Open Phys. 2019; 17:480–488 DE GRUYTER

Research Paper

Fangqin He and Lianzhong Li*

Time fractional modified KdV-type equations: Lie symmetries, exact solutions and conservation laws

https://doi.org/10.1515/phys-2019-0049
Received Jun 03, 2019; accepted Aug 01, 2019

Abstract: In the paper, we research a time fractional modified KdV-type equations. We give the symmetry reductions and exact solutions of the equations, and we investigate the convergence of the solutions. In addition, the conservation laws of the equations are constructed.

Keywords: Time fractional differential equations; Lie symmetry; Exact solution; Conservation laws

PACS: 02.30.Jr, 02.20.-a, 02.30.Ik, 04.20.Jb

1 Introduction

Nonlinear partial differential equation(NLPDE) is a kind of important mathematical model for describing the natural phenomena and mathematical physics. Over the past few years, many ordinary and partial differential equations(PDEs) were concerned by the researchers, they have obtained many good results[1 – 10]. At present, many approaches have been extensively studied for constructing exact solutions of the equation, such as the inverse scattering transformation(IST)[11], Darboux transformation method and Bäcklund transformation method[12], Hirota's bilinear method[12–14], Lie symmetry analysis[15–20], CK method[21], and so on. Now, there are more and more related researches on fractional partial differential equations(FPDEs). At the same time, those methods are also widely used in solving the precise solution of these equations[6, 24, 25]. In particular, the classical Korteweg-de Vries (Kdv) equations which play an important role in many mathematical and physical fields. In [19, 24 – 30], the authors used the power series method to solve the classical Kdv equations. Meanwhile, they sought the help of the new conservation theorem which constructed the conservation laws(Cls) for the governing equations. At the same time, fractional calculus is also very popular, it has been successfully used to explain many complex nonlinear phenomena and dynamic processes in physics, engineering, electromagnetics, viscoelasticity, and electrochemistry. Inspired by the above, we considered here to study the time-fractional modified KdV-type equations, which are presented in the following form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - u_{xxx} - uvu_x = 0, \qquad (1)$$
$$\frac{\partial^\alpha v}{\partial t^\alpha} - v_{xxx} - uvv_x = 0,$$

in Eq. (1), $0 < \alpha < 1$, $\frac{\partial^\alpha \cdot}{\partial t^\alpha}$ is the Riemann-Liouville (RL) fractional derivatives. If $\alpha = 1$, Eq. (1) becomes

$$u_t - u_{xxx} - uvu_x = 0, \qquad (2)$$
$$v_t - v_{xxx} - uvv_x = 0.$$

Eq. 2 is a modified KdV-type equations. The modified KdV-type equations is most popular mathematical models and have been extensively investigated. And it has been applied to describe the electromagnetic waves in size-quantized films, interfacial waves in two-layer liquids, and transmission lines in the Schottky barrier. It was analyzed and studied in [24], Lie symmetry analysis, exact solutions and CLs for Eq. 2 were investigated. And Eq. (1) comes from the modified KdV-type equations by replacing its time derivative with a fractional derivative. so, we will analyze and investigate the Lie symmetries, exact solutions, CLs and the convergence of the exact solutions for Eq. (1).

In this paper, to the best of our knowledge, we apply Lie method to study Eq. (1), we get the optimal system and the exact solutions for the equation, we also give the Cls for the equations via a new conservation theorem.

This article is divided into the following sections: First of all, in section 2, we introduce some essential knowledge which will be used in later chapters; in section 3, we seek the help of the Lie method which can acquire the optimal system and the symmetry reductions of Eq. (1); in section

***Corresponding Author: Lianzhong Li:** School of Science, Jiangnan University, Wuxi, 214122, China; Email: llz3497@163.com
Fangqin He: School of Science, Jiangnan University, Wuxi, 214122, China

4, on the basis of the third quarter, we calculate the exact solution of the equations; in section 5, the convergence of the exact solutions for the equations will be investigated; in section 6, we use the symmetries and adjoint equations to construct the conservation laws, there are some conclusions and discussions in the last section.

2 Preliminaries

We introduce some essential knowledge about the RL fractional derivative and the Lie symmetries in the section. Firstly, the definition of the RL fractional derivative [22, 23] is as follows:

$$D^{\alpha}f(t)=\begin{cases}\frac{d^nf}{dt^n}, & \alpha=n,\\ \frac{d}{dt^n}I^{n-\alpha}f(t), & 0\le n-1<\alpha<n,\end{cases} \tag{3}$$

n is a natural number and $I^{n-\alpha}f(t)$ is defined by

$$I^{n-\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f(s)ds, \tag{4}$$

$$n-\alpha>0,\quad I^{n-\alpha}f(t)=f(t),\quad n-\alpha=0,$$

where $\Gamma(n-\alpha)$ is the gamma function.

Let us consider the blow space-time FPDEs:

$$\begin{aligned} D_t^{\alpha}u&=F_1(t,x,u,v,u_x,v_x,\dots),(\alpha>0)\\ D_t^{\alpha}v&=F_2(t,x,u,v,u_x,v_x,\dots),(\alpha>0)\end{aligned} \tag{5}$$

next, we present the form of a one-parameter Lie group of infinitesimal transformations is as blow:

