tau_1 = \frac{\partial B(a,b)}{\partial a} = \frac{\Gamma[\mathrm{b}]\{\Gamma[\mathrm{a}+\mathrm{b}]\acute{\Gamma}[a] - \Gamma[a]\frac{\partial\Gamma[a+b]}{\partial a}\}}{(\Gamma[a+b])^2},$$
$$= B(a,b)\{\psi(a) - \psi(a+b)\},$$

$\psi(z) = \frac{1}{\Gamma[z]}\frac{\partial\Gamma[z]}{\partial z} = \frac{\acute{\Gamma}[z]}{\Gamma[z]}$is called the Psi function(Abramowitz and Stegun (1972)).

$$\therefore l_1 = -n\{\psi(a) - \psi(a+b)\} \tag{18}$$

$$+\beta\sum_{i=1}^{n}\ln\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right],$$

Then,

$$l_2=\frac{\partial l}{\partial b}=\frac{-n}{B(a,b)}\tau_2$$
$$+\sum_{i=1}^{n}\ln\left[1-\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right]^{\beta}\right],$$

Where:

$$\tau_2=\frac{\partial B(a,b)}{\partial b} \tag{19}$$
$$=\frac{\Gamma[\mathrm{a}]\{\Gamma[\mathrm{a}+\mathrm{b}]\acute{\Gamma}[b]-\Gamma[b]\frac{\partial\Gamma[a+b]}{\partial b}\}}{(\Gamma[a+b])^2},$$
$$\tau_2=B(a,b)\{\psi(b)-\psi(a+b)\},$$
$$\therefore l_2=-n\ \{\psi(b)-\psi(a+b)\}$$
$$+\sum_{i=1}^{n}\ln\left[1-\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right]^{\beta}\right],$$

Similarly:

$$l_3=\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\ln[x_i] \tag{20}$$
$$+\sum_{i=1}^{n}x_i^{-\alpha}\ln[x_i]+\alpha(\theta\lambda-1)\sum_{i=1}^{n}\frac{x_i^{-\alpha}e^{-x_i^{-\alpha}}\ln[x_i]}{(1-e^{-x_i^{-\alpha}})}$$
$$-(a\beta-1)\sum_{i=1}^{n}\frac{\theta\lambda x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}\ln[x_i]}{[1-(1-e^{-x_i^{-\alpha}})^{\theta\lambda}]}$$
$$+(b-1)\sum_{i=1}^{n}$$
$$\frac{\theta\lambda\beta\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right]^{\beta-1}\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}x_i^{-\alpha}e^{-x_i^{-\alpha}}\ln[x_i]}{\left[1-\left[1-(1-e^{-x_i^{-\alpha}})^{\theta\lambda}\right]^{\beta}\right]},$$

$$l_4=\frac{\partial l}{\partial\theta}=\frac{n}{\theta}+\lambda\sum_{i=1}^{n}\ln\left[1-e^{-x_i^{-\alpha}}\right]-(a\beta-1) \tag{21}$$
$$\sum_{i=1}^{n}\frac{\lambda\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\ln[1-e^{-x_i^{-\alpha}}]}{[1-(1-e^{-x_i^{-\alpha}})^{\theta\lambda}]}$$
$$+(b-1)\sum_{i=1}^{n}\frac{\lambda\beta\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right]^{\beta-1}}{\left[1-\left[1-(1-e^{-x_i^{-\alpha}})^{\theta\lambda}\right]^{\beta}\right]},$$

$$l_5=\frac{\partial l}{\partial\lambda}=\frac{n}{\lambda}+\theta\sum_{i=1}^{n}\ln\left[1-e^{-x_i^{-\alpha}}\right]-((a\beta-1) \tag{22}$$
$$\sum_{i=1}^{n}\frac{\theta\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\ln[\ 1-e^{-x_i^{-\alpha}}]}{[1-(1-e^{-x_i^{-\alpha}})^{\theta\lambda}]}+(b-1)\sum_{i=1}^{n}$$
$$\frac{\beta\theta\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right]^{\beta-1}\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\ln[\ 1-e^{-x_i^{-\alpha}}]}{\left[1-\left[1-(1-e^{-x_i^{-\alpha}})^{\theta\lambda}\right]^{\beta}\right]},$$

