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Research Article

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Plane Wave Reflection in a Compressible Half Space with Initial Stress

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Abstract: In this paper, the problem of wave propagation in a compressible half-space with an initial stress is considered. General discussion on the speed of wave in the presence of an initial stress is presented. Furthermore, reflection of a homogeneous plane P–wave is also studied. A special strain energy function dependent on this initial stress is used to understand the response of the materials. Explicit formulas for the reflection coefficients are also presented. General nonlinear theory and the theory of invariants are used to derive theoretical results. Graphical illustration of theoretical results for various numerical values of parameters show that initial stress has considerable bearing on the behavior of a plane wave.

Keywords: Compressible elastic material; Initial stress; Plane waves; Nonlinear elasticity

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1 Introduction

In linear theory of elasticity, the materials are mostly considered to be stress-free in their reference state [1, 2]. In real world, however, the existence of an initial stress is proven and considered important for the study of wave propagation in elastic solids. In this context, many studies are presented for incompressible materials. In this paper, a compressible material is considered with a homogeneous initial stress. The effect of this stress on the speed of plane waves is analyzed using the nonlinear theory of elasticity. In addition, the problem of plane wave reflection from the boundary of such a material is also presented with the help of reflection coefficients. Mainly P–waves are considered and various cases are outlined for existence of one or two reflected waves. This depends on the initial stress and the incidence angle. The problem is majorly applicable but not limited to earthquake waves which are used in seismology. Other applications include the study of waves in biological tissues, toughened glass etc.

The term initial stress is used here in its most general sense which includes both the cases of prestress and residual stress. Here, the source of this initial stress is irrelevant. A prestress is a kind of initial stress which has a related finite deformation from the reference configuration whereas a residual stress may not occur due to a finite deformation but may be a consequence of some manufacturing or growth process [3].

In [4], the authors studied the effect of a (homogeneous) pre-stress and finite deformation on the speed of plane waves in compressible hyperelastic materials and the reflection of plane wave from such a half-space. Few results in this paper appear similar to those cited in [4]. However, the nature of the material constants here is considerably different since these depend on the initial stress as well. Biot [5, 6] presented major studies on various problems to see the effects of initial stress on wave propagation. Also, wave motion in an infinite and initially stressed material medium for various special cases were considered by Tang [7]. For the basic equations for a residually-stressed material, the reasder is referred to [8–11]. Man and Lu [12] followed Hoger [3] and presented generalized results which can be related to Biot's work. For discussion on wave propagation in pre-stressed materials, we refer to [13–16] and references therein. For general non-linear elasticity theory, see [17, 18]. More recently, discussion on initial stress can be found in [19, 20] and references therein.

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Section 2 comprises of basic equations describing the finite deformation of a compressible elastic material and the corresponding equations which govern incremental motion superimposed on the finite deformation. In Section 3 the effect of initial stress on infinitesimal wave propagation through the acoustic tensor is discussed. The special case of isotropy and few examples of initial stress in compressible material are also given. In Section 4, the reflection of a plane wave from the boundary of the half-space is discussed and the cases for which either one or two waves are reflected are observed. Reflection coefficients are calculated in Section 5 for the specific strain energy function and the results are presented graphically to examine the behavior of reflected plane waves for an incident P–wave.

2 Basic equations, incremental deformations and invariant-based formulation

Let $\mathcal{B}_r$ represents the reference configuration of an elastic body and $\mathcal{R} = [X_1, X_2, X_3]$ represents a material point in $\mathcal{B}_r$. It is assumed that the material has an initial stress $\boldsymbol{\Theta}$ from this configuration. The material is isotropic in the absence of this initial stress. Further, it is assumed that all subsequent deformations are measured from this initially-stressed reference configuration. This initial stress is symmetric and satisfies $\operatorname{Div}\boldsymbol{\Theta} = \mathbf{0}$ due to equilibrium, in the absence of body forces. Here Div is the divergence operator used with respect to $\mathcal{B}_r$. Also, it is immaterial as to how this initial stress is generated and the term is used in its most generalized sense.

As the elastic body undergoes a finite deformation, the position vector $\mathcal{R}$ becomes $\mathcal{R} = \boldsymbol{\Psi}(\mathcal{R})$, where $\boldsymbol{\Psi}$ denotes the deformation (which is a bijection) for $\mathcal{R}$ in $\mathcal{B}_r$. Let the associated deformation gradient tensor be denoted by $\mathbf{K}$ with $\mathbf{K} = \operatorname{Grad}\Psi$, where Grad is the gradient operator applied with respect to $\mathcal{B}_r$.

Let $\mathcal{B}_t$ denotes the deformed configuration, $\mathcal{T}$ the Cauchy stress tensor and $\mathcal{S}$ the nominal stress tensor. The tensors $\mathcal{T}$ and $\mathcal{S}$ satisfy their respective equilibrium equations and the body forces are assumed as nil. The two stresses are connected through $\mathcal{T} = (\det\mathbf{K})^{-1}\mathbf{KS}$.

Let the strain energy function $\mathcal{F}$ specifies the elastic response of the elastic material with

$$\mathcal{F} = \mathcal{F}(\mathcal{C}, \boldsymbol{\Theta}), \tag{1}$$

where we make explicit dependence of $\mathcal{F}$ on $\boldsymbol{\Theta}$ and the right Cauchy-Green deformation tensor $\mathcal{C} = \mathbf{K}^{\mathrm{T}}\mathbf{K}$. It should be noted that the presence of $\boldsymbol{\Theta}$ may introduce anisotropic behavior in the material whereas the material may be initially isotropic [12]. Thus, $\boldsymbol{\Theta}$ affects the constitutive equations in a manner similar to that of the structure tensor in anisotropic elasticity.

The connections of the nominal and Cauchy stress with $\mathcal{F}$ are given by

$$\mathcal{S} = \frac{\partial\mathcal{F}}{\partial\mathbf{K}}(\mathbf{K}, \boldsymbol{\Theta}), \quad \mathcal{T} = J^{-1}\mathbf{KS} = J^{-1}\mathbf{K}\frac{\partial\mathcal{F}}{\partial\mathbf{K}}(\mathbf{K}, \boldsymbol{\Theta}), \tag{2}$$

respectively, where $J = \det\mathbf{K} > 0$. When evaluated in $\mathcal{B}_r$, these give the connection

$$\boldsymbol{\Theta} = \frac{\partial\mathcal{F}}{\partial\mathbf{K}}(\mathbf{I}, \boldsymbol{\Theta}). \tag{3}$$

Here $\mathbf{I}$ is the identity tensor.

After the finite deformation, we consider an incremental motion in the material which results in the incremental displacement, say, $\mathbf{u} = \mathbf{u}(\mathcal{R}, t)$. In a compressible material which is also initially-stressed, the equation of motion is [19]

$$\mathcal{B}_{0piqj}D_{j,pq} = \rho D_{i,tt}, \tag{4}$$

where ρ is the density of the material in $\mathcal{B}_t$. Here, $\mathcal{B}_{0piqj}$ are the components of the fourth order updated incremental elasticity tensor and is related to the elasticity tensor $\boldsymbol{\mathcal{B}}$ by [18]

$$\mathcal{B}_{0ijkl} = J^{-1}K_{i\alpha}K_{k\beta}\mathcal{B}_{\alpha j\beta l}, \tag{5}$$

where the elasticity tensor $\boldsymbol{\mathcal{B}}$ is defined as

$$\boldsymbol{\mathcal{B}} = \frac{\partial^2\mathcal{F}}{\partial\mathbf{K}\partial\mathbf{K}}, \qquad \mathcal{B}_{\alpha i\beta j} = \frac{\partial^2\mathcal{F}}{\partial K_{i\alpha}\partial K_{j\beta}}, \tag{6}$$

in its vector and component forms, respectively.