$$\begin{aligned} x^{\star}&=x+\epsilon\xi^1(x,t,u,v)+o(\epsilon^2),\\ t^{\star}&=t+\epsilon\xi^2(x,t,u,v)+o(\epsilon^2),\\ u^{\star}&=u+\epsilon\eta^1(x,t,u,v)+o(\epsilon^2),\\ rv^{\star}&=v+\epsilon\eta^2(x,t,u,v)+o(\epsilon^2),\\ \frac{\partial^{\alpha}u^{\star}}{\partial t^{\star\alpha}}&=\frac{\partial^{\alpha}u}{\partial t^{\alpha}}+\epsilon\eta^{1\alpha,t}(x,t,u,v)+o(\epsilon^2),\\ \frac{\partial^{\alpha}v^{\star}}{\partial t^{\star\alpha}}&=\frac{\partial^{\alpha}v}{\partial t^{\alpha}}+\epsilon\eta^{2\alpha,t}(x,t,u,v)+o(\epsilon^2),\\ \frac{\partial u^{\star}}{\partial x^{\star}}&=\frac{\partial u}{\partial x}+\epsilon\eta^{1x}(x,t,u,v)+o(\epsilon^2),\\ \frac{\partial v^{\star}}{\partial x^{\star}}&=\frac{\partial v}{\partial x}+\epsilon\eta^{2x}(x,t,u,v)+o(\epsilon^2),\\ \frac{\partial^3u^{\star}}{\partial x^{\star3}}&=\frac{\partial^3u}{\partial x^3}+\epsilon\eta^{1xxx}(x,t,u,v)+o(\epsilon^2),\\ \frac{\partial^3v^{\star}}{\partial x^{\star3}}&=\frac{\partial^3v}{\partial x^3}+\epsilon\eta^{2xxx}(x,t,u,v)+o(\epsilon^2),\end{aligned} \tag{6}$$

where

$$\begin{aligned} \eta^{1x}&=D_x(\eta^1)-u_xD_x(\xi^1)-u_tD_t(\xi^2),\\ \eta^{2x}&=D_x(\eta^2)-v_xD_x(\xi^1)-v_tD_t(\xi^2),\\ \eta^{1xx}&=D_x(\eta^{1x})-u_{xt}D_x(\xi^1)-u_{xx}D_t(\xi^2),\\ \eta^{2xx}&=D_x(\eta^{1x})-v_{xt}D_x(\xi^1)-v_{xx}D_t(\xi^2),\\ \eta^{1xxx}&=D_x(\eta^{1xx})-u_{xxt}D_x(\xi^1)-u_{xxx}D_t(\xi^2),\\ \eta^{2xxx}&=D_x(\eta^{2xx})-v_{xxt}D_x(\xi^1)-v_{xxx}D_t(\xi^2)\end{aligned} \tag{7}$$

and the following vector can be used to derive the associated Lie algebra,

$$\begin{aligned} V&=\xi^1(x,t,u,v)\frac{\partial}{\partial x}+\xi^2(x,t,u,v)\frac{\partial}{\partial t}\\ &+\eta^1(x,t,u,v)\frac{\partial}{\partial u}+\eta^2(x,t,u,v)\frac{\partial}{\partial v}.\end{aligned} \tag{8}$$

The V must meet the following Lie point symmetry condition, and we also figure out the coefficient function of the vector field: $\xi^1(x,t,u,v)$, $\xi^2(x,t,u,v)$, $\eta^1(x,t,u,v)$, $\eta^2(x,t,u,v)$ via the following condition.

$$pr^{(3)}v(\Delta)\mid_{\Delta=0}=0. \tag{9}$$

The invariance condition[34] gives

$$\xi^2(x,t,u,v)\mid_{t=0}=0, \tag{10}$$

the η_{α}^0 is defined by[32, 33].

$$\begin{aligned} \eta_{\alpha}^0&=\frac{\partial^{\alpha}\eta}{\partial t^{\alpha}}+\left(\eta_u-\alpha D_t(\xi^2)\right)\frac{\partial^{\alpha}u}{\partial t^{\alpha}}-u\frac{\partial^{\alpha}\eta_u}{\partial t^{\alpha}}+\mu\\ &+\sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^{\alpha}\eta_u}{\partial t^{\alpha}}-\binom{\alpha}{n+1}D_t^{n+1}(\xi^2)\right]\times D_t^{\alpha-n}(u)\\ &-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi^1)D_t^{\alpha-n}(u_x),\end{aligned} \tag{11}$$

where μ is defined by

$$\begin{aligned} \mu&=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\\ &\cdot\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}\times[-u]^r\frac{\partial^m}{\partial t^m}[u^{k-r}]\frac{\partial^{n-m+k}}{\partial t^{n-m}\partial u^k}.\end{aligned} \tag{12}$$

Next, we apply the above knowledge to analyze the Eq. (1), and the Lie symmetry and optimal system of the Eq. (1) are received in the next chapter.

3 Lie Symmetry and optimal system

We make full use of the above Lie symmetry analysis method to research the Eq. (1). Firstly, taking (6) into (1), we have that

$$\frac{\partial^{\alpha}u^{\star}}{\partial t^{\star\alpha}}-u^{\star}_{x^{\star}x^{\star}x^{\star}}-u^{\star}v^{\star}u^{\star}_{x^{\star}}=0, \tag{13}$$

$$\frac{\partial^\alpha v^*}{\partial t^{*\alpha}} - v^*_{x^*x^*x^*} - u^* v^* v^*_{x^*} = 0,$$

substituting the third prolongation $pr^{(3)}$ that we have previously obtained into the Eq. (1), we get the blow result

$$\begin{aligned} &\eta^{1\alpha,t} - uv\eta^{1x} - vu_x\eta^1 - uu_x\eta^2 - \eta^{1xxx} = 0, \\ &\eta^{2\alpha,t} - uv\eta^{2x} - vv_x\eta^1 - uv_x\eta^2 - \eta^{2xxx} = 0, \end{aligned} \tag{14}$$

considering the condition that variables $u_t, u_x, u_{xx}, u_{xt}, v_t, v_x, v_{xx}, v_{xt}, \ldots$ and $D_t^{\alpha-n}u$, $D_t^{\alpha-n}u_x, D_t^{\alpha-n}v$, $D_t^{\alpha-n}v_x$ for $n = 1, 2, \ldots$ of u are independent, substituting (7) and (11) into consideration, let each power of the derivative of u, v become 0, we have