$$l_6=\frac{\partial l}{\partial\beta}=\frac{n}{\beta}+a\sum_{i=1}^{n}\ln\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right]-(b \tag{23}$$
$$-1)\sum_{i=1}^{n}\frac{\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right]^{\beta}\ln[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}]}{\left[1-\left[1-(1-e^{-x_i^{-\alpha}})^{\theta\lambda}\right]^{\beta}\right]}.$$

equations simultaneously yields the maximum likelihood estimates (MLEs) $\hat{\underline{\theta}}=(\hat{a},\hat{b},\hat{\alpha},\hat{\theta},\hat{\lambda},\hat{\beta})$ of parameters $\underline{\theta}=(a,b,\alpha,\theta,\lambda,\beta)$ These equations cannot be solved analytically. Clearly computer facilities and numerical method are needed to obtain the parameter estimates. So that statistical software can be used to solve them numerically by means of iterative techniques such as the Newton-Raphson algorithm. The Software Package Mathcad is used for this purpose.

5 The Fisher information matrix:

The asymptotic variance-covariance matrix of parameters is obtained by inverting the information matrix with elements that are negatives of expected values of the second order derivatives of logarithms of the likelihood function. In the present situation, it seems appropriate to approximate the expected values by their maximum likelihood estimates see [9]. In this section, we derive the asymptotic variance- covariance matrix by inverting, $I_{ij}(\underline{\theta})$, that contains of the variances and covariance of estimates, By neglecting the expectation of the second derivative $I_{ij}(\underline{\theta})$=E(-$\frac{\partial^2\ln L}{\partial\theta_i\partial\theta_j}$). Unfortunately, the exact mathematical expressions for the above expectation are very difficult to obtain, Accordingly, we have the following approximate variance-covariance matrix.

$F=[I_{ij}(\underline{\theta})]^{-1}i,j=1,2,3,4,5,6$ and $\underline{\theta}=(a,b,\alpha,\theta,\lambda,\beta)$.

6 Simulation Study

Now, we conduct a Monte Carlo simulation study to assess the performance of the MLEs of the unknown parameters for the BGEF distribution. The performance of the MLEs is evaluated in terms of their average values and mean squared errors (MSEs). The Mathcad software program is used to generate 1000 samples of the BGEF distribution for different sample sizes, where n = (10;30;50;100) , and for different parameters combinations$a,b,\theta,\alpha,\lambda and\beta$, with four sets of parameter values as follows: case 1 ≡ (2;1;0.5;1.5;2.5;3) , case 2 ≡ (3;1;0.5;1.5;2.5;2) , case 3 ≡ (1;2;0.5;1.5;2.5;2) and case 4 ≡ (1;3;0.5;1.5;2.5;2). We com-