For the considered material, since the reference configuration here is assumed to be initially-stressed, the material response depends on the invariants of $\mathcal{C}$ and $\boldsymbol{\Theta}$. We may take a possible set of independent invariants given by

$$\begin{aligned} &I_1 = \operatorname{trace}(\mathcal{C}), \quad I_2 = \frac{1}{2}[I_1^2 - \operatorname{trace}(\mathcal{C}^2)], \qquad (7)\\ &I_3 = \det(\mathcal{C}), \quad I_4 = \operatorname{trace}(\boldsymbol{\Theta}),\\ &I_5 = \frac{1}{2}[I_4^2 - \operatorname{trace}(\boldsymbol{\Theta}^2)], \quad I_6 = \det(\boldsymbol{\Theta}), \quad I_7 = \operatorname{trace}(\mathcal{C}\boldsymbol{\Theta}),\\ &I_8 = \operatorname{trace}(\mathcal{C}^2\boldsymbol{\Theta}), \quad I_9 = \operatorname{trace}(\mathcal{C}\boldsymbol{\Theta}^2), \quad I_{10} = \operatorname{trace}(\mathcal{C}^2\boldsymbol{\Theta}^2). \end{aligned}$$

This set provides the (at most) 10 independent invariants of $\mathcal{C}$ and $\boldsymbol{\Theta}$. For relevant background on invariants of tensors we refer to [21, 22]. After few manipulations, it may be shown that invariants such as $\operatorname{trace}(\mathcal{C}\boldsymbol{\Theta}\mathcal{C}\boldsymbol{\Theta})$ and $\operatorname{trace}(\mathcal{C}\boldsymbol{\Theta}\mathcal{C}^2\boldsymbol{\Theta})$, etc. may be expressed in terms of $I_1, I_2, \ldots, I_{10}$. To prove this, Cayley-Hamilton theorem is applied to $\mathcal{C} + \alpha\boldsymbol{\Theta}$, where α is an arbitrary scalar. This results in

$$(\mathcal{C} + \alpha\boldsymbol{\Theta})^3 - \mathrm{trace}(\mathcal{C} + \alpha\boldsymbol{\Theta})(\mathcal{C} + \alpha\boldsymbol{\Theta})^2 + \frac{1}{2}[\{\mathrm{trace}(\mathcal{C} + \alpha\boldsymbol{\Theta})\}^2] - \mathrm{trace}\{(\mathcal{C} + \alpha\boldsymbol{\Theta})^2\}](\mathcal{C} + \alpha\boldsymbol{\Theta}) - \det(\mathcal{C} + \alpha\boldsymbol{\Theta})\mathbf{I} = \mathbf{0}. \tag{8}$$

The above equation is cubic in α and a comparison of coefficients of various powers of α shows that the coefficient of α^0 satisfies Cayley-Hamilton theorem for $\mathcal{C}$ whereas the coefficient of α^3 satisfies Cayley-Hamilton theorem for $\boldsymbol{\Theta}$. Comparing the coefficients of α gives

$$\mathcal{C}^2\boldsymbol{\Theta} + \mathcal{C}\boldsymbol{\Theta}\mathcal{C} + \boldsymbol{\Theta}\mathcal{C}^2 - \mathrm{trace}(\mathcal{C})(\mathcal{C}\boldsymbol{\Theta} + \boldsymbol{\Theta}\mathcal{C}) - \mathrm{trace}(\boldsymbol{\Theta})\mathcal{C}^2 - \left[\mathrm{trace}(\mathcal{C})\mathrm{trace}(\boldsymbol{\Theta}) - \mathrm{trace}(\mathcal{C}\boldsymbol{\Theta})\right]\mathcal{C} + \frac{1}{2}\left[(\mathrm{trace}(\mathcal{C})^2 - \mathrm{trace}(\mathcal{C}^2)\right]\boldsymbol{\Theta} - (\det\mathcal{C})\mathrm{trace}(\mathcal{C}^{-1}\boldsymbol{\Theta})\mathbf{I} = \mathbf{0}. \tag{9}$$

Multiplication of the above equation by $\boldsymbol{\Theta}$ and then taking trace gives

$$\mathrm{trace}(\mathcal{C}\boldsymbol{\Theta}\mathcal{C}\boldsymbol{\Theta}) = 2I_2I_5 + 2I_4I_8 + 2I_1I_9 + I_7^2 - 2I_{10} - 2I_1I_4I_7, \tag{10}$$

which shows the dependence of trace($\mathcal{C}\boldsymbol{\Theta}\mathcal{C}\boldsymbol{\Theta}$) on the invariants in Eq. (7). In a similar manner, multiplication of Eq. (9) by $\mathcal{C}\boldsymbol{\Theta}$ and a few mathematical steps give trace($\mathcal{C}\boldsymbol{\Theta}\mathcal{C}^2\boldsymbol{\Theta}$) in terms of the other invariants.

Evaluating the expressions in Eqs. (7) in the reference configuration, we obtain

$$I_1 = I_2 = 3, \quad I_3 = 1, \quad I_4 = I_7 = I_8 = \mathrm{trace}(\boldsymbol{\Theta}), \quad I_5 = \frac{1}{2}[I_4^2 - \mathrm{trace}(\boldsymbol{\Theta}^2)], \quad I_6 = \det(\boldsymbol{\Theta}), \quad I_9 = I_{10} = I_4^2 - 2I_5, \tag{11}$$

where $\mathbf{K} = \mathcal{C} = \mathbf{I}$.

The function $\mathcal{F}$ depends on the invariants mentioned above. Therefore

$$\frac{\partial\mathcal{F}}{\partial\mathbf{K}} = \sum_{m=1}^{10}\mathcal{F}_m\frac{\partial I_m}{\partial\mathbf{K}}, \tag{12}$$

where $\mathcal{F}_r = \partial\mathcal{F}/\partial I_r$. The component form of the updated elasticity tensor using Eq. (5), is

$$\mathcal{B}_{0piqj} = J^{-1}\left(\sum_{m=1}^{10}\mathcal{F}_m K_{p\alpha}K_{q\beta}\frac{\partial^2 I_m}{\partial K_{i\alpha}\partial K_{j\beta}} + \sum_{m,n=1}^{10}\mathcal{F}_{mn}K_{p\alpha}K_{q\beta}\frac{\partial I_m}{\partial K_{i\alpha}}\frac{\partial I_n}{\partial K_{j\beta}}\right). \tag{13}$$

In this case, Cauchy stress tensor, using Eq. (2) is

$$J\mathcal{T} = \mathbf{KS} = 2\mathcal{F}_1\mathcal{B} + 2\mathcal{F}_2\mathcal{B}^\star + 2\mathcal{F}_3I_3\mathbf{I} + 2\mathcal{F}_7\boldsymbol{\upsilon} + 2\mathcal{F}_8(\boldsymbol{\upsilon}\mathcal{B} + \mathcal{B}\boldsymbol{\upsilon}) + 2\mathcal{F}_9\boldsymbol{\upsilon}\mathcal{B}^{-1}\boldsymbol{\upsilon} + 2\mathcal{F}_{10}(\boldsymbol{\upsilon}\mathcal{B}^{-1}\boldsymbol{\upsilon}\mathcal{B} + \mathcal{B}\boldsymbol{\upsilon}\mathcal{B}^{-1}\boldsymbol{\upsilon}), \tag{14}$$

where $\boldsymbol{\upsilon} = \mathbf{K}\boldsymbol{\Theta}\mathbf{K}^T$ and $\mathbf{B}^\star = I_1\mathbf{B} - \mathbf{B}^2$. Here, $\mathbf{B} = \mathbf{KK}^{\mathrm{T}}$ is the usual left Cauchy-Green deformation tensor. In the reference configuration, the above expression gives the expression for $\boldsymbol{\Theta}$ as

$$\boldsymbol{\Theta} = 2(\mathcal{F}_1 + 2\mathcal{F}_2 + \mathcal{F}_3)\mathbf{I} + 2(\mathcal{F}_7 + 2\mathcal{F}_8)\boldsymbol{\Theta} + 2(\mathcal{F}_9 + 2\mathcal{F}_{10})\boldsymbol{\Theta}^2, \tag{15}$$

which implies that

$$\mathcal{F}_1 + 2\mathcal{F}_2 + \mathcal{F}_3 = 0, \quad 2(\mathcal{F}_7 + 2\mathcal{F}_8) = 1, \quad \mathcal{F}_9 + 2\mathcal{F}_{10} = 0 \tag{16}$$

in the reference configuration (for details see, for example, [19, 23]).

For brevity only, $\mathcal{F}$ is assumed to depend on various invariants except for I_5, I_6, I_9 and I_{10}. Therefore, Eq. $(16)_3$ is automatically satisfied. Taking into account these simplifications in Eq. (13), the components of the fourth order elasticity tensor are