$$\begin{aligned} &\xi^1_u = \xi^1_t = \xi^2_u = \xi^2_x = \eta^1_{uu} = \eta^2_{vv} = 0, \\ &\frac{\partial^\alpha \eta^1}{\partial t^\alpha} - u\frac{\partial^\alpha \eta^1_u}{\partial t^\alpha} - uv\eta^1_x - \eta^1_{xxx} = 0, \\ &\frac{\partial^\alpha \eta^2}{\partial t^\alpha} - v\frac{\partial^\alpha \eta^2_v}{\partial t^\alpha} - uv\eta^2_x - \eta^2_{xxx} = 0, \\ &-(\eta^1_u - \xi^1_x)uv - v\eta^1 - u\eta^2 - (3\eta^1_{uxx} - \xi^1_{xxx}) = 0, \\ &-(\eta^2_v - \xi^1_x)uv - v\eta^1 - u\eta^2 - (3\eta^2_{vxx} - \xi^1_{xxx}) = 0, \\ &3(\eta^1_{xu} - \xi^1_{xx}) = 0, \\ &3(\eta^2_{xv} - \xi^1_{xx}) = 0, \\ &\alpha\xi^2_t - 3\xi^1_x = 0, \\ &\binom{\alpha}{n}\frac{\partial^\alpha \eta^1_u}{\partial t^\alpha} - \binom{\alpha}{n+1} D_t^{n+1}(\xi^2) = 0, \\ &\binom{\alpha}{n}\frac{\partial^\alpha \eta^2_v}{\partial t^\alpha} - \binom{\alpha}{n+1} D_t^{n+1}(\xi^2) = 0. \end{aligned} \tag{15}$$

Solving these equations, we get:

$$\begin{aligned} \xi^1 &= c_1 x + c_2, \\ \xi^2 &= \frac{3c_1}{\alpha}t + c_3, \\ \eta^1 &= 6c_4 u, \\ \eta^2 &= -2c_1 v - 2c_4 v, \end{aligned} \tag{16}$$

where c_1, c_2, c_3, c_4 are arbitrary constants. So, four correlative vector fields are acquired from Eq. (16)

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x}, \\ V_2 &= \frac{\partial}{\partial t}, \\ V_3 &= u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}, \\ V_3 &= x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t} - 2v\frac{\partial}{\partial v}. \end{aligned} \tag{17}$$

Next, we can acquire the optimal system of the Eq. (1) via the method that has been clearly described in Refs [35]. The first step is to get the following commutator table(see Table 1) based on the commutator operators $[V_s, V_t] = V_sV_t - V_tV_s$, we get

Table 1: Lie bracket of Eq. (1)

Lie	V_1	V_2	V_3	V_4
V_1	0	0	0	V_1
V_2	0	0	0	$\frac{3}{\alpha}V_2$
V_3	0	0	0	0
V_4	$-V_1$	$-\frac{3}{\alpha}V_2$	0	0

The second step is to get the adjoint representations of the vector fields via using the commutator relations in Table 1 and the Lie series

$$\begin{aligned} \mathrm{Ad}(\exp(\epsilon V_i))\, V_j = V_j - \epsilon\left[V_i, V_j\right] \\ + \frac{1}{2}\epsilon^2\left[V_i, \left[V_i, V_j\right]\right]\ldots, \end{aligned} \tag{18}$$

we obtain the adjoint representations of the vector fields (see Table 2).

Table 2: Adjoint representation

$\mathrm{Ad}(\epsilon)$	V_1	V_2	V_3	V_4
V_1	V_1	V_2	V_3	$V_4 - \varepsilon V_1$
V_2	V_1	V_2	V_3	$V_4 - \frac{3\varepsilon}{\alpha}V_2$
V_3	V_1	V_2	V_3	V_4
V_4	$e^\varepsilon V_1$	$e^{\frac{3\varepsilon}{\alpha}}V_2$	V_3	V_4

The final step is to get the optimal system for the Eq. (1) from the adjoint representations of the vector fields and the result is as follows

$$\{V_1, V_2, V_4, V_2 \pm V_1, V_3 \pm V_1, V_3 \pm V_2, V_4 \pm V_3\}$$

4 Similarity reductions

By simple computation, the following equation

$$\frac{dx}{x} = \frac{\alpha dt}{3t} = \frac{dv}{-2v} \tag{19}$$

show the similarity variables for the infinitesimal generator V_4 given by

$$\begin{aligned} u &= f(\xi), \\ v &= t^{-\frac{2\alpha}{3}}g(\xi), \\ \xi &= xt^{-\frac{\alpha}{3}}. \end{aligned} \tag{20}$$

Summarize the above discussed in detail, Eq. (1) can be converted to a nonlinear ordinary differential equation.

We lead into the blow Erdély-Kober fractional (EK) differential operator [22] with the intention of achieving this goal

$$\left(P_{\sigma}^{\xi^2,\alpha}f\right) := \prod_{j=0}^{n-1}\left(\xi^2 + j - \frac{1}{\sigma}\frac{d}{d\xi}\right)\left(K_{\sigma}^{\xi^2+\alpha,n-\alpha}f\right)(\xi), \quad (21)$$

where

$$n = \begin{cases} [\alpha]+1, & \alpha \notin N, \\ \alpha, & \alpha \in N, \end{cases} \quad (22)$$

and the EK fractional integral operator is defined by

$$\left(K_{\sigma}^{\xi^2,\alpha}f\right) := \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^{\infty}(u-1)^{\alpha-1}u^{-(\xi^2+\alpha)}f(\xi u^{\frac{1}{\sigma}})du, \\ f(\xi), \end{cases} \quad (23)$$

let $n-1 < \alpha < n$, $n = 1, 2, 3, 4, \ldots$ According to the definition of the RL fractional derivative, we get

$$D_t^{\alpha}u = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_1^t (t-s)^{n-\alpha-1}f(xs^{-\frac{\alpha}{3}})ds\right], \quad (24)$$

setting $v = \frac{t}{s}$, then $ds = -\frac{t}{v^2}dv$, substituting it into the above equation, we get

$$D_t^{\alpha}u = \frac{\partial^n}{\partial t^n} \cdot \left[\frac{1}{\Gamma(n-\alpha)}\int_1^{\infty}(v-1)^{n-\alpha-1}t^{n-\alpha}v^{\alpha-n-1}f(\xi v^{\frac{\alpha}{3}})dv\right], \quad (25)$$

according to Eq. (25), we have

$$D_t^{\alpha}u = \frac{\partial^n}{\partial t^n}\left[t^{n-\alpha}\left(K_{\frac{3}{\alpha}}^{1,n-\alpha}f\right)(\xi)\right]. \quad (26)$$