Table 1: Average bias and average MSE of the simulated estimates

$(a,b,\alpha,\theta,\lambda,\beta)$	$(\hat{a})$		$(\hat{b})$		$(\hat{\alpha})$		$(\hat{\theta})$		$(\hat{\lambda})$		$(\hat{\beta})$	
n= 10	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
(2;1;0.5;1.5;2.5;3)	0.013	0.039	−0.180	0.015	0.077	0.247	−0.017	0.029	−0.308	0.061	−0.197	0.025
(3;1;0.5;1.5;2.5;2)	0.166	0.140	0.026	0.040	−0.016	0.076	−0.082	0.050	−0.016	0.249	−0.001	0.032
(1;2;0.5;1.5;2.5;2)	0.054	0.140	−0.052	0.065	0.023	0.053	0.017	0.247	−0.02	0.027	−0.027	0.016
(1;3;0.5;1.5;2.5;2)	0.049	0.027	−0.200	0.098	0.122	0.248	−0.074	0.028	−0.1	0.020	0.005	0.250
n= 30												
(2;1;0.5;1.5;2.5;3)	−0.022	0.004	−0.081	0.040	−0.014	0.001	0.016	0.039	−0.222	0.008	−0.132	0.010
(3;1;0.5;1.5;2.5;2)	−0.001	0.049	−0.006	0.001	0.061	0.012	−0.189	0.001	−0.035	0.147	−0.19	0.030
(1;2;0.5;1.5;2.5;2)	0.007	0.003	0.033	0.008	−0.195	0.058	−0.034	0.160	−0.192	0.031	0.007	0.013
(1;3;0.5;1.5;2.5;2)	0.011	0.005	−0.141	0.048	−0.019	0.001	0.044	0.039	0.009	0.019	−0.106	0.010
n= 50												
(2;1;0.5;1.5;2.5;3)	−0.040	0.003	−0.143	0.033	−0.043	0.002	−0.191	0.003	0.007	0.033	−0.300	0.003
(3;1;0.5;1.5;2.5;2)	−0.081	0.023	−0.035	0.002	0.034	0.004	0.012	0.020	−0.102	0.002	0.105	0.003
(1;2;0.5;1.5;2.5;2)	−0.024	0.001	0.007	0.003	−0.242	0.066	−0.144	0.002	0.124	0.002	−0.229	0.068
(1;3;0.5;1.5;2.5;2)	−0.015	0.002	−0.200	0.050	−0.046	0.002	−0.193	0.002	−0.201	0.050	−0.300	0.003
n= 100												
(2;1;0.5;1.5;2.5;3)	0.107	0.011	−0.271	0.083	−0.034	0.001	−0.199	0.040	0.006	0.001	0.012	0.005
(3;1;0.5;1.5;2.5;2)	−0.220	0.059	−0.035	0.002	0.136	0.017	−4.830	0.001	0.036	0.019	−0.199	0.039
(1;2;0.5;1.5;2.5;2)	−0.032	0.002	0.125	0.015	−0.343	0.126	0.021	0.024	−0.199	0.040	−0.004	0.001
(1;3;0.5;1.5;2.5;2)	0.117	0.014	−0.312	0.106	−0.033	0.002	−0.199	0.040	−0.004	0.002	0.013	0.001

pute the mean of the obtained estimators over all 1000 samples; the average bias and the average mean square error of simulated estimates are computed as follows:

$$Bias\,(n) = \frac{1}{1000}\sum\nolimits_{1}^{1000}(\hat{\theta} - -\theta),$$

$$MSE\,(n) = \frac{1}{1000}\sum\nolimits_{1}^{1000}(\hat{\theta} - -\theta)^{2}.$$

The following steps were suggested to obtain the simulation results:

1. Choose initial values for parameters (a,b,α,θ,λ,and β)
2. Choose the sample size n; n = (10;30;50;100) small, moderate and large sample sizes.
3. Generate 1000 times of random samples with size n from BGEF distribution.
4. Compute the ML estimate $\hat{\underline{\theta}}$ of $\underline{\theta}$=(a,b,α,θ,λ,β)for each of the 1000 samples
5. Compute the mean of the obtained estimators over all 1000 samples, the average bias and the average mean square error of simulated estimates.
6. Repeat steps 1 – 5 for several values of suggested four set of parameters.

The average values of bias and MSEs are provided for some selected parameter values and for different values of n in Table 1. It is noted, from Table 1, that the MSE decreases as the sample size increases. The results indicate that the estimates are stable and are more close to the true values when the sample sizes increased. Thus, the MLE method works very well to estimate the model parameters of the BGEF distribution.

7 Applications

In this section, two real data sets are used to demonstrate the flexibility and applicability of the BGEF distribution over some of different distributions. We consider some measures of goodness-of-fit Anderson-Darling (AD), Cramér-Von Mises (CvM) and Kolmogorov Smirnov (KS) statistics (with its p-value). The following statistics, Akaike information criterion (AIC), Bayesian information criterion (BIC), the corrected Akaike information criterion (CAIC) are also used. For the data sets, we shall compare the fits of the BGEF model with other models: the Exponentiated Frechet (EF) and generalized Exponentiated Frechet (GEF) distributions. The smaller these statistics are, the better the fit is.however, the better distribution corresponds to the smaller values of AIC, BIC, CAIC, A, W and KS criteria.

The First Data Set

The first real data set represents the survival times, in weeks, of 33 patients suffering from acute Myelogeneous Leukaemia. These data have been analysed by [10]. The

Table 2: ML estimates and their standard error (SE) (in parentheses) for the first data set

Model	Estimated Parameters					
	$(\hat{a})$	$(\hat{b})$	$(\hat{\alpha})$	$(\hat{\theta})$	$(\hat{\lambda})$	$(\hat{\beta})$
BGEF	0.036	0.0084	0.913	15.369	1.076	0.996
	(0.00854)	(17.699)	(4.76)	(0.462)	(2.578)	(0.198)
EF	- - -	- - -	0.561	0.778	- - -	- - -
			(14.561)	(0.611)		
GEF	- - -	- - -	5.158	86.306	25.057	0.158
			(3.631)	(366.672)	(94.904)	(0.169)

Table 3: Model selection criteria for the first data set. The p value of KS, W and A test statistics are put in parenthesis.