$$\begin{aligned} J\mathcal{B}_{0piqj} ={}& 2(\mathcal{F}_1 + I_1\mathcal{F}_2)\mathcal{B}_{pq}\Delta_{ij} + 2\mathcal{F}_2[2\mathcal{B}_{pi}\mathcal{B}_{qj} - \mathcal{B}_{iq}\mathcal{B}_{jp} - \Delta_{ij}\mathcal{B}_{p\gamma}\mathcal{B}_{\gamma q} - \mathcal{B}_{pq}\mathcal{B}_{ij}] + 2\mathcal{F}_3I_3(2\Delta_{ip}\Delta_{jq} \\ &- \Delta_{iq}\Delta_{jp}) + 2\mathcal{F}_7\upsilon_{pq}\Delta_{ij} + 2\mathcal{F}_8[\upsilon_{pq}\mathcal{B}_{ij} + \upsilon_{p\gamma}\mathcal{B}_{\gamma q}\Delta_{ij} + \mathcal{B}_{p\gamma}\upsilon_{\gamma q}\Delta_{ij} + \upsilon_{ij}\mathcal{B}_{pq} + \upsilon_{pj}\mathcal{B}_{iq} + \upsilon_{qi}\mathcal{B}_{jp}] \\ &+ 4\mathcal{F}_{11}\mathcal{B}_{ip}\mathcal{B}_{jq} + 4\mathcal{F}_{22}(I_1\mathcal{B}_{ip} - \mathcal{B}_{i\gamma}\mathcal{B}_{\gamma p})(I_1\mathcal{B}_{jq} - \mathcal{B}_{j\Delta}\mathcal{B}_{\Delta q}) + 4\mathcal{F}_{33}I_3^2\Delta_{ip}\Delta_{jq} + 4\mathcal{F}_{12}[2I_1\mathcal{B}_{ip}\mathcal{B}_{jq} \\ &- \mathcal{B}_{ip}\mathcal{B}_{j\Delta}\mathcal{B}_{\Delta q} - \mathcal{B}_{jq}\mathcal{B}_{i\gamma}\mathcal{B}_{\gamma p}] + 4\mathcal{F}_{13}I_3(\mathcal{B}_{ip}\Delta_{jq} + \mathcal{B}_{jq}\Delta_{ip}) + 4\mathcal{F}_{17}(\mathcal{B}_{ip}\upsilon_{jq} + \mathcal{B}_{jq}\upsilon_{ip}) \\ &+ 4\mathcal{F}_{18}[\mathcal{B}_{ip}(\upsilon_{j\Delta}\mathcal{B}_{\Delta q} + \mathcal{B}_{j\Delta}\upsilon_{\Delta q}) + (\upsilon_{i\gamma}\mathcal{B}_{\gamma p} + \mathcal{B}_{i\gamma}\upsilon_{\gamma p})\mathcal{B}_{jq}] + 4\mathcal{F}_{23}I_3[I_1(\mathcal{B}_{ip}\Delta_{jq} + \mathcal{B}_{jq}\Delta_{ip}) \\ &- \Delta_{ip}\mathcal{B}_{j\Delta}\mathcal{B}_{\Delta q} - \mathcal{B}_{i\gamma}\mathcal{B}_{\gamma p}\Delta_{jq}] + 4\mathcal{F}_{27}[(I_1\mathcal{B}_{ip} - \mathcal{B}_{i\gamma}\mathcal{B}_{\gamma p})\upsilon_{jq} + \upsilon_{ip}(I_1\mathcal{B}_{jq} - \mathcal{B}_{j\Delta}\mathcal{B}_{\Delta q})] \\ &+ 4\mathcal{F}_{28}[(I_1\mathcal{B}_{ip} - \mathcal{B}_{i\gamma}\mathcal{B}_{\gamma p})(\upsilon_{j\Delta}\mathcal{B}_{\Delta q} + \mathcal{B}_{j\Delta}\upsilon_{\Delta q}) + (\upsilon_{i\gamma}\mathcal{B}_{\gamma p} + \mathcal{B}_{i\gamma}\upsilon_{\gamma p})(I_1\mathcal{B}_{jq} - \mathcal{B}_{j\Delta}\mathcal{B}_{\Delta q})] \\ &+ 4\mathcal{F}_{37}I_3[\Delta_{ip}\upsilon_{jq} + \Delta_{jq}\upsilon_{ip}] + 4\mathcal{F}_{38}I_3[\Delta_{ip}(\upsilon_{j\Delta}\mathcal{B}_{\Delta q} + \mathcal{B}_{j\Delta}\upsilon_{\Delta q}) + (\upsilon_{i\gamma}\mathcal{B}_{\gamma p} + \mathcal{B}_{i\gamma}\upsilon_{\gamma p})\Delta_{jq}] + 4\mathcal{F}_{77}\upsilon_{ip}\upsilon_{jq} \\ &+ 4\mathcal{F}_{78}[\upsilon_{ip}(\upsilon_{j\Delta}\mathcal{B}_{\Delta q} + \mathcal{B}_{j\Delta}\upsilon_{\Delta q}) + (\upsilon_{i\gamma}\mathcal{B}_{\gamma p} + \mathcal{B}_{i\gamma}\upsilon_{\gamma p})\upsilon_{jq}] + 4\mathcal{F}_{88}(\upsilon_{i\gamma}\mathcal{B}_{\gamma p} + \mathcal{B}_{i\gamma}\upsilon_{\gamma p})(\upsilon_{j\Delta}\mathcal{B}_{\Delta q} + \mathcal{B}_{j\Delta}\upsilon_{\Delta q}). \end{aligned} \tag{17}$$

Since $\mathbf{K} = \mathbf{I}$ in the reference configuration and using Eq. (16) in Eq. (17), for an unconstrained compressible material, we have

$$\mathcal{B}_{0piqj} = \gamma_1(\Delta_{ij}\Delta_{pq} + \Delta_{iq}\Delta_{jp} - \Delta_{ip}\Delta_{jq}) + \gamma_2\Delta_{ip}\Delta_{jq} + \Delta_{ij}\Theta_{pq} + \gamma_3(\Delta_{ij}\Theta_{pq} + \Delta_{pq}\Theta_{ij} + \Delta_{iq}\Theta_{jp} + \Delta_{jp}\Theta_{iq}) + \gamma_4(\Delta_{ip}\Theta_{jq} + \Delta_{jq}\Theta_{ip}) + \gamma_5\Theta_{ip}\Theta_{jq}, \tag{18}$$

which represent the components of the elasticity tensor in the reference configuration. Here, we have defined

$$\begin{aligned} \gamma_1 &= 2(\mathcal{F}_1 + \mathcal{F}_2), \\ \gamma_2 &= 2(\mathcal{F}_2 + \mathcal{F}_3) + 4(\mathcal{F}_{11} + 4\mathcal{F}_{12} + 2\mathcal{F}_{13} + 4\mathcal{F}_{22} + 4\mathcal{F}_{23} + \mathcal{F}_{33}), \\ \gamma_3 &= 2\mathcal{F}_8, \\ \gamma_4 &= 4(\mathcal{F}_{17} + 2\mathcal{F}_{18} + 2\mathcal{F}_{27} + 4\mathcal{F}_{28} + \mathcal{F}_{37} + 2\mathcal{F}_{38}), \\ \gamma_5 &= 4(\mathcal{F}_{77} + 4\mathcal{F}_{78} + 4\mathcal{F}_{88}). \end{aligned} \tag{19}$$

When $\boldsymbol{\Theta} = \mathbf{0}$, Eq. (18) gives

$$\mathcal{B}_{0piqj} = \gamma_1(\Delta_{pq}\Delta_{ij} + \Delta_{iq}\Delta_{jp} - \Delta_{ip}\Delta_{jq}) + \gamma_2\Delta_{ip}\Delta_{jq}, \tag{20}$$

which is the linear theory expression of elasticity tensor.

A detailed discussion on the invariants and the derivatives of various invariants with respect to $\mathbf{K}$ can be found in [19].

3 Small amplitude wave propagation in a compressible half-space with initial stress

We now consider the updated configuration as a finitely deformed one along with a uniform initial stress. The deformation is assumed to be homogeneous. It is also assumed the principal axis of the strain (say x_3) coincides with the corresponding principal axis of the initial stress. Therefore, any subsequent infinitesimal deformation is in the (x_1, x_2)-plane which is the principal plane for the initial stress and the finite deformation. Let (D_1, D_2) be the components of displacement dependent on x_1, x_2, t and the principal initial stress components Θ_{11}, Θ_{22} in the plane. The governing equations of motion in (D_1, D_2) from Eq. (4), when expanded for $i = 1, 2$, are

$$\Gamma_{11}D_{1,11} + 2\Gamma_1 D_{2,11} + \Gamma_2 D_{1,12} + \Delta D_{2,12} + \Gamma_6 D_{1,22} + \Gamma_3 D_{2,22} = \rho_r D_{1,tt}, \text{ for } i = 1, \tag{21}$$

$$\Gamma_1 D_{1,11} + \Gamma_5 D_{2,11} + \Delta D_{1,12} + 2\Gamma_4 D_{2,12} + \Gamma_3 D_{1,22} + \Gamma_{22}D_{2,22} = \rho_r D_{2,tt}, \text{ for } i = 2, \tag{22}$$

where $\Gamma_{11}, \Gamma_{22}, \Gamma_5, \Gamma_6$, and Δ are constants depending on the material, t in subscript represents differentiation with respect to time and ρ_r is the density per unit reference volume. Here we have used

$$\mathcal{B}_{0ijji} = \mathcal{B}_{0iijj} = \mathcal{B}_{0ijij} - T_{ii}. \tag{23}$$