Continue to simplify the above equation, consider $\xi = xt^{-\frac{\alpha}{3}}$, $\varphi \in (0,\infty)$, we acquire

$$t\frac{\partial}{\partial t}\varphi(\xi) = tx\left(-\frac{\alpha}{3}\right)t^{-\frac{\alpha}{3}-1}\varphi'(\xi) = -\frac{\alpha}{3}\xi\frac{\partial}{\partial\xi}\varphi(\xi), \quad (27)$$

so,

$$\begin{aligned} &\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha}\left(K_{\frac{3}{\alpha}}^{1,n-\alpha}f\right)(\xi)\right] \quad (28)\\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\alpha-1}\left(n-\alpha-\frac{\alpha}{3}\xi\frac{\partial}{\partial\xi}\right)\left(K_{\frac{3}{\alpha}}^{1,n-\alpha}f\right)(\xi)\right]\\ &= \frac{\partial^{n-2}}{\partial t^{n-2}}\left[t^{n-\alpha-2}\left(n-\alpha-\frac{\alpha}{3}\xi\frac{\partial}{\partial\xi}\right)\left(n-\alpha-1-\frac{\alpha}{3}\xi\frac{\partial}{\partial\xi}\right) \cdot \left(K_{\frac{3}{\alpha}}^{1,n-\alpha}f\right)(\xi)\right]\\ &= \frac{\partial^{n-3}}{\partial t^{n-3}}\left[t^{n-\alpha-3}\left(n-\alpha-\frac{\alpha}{3}\xi\frac{\partial}{\partial\xi}\right)\left(n-\alpha-1-\frac{\alpha}{3}\xi\frac{\partial}{\partial\xi}\right) \cdot \left(n-\alpha-2-\frac{\alpha}{3}\xi\frac{\partial}{\partial\xi}\right)\times\left(K_{\frac{3}{\alpha}}^{1,n-\alpha}f\right)(\xi)\right]\\ &= \ldots\\ &= t^{-\alpha}\prod_{j=0}^{n-1}\left(1-\alpha+j-\frac{\alpha}{3}\xi\frac{\partial}{\partial\xi}\right)\left(K_{\frac{3}{\alpha}}^{1,n-\alpha}f\right)(\xi). \end{aligned}$$

Substituting EK fractional differential operator Eq. (21) in Eq. (28), we have

$$\begin{aligned} &t^{-\alpha}\prod_{j=0}^{n-1}\left(1-\alpha+j-\frac{\alpha}{3}\xi\frac{\partial}{\partial\xi}\right)\left(K_{\frac{3}{\alpha}}^{1,n-\alpha}f\right)(\xi) \quad (29)\\ &= t^{-\alpha}\left(P_{\frac{3}{\alpha}}^{1-\alpha,\alpha}f\right)(\xi). \end{aligned}$$

And by the same logic, we can get:

$$D_t^{\alpha}v = t^{-\frac{5\alpha}{3}}\prod_{j=0}^{n-1}\left(1-\frac{5\alpha}{3}+j -\frac{\alpha}{3}\xi\frac{\partial}{\partial\xi}\right)\left(K_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},n-\alpha}g\right)(\xi) = t^{-\frac{5\alpha}{3}}\left(P_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha}g\right)(\xi). \quad (30)$$

So, Eq. (1) can be converted to the nonlinear ordinary differential equation of fractional order, we obtain

$$\begin{aligned} &\left(P_{\frac{3}{\alpha}}^{1-\alpha,\alpha}f\right)(\xi) - f'''(\xi) - f'(\xi)f(\xi)g(\xi) = 0, \quad (31)\\ &\left(P_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha}g\right)(\xi) - g'''(\xi) - g'(\xi)g(\xi)f(\xi) = 0. \end{aligned}$$

5 Explicit analytical power series solutions

In the section, the exact explicit solution of the Eq. (31) will be obtained via using the power series method [36, 37]. Power series method is a method for solving ordinary differential equations, especially when the solution of differential equation cannot use elementary function or or its integral expression, we seek other solution, especially power series solution is an approximate solution of the commonly used. Using the power series solution and generalized power series solution can solve many important differential equation in mathematical physics, Set

$$f(\xi) = \sum_{n=0}^{\infty}a_n\xi^n, \qquad g(\xi) = \sum_{n=0}^{\infty}b_n\xi^n, \quad (32)$$

we get

$$\begin{aligned} &f'(\xi) = \sum_{n=0}^{\infty}(n+1)a_{n+1}\xi^n, \quad (33)\\ &f''(\xi) = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}\xi^n, \end{aligned}$$

$$f'''(\xi) = \sum_{n=0}^{\infty}(n+1)(n+2)(n+3)a_{n+3}\xi^n,$$
$$g'(\xi) = \sum_{n=0}^{\infty}(n+1)b_{n+1}\xi^n,$$
$$g''(\xi) = \sum_{n=0}^{\infty}(n+1)(n+2)b_{n+2}\xi^n,$$
$$g'''(\xi) = \sum_{n=0}^{\infty}(n+1)(n+2)(n+3)b_{n+3}\xi^n.$$

Substituting Eq. (32) and Eq. (33) into Eq. (31), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{\Gamma(2-\alpha+\frac{n\alpha}{3})}{\Gamma(2+\frac{n\alpha}{3})}a_n\xi^n \\ &-\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)a_{n+3}\xi^n \\ &-\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)a_{n+1-k}a_ib_{k-i}\xi^n, \\ &\sum_{n=0}^{\infty}\frac{\Gamma(2-\frac{5\alpha}{3}+\frac{n\alpha}{3})}{\Gamma(2-\frac{2\alpha}{3})}b_n\xi^n-\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)b_{n+3}\xi^n \\ &-\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)b_{n+1-k}a_ib_{k-i}\xi^n,\end{aligned} \tag{34}$$