Model	KS	W	A	BIC	CAIC	AIC
BGEF	0.0642	0.0232	0.1652	126.24	120.642	118.642
	(0.6988)	(0.7996)	(0.8998)			
EF	0.0804	0.0444	0.2882	128.45	123.406	120.804
	(0.596)	(0.698)	(0.7638)			
GEF	0.191	0.0356	0.2317	132.82	122.793	120.191
	(0.6613)	(0.785)	(0.7771)			

Table 4: ML estimates and their standard error (SE) (in parentheses) for the second data set

Model	Estimated Parameters					
	$(\hat{a})$	$(\hat{b})$	$(\hat{\alpha})$	$(\hat{\theta})$	$(\hat{\lambda})$	$(\hat{\beta})$
BGEF	1.43	0.83	5.412	7.417	0.562	1.33
	(1.44)	(2.74)	(3.52)	(6. 23)	(0.491)	(0.918)
EF	—	—	21.241	11.28	—	—
			(18.931)	(9.051)		
GEF	—	—-	8.587	10.1118	56.913	19.194
			(31.558)	(6.016)	(79.93)	(10.153)

Table 5: Model selection criteria for the second data set. The p value of KS, W and A test statistics are put in parenthesis.

Model	KS	W	A	BIC	CAIC	AIC
BGEF	0.0596	0.0506	0.6125	305.65	293.249	292.624
	(0.883)	(0.908)	(0.947)			
EF	0.0645	0.0685	0.707	310.849	298.461	297.823
	(0.832)	(0.872)	(0.918)			
GEF	0.095	0.183	0. 837	306.57	294.183	298.816
	(0.866)	(0.893)	(0.929)			

data are: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43.

Results in Table 3, indicate that the BGEF model is more suitable than the other competitive models for this data set based on the selected criteria

The Second Data Set

The second data set from [11] consisting of 100 observations on

breaking stress of carbon fibres (in Gba): 3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

It is clear from Table 5 that the BGEF distribution provides a better fit than the Exponentiated Frechet (EF) and generalized Exponentiated Frechet (GEF)for this data set.

8 Conclusion

In this paper, we present a new class of distributions, called Beta Generalized Exponentiated Frechet distribution, based on Beta -G family. The BGEF distribution generalizes the Exponentiated Frechet (EF) and generalized Exponentiated Frechet (GEF) and at the same time, provides some new models. Some properties of the BGEF distribution such as, moments, reliability and hazard rate functions are derived. The maximum likelihood estimators are obtained and simulation study is provided to compare the model performance of the estimates. An application of the BGEF distribution to two real data sets show that the new distribution can be used quite effectively to provide better fits than other types of models.

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Appendix

$$\frac{\partial^2 l}{\partial a^2} = -n\left\{\acute{\psi}\,[a] - \acute{\psi}\,[a+b]\right\} \tag{24}$$

where

$$\acute{\psi}\,[z] = \frac{\partial \psi[z]}{\partial z},$$

$$\frac{\partial^2 l}{\partial b^2} = -n\left\{\acute{\psi}\,[b] - \acute{\psi}\,[a+b]\right\} \tag{25}$$

$$\frac{\partial^2 l}{\partial a \partial b} = \frac{\partial^2 l}{\partial b \partial a} = n\left\{\acute{\psi}\,[a+b]\right\} \tag{26}$$