The various constants appearing are given by

$$\begin{aligned} &\Gamma_{ii} = J\mathcal{B}_{0iiii}, \quad \Gamma_{ij} = J\mathcal{B}_{0iijj}, \quad \Gamma_1 = J\mathcal{B}_{01112}, \\ &\Gamma_2 = J\mathcal{B}_{02111}, \quad \Gamma_3 = J\mathcal{B}_{02221}, \quad \Gamma_4 = J\mathcal{B}_{01222}, \\ &\Gamma_5 = J\mathcal{B}_{01212}, \quad \Gamma_6 = J\mathcal{B}_{02121}, \\ &\Delta = \Gamma_{12} + \Gamma_6 - T_{22} = \Gamma_{12} + \Gamma_5 - T_{11}. \end{aligned} \tag{24}$$

Here, it is supposed that $\Theta_{ij} = 0, i \neq j$ which implies $\upsilon_{ij} = 0$, for $i \neq j$ and $\boldsymbol{\upsilon}$ is coaxial with $\mathcal{B}$. This results in $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4 = 0$ and Eqs. (21-22) give

$$\Gamma_{11}D_{1,11} + \Delta D_{2,12} + \Gamma_6 D_{1,22} = \rho_r D_{1,tt}, \tag{25}$$

$$\Gamma_5 D_{2,11} + \Delta D_{1,12} + \Gamma_{22}D_{2,22} = \rho_r D_{2,tt}, \tag{26}$$

respectively. A few manipulations with Eqs. (25) and (26) give for $i \in (1, 2)$

$$\Gamma_{11}\Gamma_5 D_{i,1111} + (\Gamma_{11}\Gamma_{22} + \Gamma_5\Gamma_6 - \Delta^2)D_{i,1122} + \Gamma_{22}\Gamma_6 D_{i,2222} = \rho_r(\Gamma_{11} + \Gamma_5)D_{i,11tt} + \rho_r(\Gamma_{22} + \Gamma_6)D_{i,22tt} - \rho_r^2 D_{i,tttt}, \tag{27}$$

which is satisfied by D_1 and D_2 both.

Equation (2.7) of [4] is of the form of Eq. (27), in a different notation. However the two equations differ in nature due to the dependence of material constants on the initial stress in the latter case. For $\Theta = \mathbf{0}$ in Eq. (27), the case for an isotropic material can be retrieved as in [4].

An incremental plane wave is given by

$$(D_1, D_2) = A(s_1, s_2)e^{[ik(l_1x_1 + l_2x_2 - ct)]}, \tag{28}$$

where c is the wave speed, k is the wave number, $\boldsymbol{l} = (l_1, l_2)$ is the wave normal vector, $\mathbf{s} = (s_1, s_2)$ is the polarization vector and A is the wave amplitude.

Using Eq. (28) in Eqs. (25) and (26) gives the propagation equation

$$\mathbf{Q}(\boldsymbol{l})\mathbf{s} = \rho_r c^2\mathbf{s}, \tag{29}$$

for a compressible material. Here $\mathbf{Q}(\boldsymbol{l})$ is the acoustic tensor (see, for example, [18]). From Eq. (29), we also have

$$s_2 = \upsilon s_1, \tag{30}$$

where

$$\upsilon = \frac{\rho_r c^2 - \Gamma_{11}l_1^2 - \Gamma_6 l_2^2}{\Delta l_1 l_2} = \frac{\Delta l_1 l_2}{\rho_r c^2 - \Gamma_5 l_1^2 - \Gamma_{22}l_2^2}. \tag{31}$$

The acoustic tensor $\mathbf{Q}$ has the component form

$$Q_{ij}(\boldsymbol{l}) = \mathcal{B}_{0piqj} l_p l_q. \tag{32}$$

Using Eq. (17) in Eq. (32), we get

$$\begin{aligned}\mathbf{Q}(\boldsymbol{l}) = {} & [2(\mathcal{F}_1 + \mathcal{F}_2 I_1)\mathcal{B}^{(\boldsymbol{l})} - 2\mathcal{F}_2\mathcal{B}^{2(\boldsymbol{l})} + 2\mathcal{F}_7 \upsilon^{(\boldsymbol{l})} \\ & + 4\mathcal{F}_8(\upsilon\mathcal{B})^{(\boldsymbol{l})}]\mathbf{I} - 2(\mathcal{F}_2\mathcal{B}^{(\boldsymbol{l})} - \mathcal{F}_8\upsilon^{(\boldsymbol{l})})\mathcal{B} + 2\mathcal{F}_8 B^{(\boldsymbol{l})}\upsilon \\ & + 2(\mathcal{F}_3 I_3 + 2\mathcal{F}_{33} I_3^2)\boldsymbol{l}\otimes\boldsymbol{l} + 2(\mathcal{F}_2 + 2\mathcal{F}_{11} + 4\mathcal{F}_{12} I_1)\mathcal{B}\boldsymbol{l} \\ & \otimes \mathcal{B}\boldsymbol{l} + 2(\mathcal{F}_8 + 2\mathcal{F}_{17})(\mathcal{B}\boldsymbol{l}\otimes\upsilon\boldsymbol{l} + \upsilon\boldsymbol{l}\otimes\mathcal{B}\boldsymbol{l}) + 4\mathcal{F}_{22}\mathcal{B}^{\star}\boldsymbol{l} \\ & \otimes \mathcal{B}^{\star}\boldsymbol{l} - 4\mathcal{F}_{12}(\mathcal{B}\boldsymbol{l}\otimes\mathcal{B}^2\boldsymbol{l} + \mathcal{B}^2\boldsymbol{l}\otimes\mathcal{B}\boldsymbol{l}) \\ & + 4I_3(\mathcal{F}_{13} + \mathcal{F}_{23} I_1)(\mathcal{B}\boldsymbol{l}\otimes\boldsymbol{l} + \boldsymbol{l}\otimes\mathcal{B}\boldsymbol{l}) \\ & + 4\mathcal{F}_{18}[\mathcal{B}\boldsymbol{l}\otimes(\upsilon\mathcal{B} + \mathcal{B}\upsilon)\boldsymbol{l} + (\upsilon\mathcal{B} + \mathcal{B}\upsilon)\boldsymbol{l}\otimes\mathcal{B}\boldsymbol{l}] \\ & - 4\mathcal{F}_{23} I_3(\boldsymbol{l}\otimes\mathcal{B}^2\boldsymbol{l} + \mathcal{B}^2\boldsymbol{l}\otimes\boldsymbol{l}) + 4\mathcal{F}_{27}(\mathcal{B}^{\star}\boldsymbol{l}\otimes\upsilon\boldsymbol{l} \\ & + \upsilon\boldsymbol{l}\otimes\mathcal{B}^{\star}\boldsymbol{l}) + 4\mathcal{F}_{28}[\mathcal{B}^{\star}\boldsymbol{l}\otimes(\upsilon\mathcal{B} + \mathcal{B}\upsilon)\boldsymbol{l} + (\upsilon\mathcal{B} + \mathcal{B}\upsilon)\boldsymbol{l} \\ & \otimes \mathcal{B}^{\star}\boldsymbol{l}] + 4\mathcal{F}_{37} I_3(\boldsymbol{l}\otimes\upsilon\boldsymbol{l} + \upsilon\boldsymbol{l}\otimes\boldsymbol{l}) + 4\mathcal{F}_{38} I_3[\boldsymbol{l}\otimes(\upsilon\mathcal{B} \\ & + \mathcal{B}\upsilon)\boldsymbol{l} + (\upsilon\mathcal{B} + \mathcal{B}\upsilon)\boldsymbol{l}\otimes\boldsymbol{l}] + 4\mathcal{F}_{77}\upsilon\boldsymbol{l}\otimes\upsilon\boldsymbol{l} \\ & + 4\mathcal{F}_{78}[\upsilon\boldsymbol{l}\otimes(\upsilon\mathcal{B} + \mathcal{B}\upsilon)\boldsymbol{l} + (\upsilon\mathcal{B} + \mathcal{B}\upsilon)\boldsymbol{l}\otimes\upsilon\boldsymbol{l}] \\ & + 4\mathcal{F}_{88}(\upsilon\mathcal{B} + \mathcal{B}\upsilon)\boldsymbol{l}\otimes(\upsilon\mathcal{B} + \mathcal{B}\upsilon)\boldsymbol{l},\end{aligned} \tag{33}$$

where $B^{(\boldsymbol{l})} = \boldsymbol{l}\cdot\mathcal{B}\boldsymbol{l}$, $\upsilon^{(\boldsymbol{l})} = \boldsymbol{l}\cdot\upsilon\boldsymbol{l}$, $B^{2(\boldsymbol{l})} = \boldsymbol{l}\cdot\mathcal{B}^2\boldsymbol{l}$ and $(\upsilon B)^{(\boldsymbol{l})} = \boldsymbol{l}\cdot\upsilon\mathcal{B}\boldsymbol{l}$.