Comparing coefficients in Eq. (34), when $n=0$, we have

$$a_3 = \frac{1}{6}\left[\frac{\Gamma(2-\alpha)}{\Gamma(2)}a_0 - a_1a_0^2\right], \tag{35}$$
$$b_3 = \frac{1}{6}\left[\frac{\Gamma(2-\frac{5\alpha}{3})}{\Gamma(2-\frac{2\alpha}{3})}b_0 - b_1a_0b_0\right]$$

when $n \geq 1$, we get

$$a_{n+3} = \frac{1}{(n+1)(n+2)(n+3)}\left[\frac{\Gamma(2-\alpha+\frac{n\alpha}{3})}{\Gamma(2+\frac{n\alpha}{3})}a_n - \sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)a_{n+1-k}a_ib_{k-i}\right], \tag{36}$$

$$b_{n+3} = \frac{1}{(n+1)(n+2)(n+3)}\left[\frac{\Gamma(2-\frac{5\alpha}{3}+\frac{n\alpha}{3})}{\Gamma(2-\frac{2\alpha}{3}+\frac{n\alpha}{3})}b_n - \sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)b_{n+1-k}a_ib_{k-i}\right], \tag{37}$$

substituting (35), (36) and (37) into (32), we can get the solution in the form of power series for Eq. (32),

$$\begin{aligned}f(\xi) &= a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + \sum_{n=1}^{\infty}a_{n+3}\xi^{n+3} \\ &= a_0 + a_1\xi + a_2\xi^2 + \frac{1}{6}\left[\frac{\Gamma(2-\alpha)}{\Gamma(2)}a_0 - a_1a_0^2\right]\xi^3 \\ &+ \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)}\left[\frac{\Gamma(2-\alpha+\frac{n\alpha}{3})}{\Gamma(2+\frac{n\alpha}{3})}a_n - \sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)a_{n+1-k}a_ib_{k-i}\right]\xi^{n+3},\end{aligned} \tag{38}$$

$$\begin{aligned}g(\xi) &= b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3 + \sum_{n=1}^{\infty}b_{n+3}\xi^{n+3} \\ &= a_0 + a_1\xi + a_2\xi^2 + \frac{1}{6}\left[\frac{\Gamma(2-\frac{5\alpha}{3})}{\Gamma(2-\frac{2\alpha}{3})}b_0 - b_1a_0b_0\right]\xi^3 \\ &+ \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)}\left[\frac{\Gamma(2-\frac{5\alpha}{3}+\frac{n\alpha}{3})}{\Gamma(2-\frac{2\alpha}{3}+\frac{n\alpha}{3})}b_n - \sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)b_{n+1-k}a_ib_{k-i}\right]\xi^{n+3},\end{aligned} \tag{39}$$

where $a_1, a_2, a_3, c_1, c_2, c_3$ is constants.

Finally, the exact explicit solution for Eq. (1) is acquired as below

$$\begin{aligned}u(x,t) &= a_0 + a_1xt^{-\frac{\alpha}{3}} + a_2x^2t^{-\frac{2\alpha}{3}} + a_3x^3t^{-\alpha} + \sum_{n=1}^{\infty}a_{n+3}x^{n+3}t^{-\frac{-\alpha(n+3)}{3}} \\ &= a_0 + a_1xt^{-\frac{\alpha}{3}} + a_2x^2t^{-\frac{2\alpha}{3}} + \frac{1}{6}\left[\frac{\Gamma(2-\alpha)}{\Gamma(2)}a_0 - a_1a_0^2\right]x^3t^{-\alpha} \\ &+ \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)}\left[\frac{\Gamma(2-\alpha+\frac{n\alpha}{3})}{\Gamma(2+\frac{n\alpha}{3})}a_n - \sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)a_{n+1-k}a_ib_{k-i}\right]x^{n+3}t^{-\frac{-\alpha(n+3)}{3}}.\end{aligned} \tag{40}$$

$$\begin{aligned}v(x,t) &= b_0t^{-\frac{2\alpha}{3}} + b_1xt^{-\alpha} + b_2x^2t^{-\frac{4\alpha}{3}} + b_3x^3t^{-\frac{5\alpha}{3}} + \sum_{n=1}^{\infty}b_{n+3}x^{n+3}t^{-\frac{-\alpha(n+5)}{3}} \\ &= b_0t^{-\frac{2\alpha}{3}} + b_1xt^{-\alpha} + b_2x^2t^{-\frac{4\alpha}{3}} + \frac{1}{6}\left[\frac{\Gamma(2-\frac{5\alpha}{3})}{\Gamma(2-\frac{2\alpha}{3})}b_0 - b_1a_0b_0\right]x^3t^{-\frac{5\alpha}{3}} \\ &+ \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)}\left[\frac{\Gamma(2-\frac{5\alpha}{3}+\frac{n\alpha}{3})}{\Gamma(2-\frac{2\alpha}{3}+\frac{n\alpha}{3})}b_n - \sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)b_{n+1-k}a_ib_{k-i}\right]x^{n+3}t^{-\frac{-\alpha(n+5)}{3}}.\end{aligned} \tag{41}$$

6 Analysis of the convergence

Here, the convergence of the PS solution equation (41) for eq. (1) will be investigated. Consider eq. (36) and (37), such that

$$|a_{n+3}| \leq \left(\left| \frac{\Gamma(2-\alpha+\frac{n\alpha}{3})}{\Gamma(2+\frac{n\alpha}{3})} \right| |a_n| - \sum_{k=0}^{n}\sum_{i=0}^{k} |(n+1-k)||a_{n+1-k}||a_i||b_{k-i}| \right), \tag{42}$$
$$|b_{n+3}| \leq \left(\left| \frac{\Gamma(2-\frac{5\alpha}{3}+\frac{n\alpha}{3})}{\Gamma(2-\frac{2\alpha}{3}+\frac{n\alpha}{3})} \right| |b_n| - \sum_{k=0}^{n}\sum_{i=0}^{k} |(n+1-k)||b_{n+1-k}||a_i||b_{k-i}| \right).$$