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \sum_{i=1}^{n} x_i^{-\alpha}(\ln[x_i])^{\,2} + (\theta\lambda - 1) \tag{27}$$

$$\sum_{i=1}^{n}\Bigg(-\frac{x_i^{-2\alpha} e^{-2x_i^{-\alpha}} \ln[x_i^{-\alpha}]}{\left(1 - e^{-x_i^{-\alpha}}\right)^2} - \frac{x_i^{-2\alpha} e^{-x_i^{-\alpha}} (\ln[x_i])^2}{\left(1 - e^{-x_i^{-\alpha}}\right)}$$

$$+ \frac{x_i^{-\alpha} e^{-x_i^{-\alpha}} (\ln[x_i])^2}{\left(1 - e^{-x_i^{-\alpha}}\right)}\Bigg) + (a\beta - 1)$$

$$\sum_{i=1}^{n}\Bigg(\frac{\theta\lambda x_i^{-2\alpha} e^{-x_i^{-\alpha}} \left(1 - e^{-x_i^{-\alpha}}\right)^{\theta\lambda - 1} (\ln[x_i]\,)^2}{1 - \left(1 - e^{-x_i^{-\alpha}}\right)^{\theta\lambda}}$$

$$- \frac{\theta^2 \lambda^2 x_i^{-2\alpha} e^{-2x_i^{-\alpha}} \left(1 - e^{-x_i^{-\alpha}}\right)^{2\theta\lambda - 2} (\ln[x_i])^2}{\left(1 - \left(1 - e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^2}$$

$$- \frac{\theta\lambda\,(\theta\lambda - 1)\, x_i^{-2\alpha} e^{-2x_i^{-\alpha}} \left(1 - e^{-x_i^{-\alpha}}\right)^{\theta\lambda - 2}}{\left(1 - \left(1 - e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)}$$

$$- \frac{\theta\lambda x_i^{-\alpha} e^{-x_i^{-\alpha}} \left(1 - e^{-x_i^{-\alpha}}\right)^{\theta\lambda - 1} (\ln[x_i])^2}{\left(1 - \left(1 - e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)}\Bigg) + (b - 1)$$

$$\sum_{i=1}^{n}\Bigg(-\frac{\theta\lambda\beta x_i^{-2\alpha} e^{-x_i^{-\alpha}} \left(1 - \left(1 - e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta - 1} (\ln[x_i])^2}{\left(1 - \left(1 - \left(1 - e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}$$

$$- \frac{A^\star}{\left(1 - \left(1 - \left(1 - e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)} - \frac{B^\star}{\left(1 - \left(1 - \left(1 - e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)^2}$$

$$+\frac{C^{\star}}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}$$

$$+\frac{\beta\theta\lambda x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}(\ln[x_i])^2}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}\Bigg),$$

Where

$$A^{\star}=\beta\theta^2\lambda^2x_i^{-2\alpha}e^{-2x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda-2}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\hat{a}-2}(\beta-1)\,(\ln[x_i])^2$$

$$B^{\star}=\beta^2\theta^2\lambda^2x_i^{-2\alpha}e^{-2x_i^{-\alpha}}(\ln[x_i])^2\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda-2}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{2\beta-2}$$

$$C^{\star}=\beta\theta\lambda(\theta\lambda-1)\,x_i^{-2\alpha}e^{-2x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-2}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}(\ln[x_i])^2$$

$$\frac{\partial^2 l}{\partial\theta^2}=-\frac{n}{\theta^2}+(a\beta-1)\sum_{i=1}^{n}\left(-\frac{\lambda^2\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda}(\ln[1-e^{-x_i^{-\alpha}}])^2}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^2}-\frac{\lambda^2\left(1-e^{-x_i^{-\alpha}}\right)^{\grave{e}\lambda}(\ln[1-e^{-x_i^{-\alpha}}])^2}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)}\right)+(b-)\sum_{i=1}^{n}\Bigg(\frac{\lambda^2\beta\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}(\ln[1-e^{-x_i^{-\alpha}}])^2\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}-\frac{\beta\lambda^2(\beta-1)\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}(\ln[1-e^{-x_i^{-\alpha}}])^2}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}-\frac{\lambda^2\beta^2\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda}(\ln[1-e^{-x_i^{-\alpha}}])^2\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{2\beta-2}}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)^2}\Bigg), \tag{28}$$

$$\frac{\partial^2 l}{\partial\lambda^2}=-\frac{n}{\lambda^2}+(a\beta-1)\sum_{i=1}^{n}\left(-\frac{\theta^2\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda}(\ln[1-e^{-x_i^{-\alpha}}])^2}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^2}-\frac{\theta^2\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}(\ln[1-e^{-x_i^{-\alpha}}])^2}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)}\right)+(b-)\sum_{i=1}^{n}\Bigg(\frac{\theta^2\beta\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}(\ln[1-e^{-x_i^{-\alpha}}])^2\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}-\frac{\theta^2\beta(\beta-1)\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda}(\ln[1-e^{-x_i^{-\alpha}}])^2\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-2}}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}-\frac{\beta^2\theta^2\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda}(\ln[1-e^{-x_i^{-\alpha}}])^2\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{2\beta-2}}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)^2}\Bigg). \tag{29}$$