Since $\mathcal{B}^{2(\boldsymbol{l})} = \mathcal{B}^{(\boldsymbol{l})} = 1$, $\mathcal{B}^{\star(\boldsymbol{l})} = 2$ and $(\upsilon\mathcal{B})^{(\boldsymbol{l})} = \Theta^{(\boldsymbol{l})}$ in the reference configuration, we have from Eq. (33)

$$\begin{aligned}\mathbf{Q}(\boldsymbol{l}) = {} & (\gamma_1 + (1+\gamma_3)\Theta^{(\boldsymbol{l})})\mathbf{I} + \gamma_2\boldsymbol{l}\otimes\boldsymbol{l} + \gamma_3\boldsymbol{\Theta} \\ & + (\gamma_3 + \gamma_4)(\boldsymbol{l}\otimes\boldsymbol{l\Theta} + \boldsymbol{\Theta l}\otimes\boldsymbol{l}) + \gamma_5\boldsymbol{\Theta l}\otimes\boldsymbol{\Theta l},\end{aligned} \tag{34}$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and γ_5 are given by Eq. (19).

For compressible materials in deformed configuration, the strong ellipticity condition is given by

$$Q_{ij}s_i s_j = \mathcal{B}_{0piqj} l_p l_q s_i s_j > 0, \tag{35}$$

for all non-zero arbitrary $\mathbf{s}$, $\boldsymbol{l}$.

Furthermore from Eq. (29), we have

$$\rho_r c^2 = (\mathbf{Q}(\boldsymbol{l})\mathbf{s})\cdot\mathbf{s} = \mathcal{B}_{0piqj} l_p l_q s_i s_j \tag{36}$$

and Eq. (35) thus ensures positive values for $\rho_r c^2$. However, c can be of either signature.

From Eqs. (29) and (34), it follows, for arbitrary $\mathcal{S}$ and $\boldsymbol{l}$,

$$\begin{aligned}\rho_r c^2 = {} & \Gamma_1 + (1+\Gamma_3)\Theta^{(\boldsymbol{l})} + \Gamma_2(\boldsymbol{l}\cdot\mathbf{s})^2 + \Gamma_3(\mathbf{s}\cdot\boldsymbol{\Theta}\mathbf{s}) \\ & + 2(\Gamma_3 + \Gamma_4)(\mathbf{s}\cdot\boldsymbol{\Theta l})(\boldsymbol{l}\cdot\mathbf{s}) + \Gamma_5(\mathbf{s}\cdot\boldsymbol{\Theta l})^2.\end{aligned} \tag{37}$$

For a particular choice of $\boldsymbol{l}$, the wave speed is calculated by

$$|(\mathbf{Q}(\boldsymbol{l}) - \rho_r c^2\mathbf{I}| = 0, \tag{38}$$

which is the characteristic equation. Here, $\mathbf{I}$ is the identity matrix. This gives a quadratic equation in $\rho_r c^2$, namely

$$(\rho_r c^2)^2 - C_1\rho_r c^2 + C_2 = 0, \tag{39}$$

where

$$\begin{aligned} & C_1 = Q_{11} + Q_{22}, \quad C_2 = Q_{11}Q_{22} - Q_{12}Q_{21}, \\ & Q_{11} = \Gamma_{11}l_1^2 + \Gamma_6 l_2^2, \quad Q_{12} = Q_{21} = \Delta l_1 l_2, \\ & Q_{22} = \Gamma_5 l_1^2 + \Gamma_{22} l_2^2.\end{aligned} \tag{40}$$

Another approach to obtain a quadratic equation for $\rho_r c^2$ is by using Eq. (28) into Eq. (27), given as

$$\begin{aligned} & (\rho_r c^2)^2 - [(\Gamma_{11} + \Gamma_5)l_1^2 + (\Gamma_{22} + \Gamma_6)l_2^2]\rho_r c^2 \\ & + \Gamma_{11}\Gamma_5 l_1^4 + 2\beta l_1^2 l_2^2 + \Gamma_{22}\Gamma_6 l_2^4 = 0,\end{aligned} \tag{41}$$

where

$$2\beta = \Gamma_{11}\Gamma_{22} + \Gamma_5\Gamma_6 - \Delta^2. \tag{42}$$

Two positive solutions for $\rho_r c^2$ are obtained from Eq. (41) if and only if

$$\Gamma_{11} > 0, \quad \Gamma_{22} > 0, \quad \Gamma_5 > 0, \quad \Gamma_6 > 0, \tag{43}$$

and

$$\beta > -(\Gamma_{11}\Gamma_{22}\Gamma_5\Gamma_6)^{1/2}, \tag{44}$$

which are also quoted in a similar manner in [4] for prestressed isotropic material. However, here these conditions imply bounds on the values of the principal stress components.

4 Reflection from a plane boundary

We consider a material body in its finitely deformed configuration such that the half-space is $x_2 < 0$ and $x_2 = 0$ is the boundary. On the boundary, vanishing of incremental dead load requires the incremental traction components

$\dot{\mathcal{S}}_{021} = 0 = \dot{\mathcal{S}}_{022}$, where $\dot{\mathcal{S}}_{0pi}D_k = \mathcal{B}_{0piqj}D_{j,q}D_k$. Therefore, after using Eq. (24), the boundary equations at $x_2 = 0$ are

$$\dot{\mathcal{S}}_{021} = D_{1,2} + D_{2,1} = 0, \tag{45}$$

$$\dot{\mathcal{S}}_{022} = \Gamma_{12}D_{1,1} + \Gamma_{22}D_{2,2} = 0. \tag{46}$$

A homogeneous plane wave is assumed to propagate in the half-space and analogous to Eq. (28), this wave is of the form

$$\mathbf{U} = A\mathbf{s}e^{[ik(\boldsymbol{l}\cdot\mathbf{x} - ct)]}. \tag{47}$$

Let the direction of propagation of the incident wave be $\boldsymbol{l} = (l_1, l_2)$, the polarization vector $\mathbf{s} = (s_1, s_2)$, θ the angle of incidence and c the speed of the incident wave. Due to this incident wave, depending on the material properties and the deformation, there may exist two (homogeneous) reflected plane waves or just one reflected plane wave accompanied by a surface wave. These possibilities are discussed in the following sections. Let k denote the wave number of the incident wave. The first reflected wave travels at the same wave speed as the incident wave and makes the angle θ with the boundary. For the second reflected wave, k' and c' represent the wave number and speed, respectively.

For $x_2 < 0$, the total displacement for two reflected plane waves is

$$\mathbf{U} = A\mathbf{s}e^{[ik(\boldsymbol{l}\cdot\mathbf{x}-ct)]} + A\mathcal{R}\mathbf{s}^{-}e^{[ik(\boldsymbol{l}^{-}\cdot\mathbf{x}-ct)]} + A\mathcal{R}'\mathbf{s}'e^{[ik'(\boldsymbol{l}'\cdot\mathbf{x}-c't)]}, \tag{48}$$

where $\mathcal{R}, \mathcal{R}'$ are the reflection coefficients associated with the first and the second reflected wave, respectively. Moreover, let $\mathbf{s}^{-}, \boldsymbol{l}^{-}$ be the polarization vector and the direction of propagation of the first reflected wave and $\mathbf{s}', \boldsymbol{l}'$ be the polarization vector and the direction of travel of the second reflected wave, respectively. For the compatibility of these three waves, they should bear the same frequency and hence $kc = k'c'$. Also, due to the traction-free boundary conditions, $kl_1 = k'l_1'$ and therefore

$$c'l_1 = cl_1', \tag{49}$$

which is a statement of Snell's law. Using Eq. (41), it is found that for the second reflected wave that either $l_2^{-2} = l_2^2$ or

$$2\beta l_1^2 - (\Gamma_{22} + \Gamma_6)\rho_r c^2 + \Gamma_{22}\Gamma_6(l_2^{-2} + l_2^2) = 0, \tag{50}$$

which gives possible values of l_2^- in terms of l_2. We may take $l_2^- = -l_2$ for the second reflected wave, without loss of generality.

We may take $\mathbf{s}^- = (s_1, -s_2)$, while any difference in sign can be adjusted within $\mathcal{R}$. From Eq. (30), we have $s_2 = \upsilon s_1$ for the incident wave withυ given by Eq. (31). Similarly, we have $s_2' = \upsilon' s_1'$, with υ' defined analogously to υ, given by

$$\upsilon' = (\rho_r c^2 - \Gamma_{11}l_1^2 - \Gamma_6 l_2'^2)/\Delta l_1 l_2', \tag{51}$$

for the second reflected wave.