It is known that $\frac{|\Gamma(n)|}{|\Gamma(m)|} < 1$, for arbitrary n and m. Thus eq. (42) becomes

$$|a_{n+3}| \leq M\left(|a_n| - \sum_{k=0}^{n}\sum_{i=0}^{k} |(n+1-k)||a_{n+1-k}||a_i||b_{k-i}| \right), \tag{43}$$
$$|b_{n+3}| \leq N\left(|b_n| - \sum_{k=0}^{n}\sum_{i=0}^{k} |(n+1-k)||b_{n+1-k}||a_i||b_{k-i}| \right).$$

where $M, N = \max\{e_1, e_2, e_3\}$, where e_1, e_2, e_3 are arbitrary constants. Take into consideration another PS given as

$$C(\xi) = \sum_{n=0}^{\infty} c_n \xi^n, \tag{44}$$
$$B(\xi) = \sum_{n=0}^{\infty} d_n \xi^n$$

and let $c_i = |a_i|$, $d_i = |b_i|$, $i = 0, 1, 2, \ldots$ Then, we can have

$$|c_{n+3}| = M\left(c_n - \sum_{k=0}^{n}\sum_{i=0}^{k} (n+1-k)c_{n+1-k}c_i d_{k-i} \right), \tag{45}$$
$$|d_{n+3}| = N\left(d_n - \sum_{k=0}^{n}\sum_{i=0}^{k} (n+1-k)d_{n+1-k}c_i d_{k-i} \right).$$

Therefore, it is easily seen that $|a_n| \leq c_n$, $|b_n| \leq d_n$, $n = 0, 1, \ldots$. Furthermore, the series $C(\xi) = \sum_{n=0}^{\infty} c_n\xi^n$ and $B(\xi) = \sum_{n=0}^{\infty} d_n\xi^n$ are majorant series of eq. (32). We therefore confirm that the series $C(\xi)$ and $B(\xi)$ have a positive radius of convergence. By some calculations, we have

$$C(\xi) = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + M\left(\sum_{n=0}^{\infty} c_n - \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k} (n+1-k)c_{n+1-k}c_i d_{k-i} \right)\xi^{n+3}, \tag{46}$$
$$B(\xi) = d_0 + d_1\xi + d_2\xi^2 + d_3\xi^3 + M\left(\sum_{n=0}^{\infty} d_n - \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k} (n+1-k)d_{n+1-k}c_i d_{k-i} \right)\xi^{n+3}.$$

Let us take into consideration an implicit functional system with regard to ξ as follows:

$$F(\xi, C) = C - c_0 - c_1\xi - c_2\xi^2 - c_3\xi^3 - M\xi^3\left(C - C'CB\right), \tag{47}$$
$$G(\xi, B) = B - d_0 - d_1\xi - d_2\xi^2 - d_3\xi^3 - N\xi^3\left(B - B'BC\right).$$

since F and G are analytic in a neighborhood of $(0, c_0)$ and $(0, d_0)$, where $F(0, d_0) = 0$, $G(0, c_0) = 0$ and $\frac{\partial}{\partial C}F(0, C_0) \neq 0$, $\frac{\partial}{\partial B}G(0, d_0) \neq 0$. Then, by the implicit function theorem [40], the convergence is given.

7 Conservation laws

In this part, we solve the adjoint equation and Cls by using the related formula, most of the specific knowledge about Cls has been presented in [37–39]. The form of the Lagrangian is as blow

$$L = p(x,t)(D_t^\alpha u - u_{xxx} - uvu_x) + q(D_t^\alpha v - v_{xxx} - uvv_x). \tag{48}$$

In the above equation $p(x,t)$, $q(x,t)$ are another dependent variable. The Euler-Lagrange operator [39] are

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^\alpha)^* \frac{\partial}{\partial D_t^\alpha u} - D_x\frac{\partial}{\partial u_x} - D_{xxx}\frac{\partial}{\partial u_{xxx}}, \tag{49}$$
$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} + (D_t^\alpha)^* \frac{\partial}{\partial D_t^\alpha v} - D_x\frac{\partial}{\partial v_x} - D_{xxx}\frac{\partial}{\partial v_{xxx}},$$

where $(D_t^\alpha)^*$ is the adjoint operator of (D_t^α), the adjoint equations are given by

$$F_1^\star = \frac{\delta L}{\delta u} = 0, \tag{50}$$
$$F_2^\star = \frac{\delta L}{\delta v} = 0,$$

combining the above equations, we get:

$$F_1^\star = \frac{\delta L}{\delta u} = p(D_t^\alpha)^* - qvv_x + p_x uv + puv_x + p_{xxx} = 0. \tag{51}$$
$$F_2^\star = \frac{\delta L}{\delta v} = q(D_t^\alpha)^* - puu_x + qu_x v + q_x uv + q_{xxx} = 0$$

the adjoint equations of Eq. (1) can be write as below:

$$p(D_t^\alpha)^* - qvv_x + p_x uv + puv_x + p_{xxx}|_{q=\phi^1(x,t,u,v)} \tag{52}$$

$$= \lambda_1(D_t^\alpha u - u_{xxx} - uvu_x).$$
$$q(D_t^\alpha)^* - puu_x + qu_xv + q_xuv + q_{xxx}|_{p=\phi^2(x,t,u,v)}$$
$$= \lambda_2(D_t^\alpha v - v_{xxx} - uvv_x)$$