Similarly:

$$\frac{\partial^2 l}{\partial\beta^2}=\frac{-n}{\beta^2}+(b-1)\sum_{i=1}^{n}\left(-\frac{\left(ln\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right]\right)^2\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{2\beta}}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)^2}-\frac{(ln[1-e^{-x_i^{-\alpha}}])^2\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}\right), \tag{30}$$

$$\frac{\partial^2 l}{\partial a\partial\alpha}=\frac{\partial^2 L}{\partial\alpha\partial a}=\beta\sum_{i=1}^{n}\left(\frac{\theta\lambda x_i^{-\alpha}e^{-x_i^{-\alpha}}\ln[x_i]\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)}\right), \tag{31}$$

$$\frac{\partial^2 l}{\partial a\partial\theta}=\frac{\partial^2 l}{\partial\theta\partial a}=\beta\sum_{i=1}^{n}\left(-\frac{\lambda\ln\left[1-e^{-x_i^{-\alpha}}\right]\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)}\right), \tag{32}$$

$$\frac{\partial^2 l}{\partial a\partial\lambda}=\frac{\partial^2 l}{\partial\lambda\partial a}=\beta\sum_{i=1}^{n}\left(-\frac{\theta\ln\left[1-e^{-x_i^{-\alpha}}\right]\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)}\right), \tag{33}$$

$$\frac{\partial^2 l}{\partial a\partial\beta}=\frac{\partial^2 l}{\partial\beta\partial a}=\sum_{i=1}^{n}\ln\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right], \tag{34}$$

$$\frac{\partial^2 l}{\partial b\partial\alpha}=\frac{\partial^2 l}{\partial\alpha\partial b}=\sum_{i=1}^{n} \tag{35}$$

$$\left(-\frac{\beta\theta\lambda x_i^{-\alpha}e^{-x_i^{-\alpha}}\ln[x_i]\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}\right),$$

$$\frac{\partial^2 l}{\partial b\partial\theta}=\frac{\partial^2 l}{\partial\theta\partial b}=\sum_{i=1}^{n}\left(\frac{\hat{a}\lambda\ln\left[1-e^{-x_i^{-\alpha}}\right]\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}\right), \tag{36}$$

$$\frac{\partial^2 l}{\partial b\partial\lambda}=\frac{\partial^2 l}{\partial\lambda\partial b}=\sum_{i=1}^{n}\left(\frac{\beta\theta\ln\left[1-e^{-x_i^{-\alpha}}\right]\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}\right), \tag{37}$$

$$\frac{\partial^2 l}{\partial b\partial\beta}=\frac{\partial^2 l}{\partial\beta\partial b}=\sum_{i=1}^{n}\left(-\frac{\ln\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right]\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}\right), \tag{38}$$

and

$$\frac{\partial^2 l}{\partial\alpha\partial\theta}=\frac{\partial^2 l}{\partial\theta\partial\alpha}=\lambda\sum_{i=1}^{n}\left(-\frac{x_i^{-\alpha}e^{-x_i^{-\alpha}}\ln[x_i]}{\left(1-e^{-x_i^{-\alpha}}\right)}\right) \tag{39}$$
$$+(a\beta-1)\sum_{i=1}^{n}\Bigg(\frac{\lambda x_i^{-\alpha}e^{-x_i^{-\alpha}}\ln[x_i]\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)}$$
$$+\frac{\lambda^2\theta x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda-1}\ln\left[1-e^{-x_i^{-\alpha}}\right]\ln[x_i]}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^2}$$
$$+\frac{\lambda^2\theta x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}\ln\left[1-e^{-x_i^{-\alpha}}\right]\ln[x_i]}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)}\Bigg)$$
$$+(b-1)\sum_{i=1}^{n}$$
$$\Bigg(-\frac{\lambda\beta x_i^{-\alpha}e^{-x_i^{-\acute{a}}}\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}\ln[x_i]}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}$$
$$-\frac{A^\star}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}$$
$$+\frac{B^\star}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}$$
$$+\frac{C^\star}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)^2}\Bigg),$$