Since the polarization vectors $\mathcal{S}$ and $\mathbf{m}'$ are assumed to be unit vectors, we set

$$s_1 = 1/(1+\upsilon^2)^{\frac{1}{2}}, \quad s_1' = 1/(1+\upsilon'^2)^{\frac{1}{2}}. \tag{52}$$

4.1 The reflection coefficients

With Eq. (48) and the boundary conditions Eq. (45-46), we get the reflection coefficients $\mathcal{R}, \mathcal{R}'$, which leads to

$$\mathcal{R} = \frac{l_2'p'q - l_2pq'}{l_2'p'q + l_2pq'}, \tag{53}$$

$$\mathcal{R}' = -\frac{2pq}{l_2'p'q + l_2pq'}\frac{s_1l_2'}{s_1'}, \tag{54}$$

where p, q, p', q' are defined by

$$p = \Delta\Gamma_{12}l_1^2 + \Gamma_{22}(\rho_r c^2 - \Gamma_{11}l_1^2 - \Gamma_6 l_2^2), \quad q = \Gamma_6\Gamma_{12}l_2^2 + (\Delta - \Gamma_{12})(\rho_r c^2 - \Gamma_{11}l_1^2), \tag{55}$$

$$p' = \Delta\Gamma_{12}l_1^2 + \Gamma_{22}(\rho_r c^2 - \Gamma_{11}l_1^2 - \Gamma_6 l_2'^2), \quad q' = \Gamma_6\Gamma_{12}l_2'^2 + (\Delta - \Gamma_{12})(\rho_r c^2 - \Gamma_{11}l_1^2), \tag{56}$$

and $\frac{s_1 l_2'}{s_1'}$ is given by

$$\frac{s_1l_2'}{s_1'} = l_2\frac{(\Delta^2 l_1^2 l_2'^2 + (\rho_r c^2 - \Gamma_{11}l_1^2 - \Gamma_6 l_2'^2)^2)^{\frac{1}{2}}}{(\Delta^2 l_1^2 l_2^2 + (\rho_r c^2 - \Gamma_{11}l_1^2 - \Gamma_6 l_2^2)^2)^{\frac{1}{2}}}. \tag{57}$$

Using the connection $l_2 = l_1 \tan\theta$,
Eq. (41) becomes

$$(\Gamma_{11}\Gamma_5 + 2\beta\tan^2\theta + \Gamma_{22}\Gamma_6\tan^4\theta)l_1^4 - \rho_r c^2[\Gamma_{11} + \Gamma_5 + (\Gamma_{22} + \Gamma_6)\tan^2\theta]l_1^2 + (\rho_r c^2)^2 = 0, \tag{58}$$

which expresses l_1 in terms of the angle θ that is the direction of the wave normal.

Differentiating Eq. (41) with respect to $\frac{\Gamma_6 l_2^2}{\rho_r c^2}$ and multiplying both sides by l_2^2, we obtain

$$2\beta l_1^2 l_2^2 + (\Gamma_{22}\Gamma_6(l_2^4 + l_2^2 l_2'^2)) - (\Gamma_{22} + \Gamma_6)l_2^2\rho_r c^2 = 0. \tag{59}$$

Subtracting Eq. (59) from Eq. (41) we find

$$\Gamma_{22}\Gamma_6 l_2^2 l_2'^2 = (\Gamma_{11}l_1^2 - \rho_r c^2)(\Gamma_5 l_1^2 - \rho_r c^2), \tag{60}$$

which expresses l_2' in terms of θ. Hence, $\mathcal{R}, \mathcal{R}'$ can be expressed as a function of θ explicitly. Considering the connection $l_2' = l_1\tan\theta'$ and the notation

$$\upsilon = \tan\theta, \quad \upsilon' = \tan\theta', \tag{61}$$

Equation (58) can be rewritten as

$$(\Gamma_{11}\Gamma_5 + 2\beta\upsilon^2 + \Gamma_{22}\Gamma_6\upsilon^4)l_1^4 - \rho_r c^2[\Gamma_{11} + \Gamma_5 + \Gamma_{22} + \upsilon^2]l_1^2 + (\rho_r c^2)^2 = 0. \tag{62}$$

Also, with this notation, Eq. (60) becomes

$$\upsilon^2\upsilon'^2 = \frac{(\Gamma_{11}l_1^2 - \rho_r c^2)(\Gamma_5 l_1^2 - \rho_r c^2)}{\Gamma_{22}\Gamma_6 l_1^4}. \tag{63}$$

Since $\{\Gamma_{11}, \Gamma_5\} > 0$, for a given value of the angle of incidence, real υ' values exist given

$$\text{either } \rho_r c^2 \le \min(\Gamma_{11} l_1^2, \Gamma_5 l_1^2) \quad \text{or } \max(\Gamma_{11} l_1^2, \Gamma_5 l_1^2) \le \rho_r c^2. \tag{64}$$

Using $l_2 = l_1 \tan\theta$ and $l_2' = l_1 \tan\theta'$, Eqs. (53-57) become

$$\mathcal{R} = \frac{\upsilon' p' q - \upsilon p q'}{\upsilon' p' q + \upsilon p q'}, \tag{65}$$

$$\mathcal{R}' = -\frac{2pq}{\upsilon' p' q + \upsilon p q'} \frac{s_1 \upsilon'}{s_1'}, \tag{66}$$

where

$$p = \Delta\Gamma_{12} l_1^2 + \Gamma_{22}(\rho_r c^2 - \Gamma_{11} l_1^2 - \Gamma_6 l_1^2 \upsilon^2), \quad q = \Gamma_6 \Gamma_{12} l_1^2 \upsilon^2 + (\Delta - \Gamma_{12})(\rho_r c^2 - \Gamma_{11} l_1^2), \tag{67}$$

$$p' = \Delta\Gamma_{12} l_1^2 + \Gamma_{22}(\rho_r c^2 - \Gamma_{11} l_1^2 - \Gamma_6 l_1^2 \upsilon'^2), \quad q' = \Gamma_6 \Gamma_{12} l_1^2 \upsilon'^2 + (\Delta - \Gamma_{12})(\rho_r c^2 - \Gamma_{11} l_1^2), \tag{68}$$

and

$$\frac{s_1 \upsilon'}{s_1'} = \upsilon \sqrt{\frac{\Delta^2 l_1^4 \upsilon'^2 + (\rho_r c^2 - \Gamma_{11} l_1^2 - \Gamma_6 l_1^2 \upsilon'^2)^2}{\Delta^2 l_1^4 \upsilon^2 + (\rho_r c^2 - \Gamma_{11} l_1^2 - \Gamma_6 l_1^2 \upsilon^2)^2}}. \tag{69}$$

5 Numerical results and discussion

5.1 Compressible hyperelastic material with a homogeneous initial stress

We now choose a prototype function $\mathcal{F}$ to represent the response of a compressible elastic material with initial stress given by

$$\mathcal{F} = \frac{\mu}{2}(I_3^{-2/3} I_1 + I_3 - 3) + (\lambda + \frac{2}{3}\mu)(I_3 - 1)^2 + \frac{\overline{\mu}}{2}(I_7 - I_4)^2 + \frac{1}{2}(I_7 - I_4), \tag{70}$$

where μ, λ are Lame's parameters and $\overline{\mu}$ is a material constant (with dimensions of $(stress)^{-1}$). Equation (72) gives behavior of Neo-Hookean-type solid for $\Theta = 0$. Various other models are suggested in [24] which satisfy an additional condition on the form this function. The condition derived in [24] is plausible, however, this simple strain energy function is used for brevity and illustration of the effect of initial stress on wave propagation in a compressible elastic solid. For convenience, we introduce the notations

$$\mathcal{F}^\star = \mathcal{F}/\mu, \quad \lambda^\star = \lambda/\mu, \quad \Theta_{11}^\star = \Theta_{11}/\mu, \quad \Theta_{22}^\star = \Theta_{22}/\mu, \quad I_7^\star = I_7/\mu, \quad I_4^\star = I_4/\mu, \quad \mu^\star = \overline{\mu}\mu. \tag{71}$$

Using the above notations in (70), the strain energy function can be rewritten as

$$\mathcal{F}^\star = \frac{1}{2}\left(I_3^{-2/3} I_1 + I_3 - 3\right) + \left(\lambda^\star + \frac{2}{3}\right)(I_3 - 1)^2 + \frac{\mu^\star}{2}(I_7^\star - I_4^\star)^2 + \frac{1}{2}(I_7^\star - I_4^\star), \tag{72}$$

where $\mathcal{F}_i^\star = \partial\mathcal{F}^\star/\partial I_i$ for $i = \{1, 3\}$ and $\mathcal{F}_i^\star = \partial\mathcal{F}^\star/\partial I_i^\star$ for $i = \{4, 7\}$.