where

$$\begin{aligned}
q_x &= \phi_x^1 + \phi_u^1 u_x + \phi_v^1 v_x \\
p_x &= \phi_x^2 + \phi_u^2 u_x + \phi_v^2 v_x \\
q_{xxx} &= \phi_{xxx}^1 + 3\phi_{uuv}^1 u_x^2 v_x + 3\phi_{uvv}^1 u_x v_x^2 + 3\phi_{uu}^1 u_x u_{xx} \\
&+ 3\phi_{uv}^1 u_x v_{xx} + 3\phi_{uv}^1 u_{xx} v_x + 3\phi_{vv}^1 v_x v_{xx} + 6\phi_{xuv}^1 u_x v_x \\
&+ 3\phi_{xxu}^1 u_x + 3\phi_{xxv}^1 v_x + 3\phi_{xu}^1 u_{xx} + 3\phi_{xv}^1 v_{xx} + \phi_u^1 u_{xxx} \\
&+ \phi_v^1 v_{xxx} + 3\phi_{xuu}^1 u_x^2 + 3\phi_{xvv}^1 v_x^2 + \phi_{uuu}^1 u_x^3 + \phi_{vvv}^1 v_x^3 \\
q_{xxx} &= \phi_{xxx}^2 + 3\phi_{uuv}^2 u_x^2 v_x + 3\phi_{uvv}^2 u_x v_x^2 + 3\phi_{uu}^2 u_x u_{xx} \\
&+ 3\phi_{uv}^2 u_x v_{xx} + 3\phi_{uv}^2 u_{xx} v_x + 3\phi_{vv}^2 v_x v_{xx} + 6\phi_{xuv}^2 u_x v_x \\
&+ 3\phi_{xxu}^2 u_x + 3\phi_{xxv}^2 v_x + 3\phi_{xu}^2 u_{xx} + 3\phi_{xv}^2 v_{xx} + \phi_u^2 u_{xxx} \\
&+ \phi_v^2 v_{xxx} + 3\phi_{xuu}^2 u_x^2 + 3\phi_{xvv}^2 v_x^2 + \phi_{uuu}^2 u_x^3 + \phi_{vvv}^2 v_x^3
\end{aligned}$$

Next, we consider x, t and $u(x,t)$, $v(x,t)$, we get

$$\overline{X} + D_t(\xi^2)l + D_x(\xi^1)l = W_1\frac{\delta}{\delta u} + W_2\frac{\delta}{\delta v} + D_tN^t + D_xN^x. \quad (53)$$

In Eq. (53) l is the identity operator, $\frac{\delta}{\delta u}$ is the Euler-Lagrangian operator, N^x and N^t are the Noether operators, $\overline{X}$ is as blow

$$\begin{aligned}
\overline{X} &= \xi^2\frac{\partial}{\partial t} + \xi^1\frac{\partial}{\partial x} + \eta^1\frac{\partial}{\partial u} + \eta^2\frac{\partial}{\partial v} + \eta^{1\alpha,t}\frac{\partial}{\partial D_t^\alpha u} \quad (54) \\
&+ \eta^{2\alpha,t}\frac{\partial}{\partial D_t^\alpha v} + \eta^{1x}\frac{\partial}{\partial u_x} + \eta^{2x}\frac{\partial}{\partial v_x} + \eta^{1xxx}\frac{\partial}{\partial u_{xxx}} \\
&+ \eta^{2xxx}\frac{\partial}{\partial v_{xxx}},
\end{aligned}$$

and the W_1, W_2 are defined by

$$\begin{aligned}
W_1 &= \eta^1 - \xi^2 u_t - \xi^1 u_x, \quad (55) \\
W_2 &= \eta^2 - \xi^2 v_t - \xi^1 v_x.
\end{aligned}$$

To the generator V_4, the corresponding Lie characteristic function can be represented as

$$\begin{aligned}
W_1 &= -\frac{3t}{\alpha}u_t - xu_x, \quad (56) \\
W_2 &= -2v - -\frac{3t}{\alpha}v_t - xv_x.
\end{aligned}$$

The operator N^t is defined by [37]

$$\begin{aligned}
N^t &= \xi^2 + \sum_{k=0}^{n-1}(-1)^k {}_oD^{\alpha-1-k}(W_1)D_t^k\frac{\partial}{\partial\, {}_oD_t^\alpha u} \quad (57) \\
&- (-1)^n J\left(W_1, D_t^n\frac{\partial}{\partial\, {}_oD_t^\alpha u}\right) \\
&+ \sum_{k=0}^{n-1}(-1)^k {}_oD^{\alpha-1-k}(W_2)D_t^k\frac{\partial}{\partial\, {}_oD_t^\alpha v} \\
&- (-1)^n J\left(W_2, D_t^n\frac{\partial}{\partial\, {}_oD_t^\alpha v}\right)
\end{aligned}$$

with J given by

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T\frac{f(\xi^2,x)g(\mu,x)}{(\mu-\xi^2)^{\alpha+1-n}}d\mu dt. \quad (58)$$

For Eq. (1), the operator N^x is defined by

$$\begin{aligned}
N^x &= \xi^1 + W_1\frac{\delta}{\delta u_x} + D_x(W_1)\left[\frac{\delta}{\delta u_{xx}}\right] \quad (59) \\
&+ D_{xx}(W_1)\left[\frac{\delta}{\delta u_{xxx}}\right] + W_2\frac{\delta}{\delta v_x} \\
&+ D_x(W_2)\left[\frac{\delta}{\delta v_{xx}}\right] + D_{xx}(W_2)\left[\frac{\delta}{\delta v_{xxx}}\right] \\
&= \xi^1 + W_1\left[\frac{\partial}{\partial u_x} + D_{xx}\frac{\partial}{\partial u_{xxx}}\right] \\
&+ D_x(w_1)\left[\frac{\partial}{\partial u_{xx}} - D_x\frac{\partial}{\partial u_{xxx}}\right] + D_{xx}(W_1)\left[\frac{\partial}{\partial u_{xxx}}\right] \\
&+ W_2\left[\frac{\partial}{\partial v_x} + D_{xx}\frac{\partial}{\partial v_{xxx}}\right] + D_x(w_2)\left[\frac{\partial}{\partial v_{xx}} - D_x\frac{\partial}{\partial v_{xxx}}\right] \\
&+ D_{xx}(W_2)\left[\frac{\partial}{\partial v_{xxx}}\right].
\end{aligned}$$

Substituting (1) into (53), we get:

$$(\overline{X}L + D_t(\xi^2)L + D_x(\xi^1)L)\,|_{Eq.(1)} = 0, \quad (60)$$

hence, the form of the Cls for Eq. (1) can be written as

$$D_t(N^tL) + D_x(N^xL) = 0. \quad (61)$$

Next, according to the basic definitions present above, we acquire the Cls for Eq. (1), and divide into the following cases to discuss:

Case 1. For $\alpha \in (0,1)$, the components of the conserved vector are

$$\begin{aligned}
C_i^t &= N^tL = \xi^2L + (-1)^0\, {}_oD^{\alpha-1}(W_1)D_t^0\frac{\partial L}{\partial\, {}_oD_t^\alpha u} - (-1)^1 \\
&\times J\left(W_1, D_t^1\frac{\partial L}{\partial\, {}_oD_t^\alpha u}\right) + (-1)^0\, {}_oD^{\alpha-1}(W_2)D_t^0\frac{\partial L}{\partial\, {}_oD_t^\alpha v} \\
&- (-1)^1 \times J\left(W_2, D_t^1\frac{\partial L}{\partial\, {}_oD_t^\alpha v}\right) = {}_oD^{\alpha-1}(W_1)p \\
&+ J(W_1, p_t) + {}_oD^{\alpha-1}(W_2)q + J(W_2, q_t),
\end{aligned}$$

$$\begin{aligned}
C_i^x &= N^xL = \xi^1L + W_1\left[\frac{\partial L}{\partial u_x} + D_{xx}\frac{\partial L}{\partial u_{xxx}}\right] \\
&+ D_x(w_1)\left[\frac{\partial L}{\partial u_{xx}} - D_x\frac{\partial L}{\partial u_{xxx}}\right] + D_{xx}(W_1)\left[\frac{\partial L}{\partial u_{xxx}}\right] \\
&+ W_2\left[\frac{\partial L}{\partial v_x} + D_{xx}\frac{\partial L}{\partial v_{xxx}}\right] + D_x(w_2)\left[\frac{\partial L}{\partial v_{xx}} - D_x\frac{\partial L}{\partial v_{xxx}}\right]
\end{aligned}$$

$$+ D_{xx}(W_2)\left[\frac{\partial L}{\partial v_{xxx}}\right] = -W_1(quv + q_{xx}) + q_x D_x(W_1)$$
$$- qD_{xx}(W_1) - W_2(puv + p_{xx}) + p_x D_x(W_2) - pD_{xx}(W_2),$$

where the functions W_1, W_2 are given by

$$W_1 = -\frac{3t}{\alpha}u_t - xu_x,$$
$$W_2 = -2v - -\frac{3t}{\alpha}v_t - xv_x.$$

Case 2. When $\alpha \in (1, 2)$, the components of the conserved vector are

$$C_i^t = N^t L = \xi^2 L + (-1)^0\, {}_oD^{\alpha-1}(W_1)D_t^0 \frac{\partial L}{\partial\, {}_oD_t^\alpha u}$$
$$+ (-1)^1\, {}_oD^{\alpha-2}(W_1)D_t^1 \frac{\partial L}{\partial\, {}_oD_t^\alpha u} - (-1)^1$$
$$\times J\left(W_1, D_t^1 \frac{\partial L}{\partial\, {}_oD_t^\alpha u}\right) - (-1)^2 \times J\left(W_1, D_t^2 \frac{\partial L}{\partial\, {}_oD_t^\alpha u}\right)$$
$$+ (-1)^0\, {}_oD^{\alpha-1}(W_2)D_t^0 \frac{\partial L}{\partial\, {}_oD_t^\alpha v}$$
$$+ (-1)^1\, {}_oD^{\alpha-2}(W_2)D_t^1 \frac{\partial L}{\partial\, {}_oD_t^\alpha v} - (-1)^1$$
$$\times J\left(W_2, D_t^1 \frac{\partial L}{\partial\, {}_oD_t^\alpha v}\right) - (-1)^2 \times J\left(W_2, D_t^2 \frac{\partial L}{\partial\, {}_oD_t^\alpha v}\right)$$
$$= {}_oD^{\alpha-1}(W_1)p - {}_oD^{\alpha-2}(W_1)p_t + J(W_1, p_t)$$
$$- J(W_1, p_{tt}) + {}_oD^{\alpha-1}(W_2)q - {}_oD^{\alpha-2}(W_2)q_t + J(W_2, q_t)$$
$$- J(W_2, q_{tt}),$$

$$C_i^x = N^x L = \xi^1 L + W_1\left[\frac{\partial L}{\partial u_x} + D_{xx}\frac{\partial L}{\partial u_{xxx}}\right]$$
$$+ D_x(w_1)\left[\frac{\partial L}{\partial u_{xx}} - D_x\frac{\partial L}{\partial u_{xxx}}\right] + D_{xx}(W_1)\left[\frac{\partial L}{\partial u_{xxx}}\right]$$
$$+ W_2\left[\frac{\partial L}{\partial v_x} + D_{xx}\frac{\partial L}{\partial v_{xxx}}\right] + D_x(w_2)\left[\frac{\partial L}{\partial v_{xx}} - D_x\frac{\partial L}{\partial v_{xxx}}\right]$$
$$+ D_{xx}(W_2)\left[\frac{\partial L}{\partial v_{xxx}}\right] = -W_1(quv + q_{xx}) + q_x D_x(W_1)$$
$$- qD_{xx}(W_1) - W_2(puv + p_{xx}) + p_x D_x(W_2) - pD_{xx}(W_2),$$

where the functions W_1, W_2 are given by

$$W_1 = -\frac{3t}{\alpha}u_t - xu_x,$$
$$W_2 = -2v - -\frac{3t}{\alpha}v_t - xv_x.$$

8 Concluding remarks

In the paper, we studied the time fractional equations which were the extension of the mkdv equations. Firstly, the Lie symmetries and optimal systems of the equations are completely presented, and the equations were reduced to the nonlinear ordinary differential equations of fractional order. Then we get explicit solutions of the equations by applying the power series approach. And we study the convergence of the exact solutions for the equations. Finally, we use the symmetries and adjoint equations to construct the conservation laws of the governing equations.

Acknowledgement: Project supported by the Natural Science Foundation of Jiangsu Province under Grant No. BK20170171

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