where

$$A^\star=\lambda^2\beta\theta x_i^{-\alpha}e^{-x_i^{-\alpha}}\ln\left[1-e^{-x_i^{-\alpha}}\right]\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}\ln[x_i]$$

$$B^\star=\lambda^2\beta(\beta-1)\theta x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda-1}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-2}\ln[x_i]$$

$$C^\star=\lambda^2\beta^2\theta x_i^{-\alpha}e^{-x_i^{-\alpha}}\ln\left[1-e^{-x_i^{-\alpha}}\right]\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda-1}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{2\beta-2}\ln\left[1-e^{-x_i^{-\alpha}}\right]\ln[x_i]$$

$$\frac{\partial^2 l}{\partial\alpha\partial\lambda}=\frac{\partial^2 l}{\partial\lambda\partial\alpha}=\theta\sum_{i=1}^{n}\left(-\frac{x_i^{-\alpha}e^{-x_i^{-\alpha}}\ln[x_i]}{\left(1-e^{-x_i^{-\alpha}}\right)}\right) \tag{40}$$
$$+(a\beta-1)\sum_{i=1}^{n}\Bigg(\frac{\theta x_i^{-\alpha}e^{-x_i^{-\alpha}}\ln[x_i]\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)}$$
$$+\frac{\lambda\theta^2 x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda-1}\ln\left[1-e^{-x_i^{-\alpha}}\right]\ln[x_i]}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^2}$$
$$+\frac{\lambda\theta^2 x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}\ln\left[1-e^{-x_i^{-\alpha}}\right]\ln[x_i]}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)}\Bigg)+(b-1)\sum_{i=1}^{n}$$
$$\Bigg(-\frac{\beta\theta x_i^{-\alpha}e^{-x_i^{-\alpha}}\ln[x_i]\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}}$$
$$-\frac{A^\star}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}}+\frac{B^\star}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}}$$
$$+\frac{C^\star}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)^2}\Bigg),$$

where:

$$A^\star=\beta\theta^2\lambda x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}\ln\left[1-e^{-x_i^{-\alpha}}\right]\ln[x_i]$$

$$B^\star=\beta\theta^2\lambda(\beta-1)x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda-1}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-2}\ln\left[1-e^{-x_i^{-\alpha}}\right]\ln[x_i]$$

$$C\star=\beta^2\theta^2\lambda x_i^{-\alpha}e^{-x_i^{-\alpha}\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda-1}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{2\beta-2}}\ln\left[1-e^{-x_i^{-\alpha}}\right]\ln[x_i]$$

$$\frac{\partial^2 l}{\partial\alpha\partial\beta}=\frac{\partial^2 l}{\partial\beta\partial\alpha}\tag{41}$$

$$=a\sum_{i=1}^{n}\left(\frac{\theta\lambda x_i^{-\alpha}e^{-x_i^{-\alpha}}\ln[x_i]\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)}\right)+(b-1)\sum_{i=1}^{n}$$

$$\left(-\frac{A\star}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}-\frac{B\star}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)^2}-\frac{C\star}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}\right),$$

where

$$A\star=\theta\lambda x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}\ln\left[1-e^{-x_i^{-\alpha}}\right]\ln[x_i]$$

$$B\star=\theta\beta\lambda x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{2\beta-1}\ln\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right]\ln[x_i]$$

$$C\star=\theta\beta\lambda x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}\ln\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right]\log[x_i]$$

$$\frac{\partial^2 l}{\partial\theta\partial\alpha}=\frac{\partial^2 l}{\partial\alpha\partial\theta}=\lambda\sum_{i=1}^{n}\left(-\frac{x_i^{-\alpha}e^{-x_i^{-\alpha}}\ln[x_i]}{\left(1-e^{-x_i^{-\alpha}}\right)}\right)+(a\beta-1)\sum_{i=1}^{n}\tag{42}$$

$$\left(\frac{\lambda x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}\ln[x_i]}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)}+\frac{\theta\lambda^2x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda-1}\ln\left[1-e^{-x_i^{-\alpha}}\right]\ln[x_i]}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^2}+\frac{\theta\lambda^2x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}\ln\left[1-e^{-x_i^{-\alpha}}\right]\ln[x_i]}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)}\right)+(b-1)\sum_{i=1}^{n}$$