For the chosen material, in the deformed configuration after using Eqs. (17), (24) and (72) accordingly alongwith the assumption $\Theta_{ij} = 0, i \neq j$, we get

$$\Gamma_{11} = a\lambda_1^4(\Theta_1^\star)^2 + b\lambda_1^2\Theta_1^\star + c_1, \tag{73}$$

$$\Gamma_{22} = a\lambda_2^4(\Theta_2^\star)^2 + b\lambda_2^2\Theta_2^\star + c_2, \tag{74}$$

$$\Gamma_{12} = a\lambda_1^2\lambda_2^2(\Theta_1^\star)(\Theta_2^\star) + c_3, \tag{75}$$

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4 = 0, \tag{76}$$

$$\Gamma_5 = I_3^{-2/3}\lambda_1^2 + b\lambda_1^2\Theta_1^\star, \tag{77}$$

$$\Gamma_6 = I_3^{-2/3}\lambda_2^2 + b\lambda_2^2\Theta_2^\star, \tag{78}$$

$$\Delta = a\lambda_1^2\lambda_2^2(\Theta_1^\star)(\Theta_2^\star) + c_3 + I_3^{-2/3}\lambda_1^2 + b\lambda_2^2\Theta_2^\star, \tag{79}$$

where

$$a = 4\mu^\star, b = 2\mu^\star(I_7^\star - I_4^\star) + 1, \tag{80}$$

$$c_1 = 2(\lambda^\star + \frac{2}{3})(5I_3^2 - I_3) + (\frac{14}{9}I_1 - \frac{5}{3}\lambda_1^2)I_3^{-2/3} + I_3, \tag{81}$$

$$c_2 = 2(\lambda^\star + \frac{2}{3})(5I_3^2 - I_3) + (\frac{14}{9}I_1 - \frac{5}{3}\lambda_2^2)I_3^{-2/3} + I_3, \tag{82}$$

$$c_3 = 8(\lambda^\star + \frac{2}{3})(2I_3^2 - I_3) + 2I_3 + \frac{4}{3}(2 - (\lambda_1^2 + \lambda_2^2))I_3^{-2/3}I_1. \tag{83}$$

In the reference configuration, $b = 1, c_1 = c_2 = 8\lambda^\star + \frac{28}{3}, c_3 = 8\lambda^\star + \frac{22}{3}$, the expressions (73-79) reduce to

$$\Gamma_{11} = 4\mu^\star\Theta_1^{\star 2} + \Theta_1^\star + c_1, \quad \Gamma_{22} = 4\mu^\star\Theta_2^{\star 2} + \Theta_2^\star + c_2, \quad \Gamma_{12} = 4\mu^\star\Theta_1^\star\Theta_2^\star + c_3, \quad \Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4 = 0, \quad \Gamma_5 = 1 + \Theta_1^\star, \quad \Gamma_6 = 1 + \Theta_2^\star, \tag{84}$$

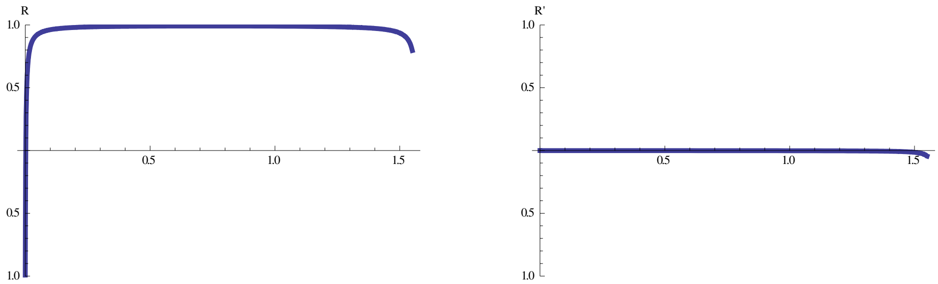

Figure 1: Plot of the reflection coefficients $\mathcal{R}$ and $\mathcal{R}'$ versus angle of incidence θ for $\lambda^\star = 0.2$, $\mu^\star = 10$ and $\Theta_1^\star = 300$, $\Theta_2^\star = 0$.

$$\Delta = 4\mu^\star\Theta_1^\star\Theta_2^\star + 8\lambda^\star + \frac{25}{3} + \Theta_2^\star.$$

In deformed configuration, the strong ellipticity conditions ($43_{3,4}$) require

$$\{\Theta_1^\star, \Theta_2^\star\} > \frac{-I_3^{-2/3}}{2\mu^\star(I_7^\star - I_4^\star) + 1}, \tag{85}$$

which gives the lower bound for the values of the principal initial stress components. Further, we additionally require from inequalities ($43_{1,2}$)

$$a(\lambda_1^2\Theta_1^\star)^2 + b\lambda_1^2\Theta_1^\star + c_1 > 0, \tag{86}$$
$$a(\lambda_2^2\Theta_2^\star)^2 + b\lambda_2^2\Theta_2^\star + c_2 > 0,$$

which hold if

$$(\sqrt{b^2 - 4ac_1} - 1)/2a\lambda_1^2 < \Theta_1^\star \tag{87}$$
$$< -(\sqrt{b^2 - 4ac_1} + 1)/2a\lambda_1^2,$$
$$(\sqrt{b^2 - 4ac_2} - 1)/2a\lambda_2^2 < \Theta_2^\star < -(\sqrt{b^2 - 4ac_2} + 1)/2a\lambda_2^2,$$

respectively.

5.2 Incident *P*–wave in the reference configuration

Considering a special class of materials where $2\beta = \Gamma_{11}\Gamma_6 + \Gamma_{22}\Gamma_5$, Eq. (41) gives

$$(\Gamma_5 l_1^2 + \Gamma_6 l_2^2 - \rho c^2)(\Gamma_{11} l_1^2 + \Gamma_{22} l_2^2 - \rho c^2) = 0. \tag{88}$$

For an incident P–wave using Eq. (88), we obtain

$$\Gamma_{11} l_1^2 + \Gamma_{22} l_2^2 = \rho c^2. \tag{89}$$

After using Eq. (89) in Eq. (63) we deduce

$$\upsilon' = -\sqrt{\frac{\Gamma_{22}\upsilon^2 + \Gamma_{11} - \Gamma_5}{\Gamma_6}}. \tag{90}$$

For $\upsilon' > 0$, we should have $\upsilon^2 > \frac{\Gamma_5 - \Gamma_{11}}{\Gamma_{22}}$.

Hence, in Eqs. (65) and (66), p, q, p', q' are accordingly defined as

$$p = \Delta\Gamma_{12} l_1^2 - \Gamma_{22}\Gamma_6 l_1^2 \upsilon^2 + \Gamma_{22}^2 l_2^2, \tag{91}$$
$$q = \Gamma_6\Gamma_{12} l_1^2 \upsilon^2 + \Delta\Gamma_{22} l_2^2 - \Gamma_{12}\Gamma_{22} l_2^2,$$

$$p' = \Delta\Gamma_{127} l_1^2 - \Gamma_{22}\Gamma_6 l_1^2 \upsilon'^2 + \Gamma_{22}^2 l_2^2, \tag{92}$$
$$q' = \Gamma_6\Gamma_{12} l_1^2 \upsilon'^2 + \Delta\Gamma_{22} l_2^2 - \Gamma_{12}\Gamma_{22} l_2^2,$$

and $\frac{s_1\upsilon'}{s_1'}$ is given by

$$\frac{s_1\upsilon'}{s_1'} = \upsilon\sqrt{\frac{\Delta^2 l_1^4 \upsilon'^2 + (\Gamma_{22} l_2^2 - \Gamma_6 l_1^2 \upsilon'^2)^2}{\Delta^2 l_1^4 \upsilon^2 + (\Gamma_{22} l_2^2 - \Gamma_6 l_1^2 \upsilon^2)^2}}. \tag{93}$$

5.3 The reflection coefficient $\mathcal{R}'$

From Eqs. (45-46) and Eq. (48), we get

$$[s_1 l_2(1 - \mathcal{R}) + s_1' l_2' \mathcal{R}'] + [s_2 l_1(1 - \mathcal{R}) + s_2' l_1 R'] = 0, \tag{94}$$

$$\Gamma_{12}[s_1 l_1(1 + \mathcal{R}) + s_1' l_1 R'] \tag{95}$$
$$+ \Gamma_{22}[(1 + \mathcal{R}) s_2 l_2 + s_2' l_2' \mathcal{R}'] = 0.$$

We now use the above expressions to look at the conditions for vanishing of the reflection coefficient $\mathcal{R}'$. It is noted that for $\theta = 0$ or $\theta = \frac{\pi}{2}$, $\mathcal{R}' = 0$. Also, from the boundary conditions, it is easy to see that if $\mathcal{R}' = 0$ then either $\mathcal{R} = 1$ (see Figure 1) with We now use the above expressions to look at the conditions for vanishing of the reflection coefficient $\mathcal{R}'$. It is noted that for $\theta = 0$ or $\theta = \frac{\pi}{2}$, $\mathcal{R}' = 0$. Also, from the boundary conditions, it is easy to see that if $\mathcal{R}' = 0$ then either $\mathcal{R} = 1$ (see Figure 1) with

$$\Gamma_{12} s_1 l_1 + \Gamma_{22} s_2 l_2 = 0, \tag{96}$$

or $\mathcal{R} = -1$ (see Figure 2) with

$$s_1 l_2 + s_2 l_1 = 0. \tag{97}$$

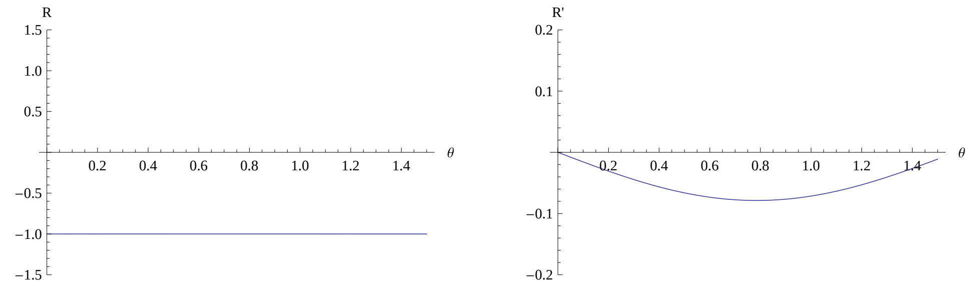

Figure 2: Plot of the reflection coefficients $\mathcal{R}$ and $\mathcal{R}'$ for $\lambda^\star = 80$, $\mu^\star = 1$, $\Theta_1^\star = 0.05 = \Theta_2^\star$.