$$\left(-\frac{\lambda\beta x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}\ln[x_i]}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}-\frac{A\star}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}+\frac{B\star}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}+\frac{C\star}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)^2}\right),$$

where

$$A\star=\theta\beta\lambda^2x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda-1}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}\ln\left[1-e^{-x_i^{-\alpha}}\right]\ln[x_i]$$

$$B\star=\lambda^2\beta\theta(\beta-1)x_i^{-\alpha}e^{-x_i^{-\alpha}}\ln\left[1-e^{-x_i^{-\alpha}}\right]\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda-1}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-2}\ln[x_i]$$

$$C\star=\beta^2\lambda^2\theta x_i^{-\alpha}e^{-x_i^{-\alpha}}\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda-1}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{2\beta-2}\ln\left[1-e^{-x_i^{-\alpha}}\right]\ln[x_i]$$

$$\frac{\partial^2 l}{\partial\theta\partial\lambda}=\frac{\partial^2 l}{\partial\lambda\partial\theta}=\sum_{i=1}^{n}\ln\left[1-e^{-x_i^{-\alpha}}\right]+(a\beta-1)\sum_{i=1}^{n}\tag{43}$$

$$\left(-\frac{\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\ln\left[1-e^{-x_i^{-\alpha}}\right]}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)}-\frac{\theta\lambda\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda}\left(\ln\left[1-e^{-x_i^{-\alpha}}\right]\right)^2}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^2}-\frac{\theta\lambda\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\left(\ln\left[1-e^{-x_i^{-\alpha}}\right]\right)^2}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)}\right)+(b-1)\sum_{i=1}^{n}$$

$$\left(\frac{\beta\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}\ln\left[1-e^{-x_i^{-\alpha}}\right]}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}+\frac{\beta\theta\lambda\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}\left(\ln\left[1-e^{-x_i^{-\alpha}}\right]\right)^2}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}-\frac{A\star}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}-\frac{\beta^2\theta\lambda\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{2\beta-2}\left(\ln\left[1-e^{-x_i^{-\alpha}}\right]\right)^2}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)^2}\right),$$

where

$$A\star=\beta\theta\lambda(\beta-1)\left(1-e^{-x_i^{-\alpha}}\right)^{2\theta\lambda}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-2}$$

$$(\ln\left[1-e^{-x_i^{-\alpha}}\right])^2$$

$$\frac{\partial^2 l}{\partial\theta\partial\beta}=\frac{\partial^2 l}{\partial\beta\partial\theta}=a\sum_{i=1}^{n}(-\frac{\lambda\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\ln\left[1-e^{-x_i^{-\alpha}}\right]}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)} \tag{44}$$

$$+(b-1)\sum_{i=1}^{n}$$

$$(\frac{\lambda\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}\ln\left[1-e^{-x_i^{-\alpha}}\right]}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}$$

$$+\frac{\beta\lambda\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{2\beta-1}\ln\left[1-e^{-x_i^{-\alpha}}\right]}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)^2}$$

$$+\frac{A^\star}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}),$$

where

$$A^\star=\beta\lambda\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}$$
$$\ln\left[1-e^{-x_i^{-\alpha}}\right]\ln\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right]$$

$$\frac{\partial^2 l}{\partial\lambda\partial\beta}=\frac{\partial^2 l}{\partial\beta\partial\lambda}=a\sum_{i=1}^{n}(-\frac{\theta\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\ln\left[1-e^{-x_i^{-\alpha}}\right]}{\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)} \tag{45}$$

$$+(b-1)\sum_{i=1}^{n}($$

$$\frac{\theta\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}\ln\left[1-e^{-x_i^{-\alpha}}\right]}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}$$

$$+\frac{\beta\theta\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{2\beta-1}\ln\left[1-e^{-x_i^{-\alpha}}\right]}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)^2}$$

$$+\frac{A^\star}{\left(1-\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta}\right)}).$$

where

$$A^\star=\beta\theta\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\left(1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right)^{\beta-1}\ln\left[1-e^{-x_i^{-\alpha}}\right]$$
$$\ln\left[1-\left(1-e^{-x_i^{-\alpha}}\right)^{\theta\lambda}\right]$$