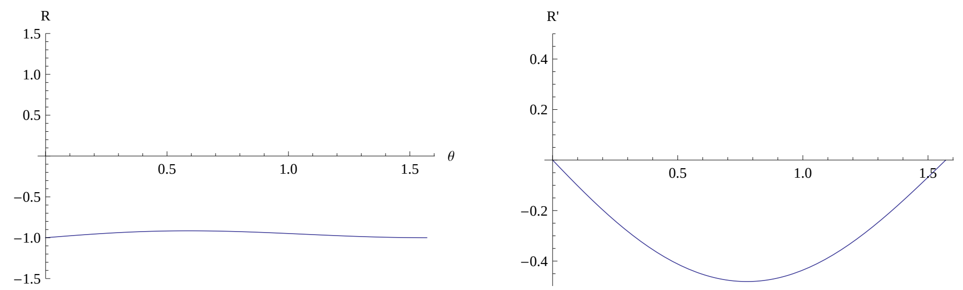

Figure 3: $\mathcal{R}$ and $\mathcal{R}'$ for $\lambda^\star = 5$, $\mu^\star = 50$ and $\Theta_1^\star = 0.2$.

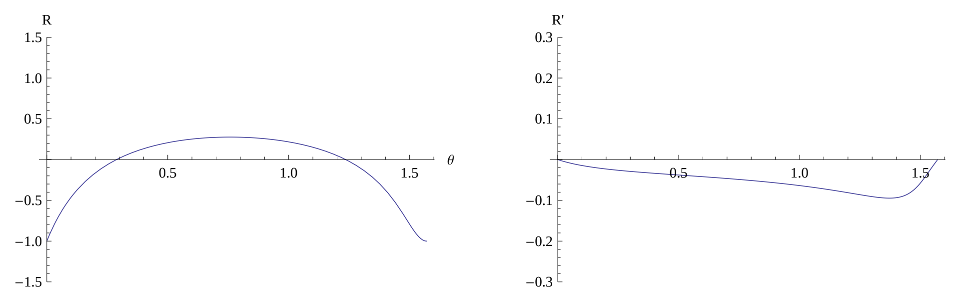

Figure 4: $\mathcal{R}$ and $\mathcal{R}'$ for $\lambda^\star = 10$, $\mu^\star = 25$ and $\Theta_1^\star = 0.2$.

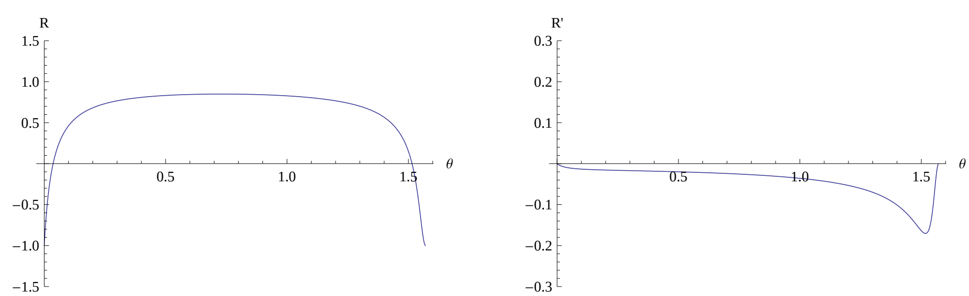

Figure 5: $\mathcal{R}$ and $\mathcal{R}'$ for $\lambda^\star = 1$, $\mu^\star = 20$ and $\Theta_1^\star = 10$.

In general, these conditions depend on the incidence angle and the principal initial stresses. However, from Eq. (96) for $\mathcal{R} = 1$, an incident SV wave can exist if material properties are such that $\Gamma_{12} = \Gamma_{22}$. It may be noted that $\mathcal{R} = 1$ happens for larger values and $\mathcal{R} = -1$ for very small values of initial stress components. For the case of $\mathcal{R} = -1$ in the case of a pre-stressed material[4], an incident P–wave is admissible if $T_{22} = 2\Gamma_6$ and if the material properties allow such a wave. However, in this case from Eq. (97), an incident P–wave is admissible for every such material where $\Gamma_6 \neq 0$.

Figures (3-7) are the graphical representation of Eqs. (65-66) for various particular values of the material constants and (uniaxial case) non-zero principal stress $\Theta_1^\star$. The behavior of two reflected waves is shown. A reflected SV–wave accompanies a reflected P–wave in most of the

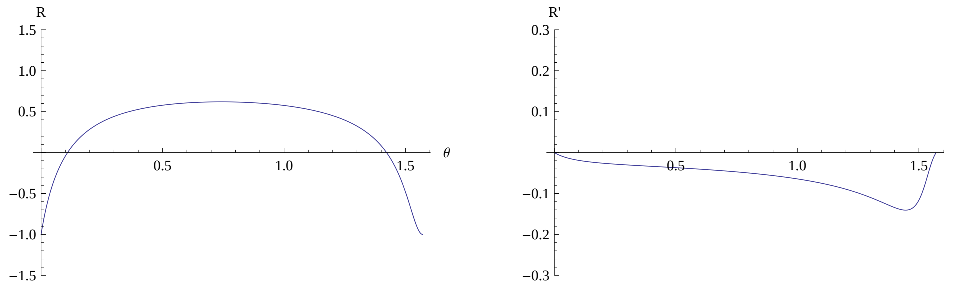

Figure 6: $\mathcal{R}$ and $\mathcal{R}'$ for $\lambda = 20$, $\mu^{\star} = 80$ and $\Theta_1^{\star} = 30$.

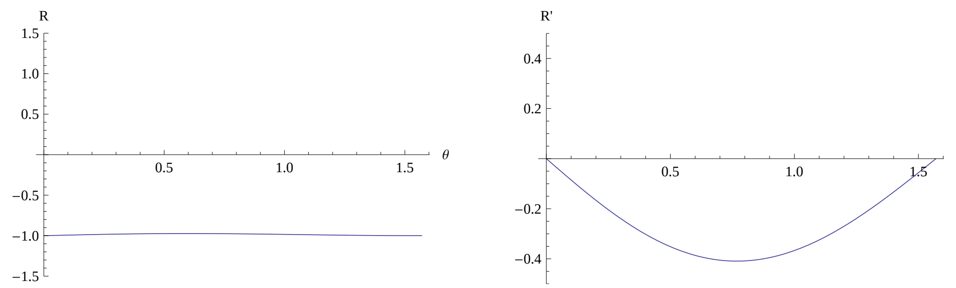

Figure 7: $\mathcal{R}$ and $\mathcal{R}'$ for $\lambda = 10$, $\mu^{\star} = 25$ and $\Theta_1^{\star} = 0$.

cases. The behavior of wave velocity is similar to that observed in [4] for various particular choices of the parameters and the results are in good accordance with those in [4] for pre-stressed compressible material.

of parameters. However, the amplitude of this wave may vanish in certain cases which depend on the angle of incidence θ. Other conditions depend on the material parameters as discussed in Section 5.3.

6 Conclusions

In this paper, using the nonlinear theory of elasticity, formulation is presented for compressible hyperelastic materials when the material is initially stressed in its reference configuration. The speed of wave in such a material is affected by the presence of this stress. The components of the initial stress are incorporated in the components of the fourth order elasticity tensor instead of introducing separate terms as is done in linear elasticity following [5]. It is assumed that the stored energy function depends on the invariants of the deformation as well initial stress tensor components. It is found that the wave speed depends considerably on the principal initial stresses and is required to satisfy the strong ellipticity conditions.

In particular, a study is presented to understand the effect of a homogeneous initial stress on the reflection of an incident P–wave. A prototype (so-called) stored energy function for a compressible material is used to elaborate the theoretical results and graphs are presented to observe the effect of initial stress on the reflection of a P– wave. It is found that a reflected SV exists in most cases of choices

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