Research Article

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Solitary Wave Solution of Nonlinear PDEs Arising in Mathematical Physics

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Abstract: The solution of nonlinear mathematical models has much importance and in soliton theory its worth has increased. In the present article, we have investigated the Caudrey-Dodd-Gibbon and Pochhammer-Chree equations, to discuss the physics of these equations and to attain soliton solutions. The $\exp(-\phi(\zeta))$-expansion technique is used to construct solitary wave solutions. A wave transformation is applied to convert the problem into the form of an ordinary differential equation. The drawn-out novel type outcomes play an essential role in the transportation of energy. It is noted that in the study, the approach is extremely reliable and it may be extended to further mathematical models signified mostly in nonlinear differential equations.

Keywords: Exp$(-\phi(\zeta))$-expansion technique; Caudrey-Dodd-Gibbon equation; Pochhammer-Chree equation; Homogenous principal; Solitary wave solution

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1 Introduction

Recently solitary wave theory got some great improvements. Soliton wave occurrence interested a number of researchers for its comprehensive applications in engineering, and mathematical physics. First, J S Russell (by profession an engineer) contemplated the solitary wave in 1834. In the form of differential equations, various physical occurrences in nature are modeled. Scientists have made great efforts to determine the solution of such differential equations. Different approaches have been used to determine soliton solutions. Modeling of various physical, biochemical and biological occurrences are in forms of nonlinear PDEs. The goal is to obtain exact soliton solutions for mathematically modelled differential equations. Different mathematical techniques have been developed. For the observation of physical activities of the problem, exact solutions are vital. We have more applications and the ability to examine the number of properties of the mathematical model by utilizing the exact solution.

Nonlinear equations play a very vital part in a variety of engineering and scientific arenas, such as, heat flow, quantum mechanics, solid state physics, chemical kinematics, fluid mechanics, optical fibers, plasma physics, the wave proliferation phenomena and proliferation of shallow water waves.

Therefore, different techniques for finding exact solutions are used for a diversified field of partial differential equations for example, the homogeneous balance technique [15, 25], Hirota's bilinear approach [7, 8], the auxiliary equivalence technique [14], the trial task technique [9], the jacobi elliptic task system [5], the tanh-function technique [2], method of sine- cosine [17], the truncated Painleve expansion technique [18], the variational iteration method (VIM) [1], the exp-function technique [3, 6], the (G'/G)-expansion approach [4, 13, 16, 23, 24, 26], and the exact soliton solution [10, 11, 27]. Several theoretical and experimental research for solitons is described in, [28–32]. For the exact solution some novel results and computational methods involving the travelling-wave transformation are described in, [12, 19–22]. In recent years, soliton theory has attracted the attention of scientists from different areas of research as [33–43].

In this paper, our basic aim is to discuss the physics of the nonlinear Caudrey-Dodd-Gibbon and Pochhammer-Chree equations, and also to obtain soliton like solutions of these equations through an established method that is notable in literature as the $\exp(-\phi(\zeta))$-expansion technique. From the technique, the solution procedure is quite simple and all types of nonlinear evolution equations are

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easily expanded. This technique simply emphasises that to obtain the solution of the PDE that is indicated in the kind of polynomial in $\exp(-\phi(\zeta))$, $\phi(\zeta)$ must satisfies the ODE:

$$\phi'(\zeta) = \exp(-\phi(\zeta)) + \mu \exp(\phi(\zeta)) + \lambda. \tag{1}$$

We know $\zeta = x + y + z - \omega t$.

By the homogenous principle, the degree of the polynomial is obtained. We obtain a set of algebraic equations by balancing the highest order derivative with nonlinear terms.

The article is divided in to various segments. First, we study the analysis of the method. Then we discuss the physics of nonlinear equations for the application of the $\exp(-\phi(\zeta))$-expansion method. We conclude by discussing the results.

2 *Exp*$(-\phi(\zeta))$-expansion Technique

In general, a nonlinear partial differential equation (NPDE) can be written as:

$$P\left(\eta, \eta_x, \eta_y, \eta_z, \eta_{xx}, \eta_{xy}, \eta_{xz}, \ldots\right) = 0. \tag{2}$$

Where $\eta(x, y, z, t)$ is unknown, P is the polynomial in $\eta(x, y, z, t)$ and different derivatives of $\eta(x, y, z, t)$ involving nonlinear terms and the highest order differential. Using the $\exp(-\phi(\zeta))$-expansion method, we follow the detailed steps given below:

Step 1: Invoking a wave transformation:

$$\zeta = x + y + z - \omega t. \tag{3}$$

Here ω represents wave speed. Applying the wave transformation into equation (2) we get an ODE:

$$Q(\eta, \eta', \eta'', \eta''', \ldots) = 0. \tag{4}$$

In equation (4) the prime denotes the derivative w.r.t ζ. If it is needed, integrate equation (4) and set the constant of integration equal to zero.

Step 2: The solution of equation (4) is expressed in the form of a polynomial in $\exp(-\phi(\zeta))$ as:

$$\eta(\zeta) = a_n(\exp(-\phi(\zeta)))^n + a_{n-1}(\exp(-\phi(\zeta)))^{n-1} + \cdots, \tag{5}$$

In equation (5) $a_n, a_{n-1}, \ldots$ are arbitrary constants which are to be evaluated such that $a_n \neq 0$. Also $\phi(\zeta)$ satisfies equation (1).

Step 3: Calculate the value of n by using the homogeneous balance principle. There are five cases and in all these cases c_1 is a constant of integration.

Case 1: For $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$,

$$\phi(\zeta) = \ln\left\{\frac{1}{2\mu}\left(-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\zeta + c_1)\right) - \lambda\right)\right\}. \tag{6}$$

Case 2: For $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$,

$$\phi(\zeta) = \ln\left\{\frac{1}{2\mu}\left(-\lambda + \sqrt{4\mu - \lambda^2}\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\zeta + c_1)\right)\right)\right\}. \tag{7}$$

Case 3: For $\lambda \neq 0$ and $\mu = 0$,

$$\phi(\zeta) = -\ln\left\{\frac{\lambda}{(\exp(\lambda(\zeta + c_1)) - 1)}\right\}. \tag{8}$$

Case 4: For $\lambda^2 - 4\mu = 0$ and $\mu \neq 0$, $\lambda \neq 0$,

$$\phi(\zeta) = \ln\left\{\frac{2(\lambda(\zeta + c_1)) + 2}{\lambda^2(\zeta + c_1)}\right\}. \tag{9}$$

Case 5: For $\lambda = 0$ and $\mu = 0$,

$$\phi(\zeta) = \ln(\zeta + c_1). \tag{10}$$

Step 4: Insert the polynomial given in equation (5) into equation (4) and then use equation (1). The left hand side of the nonlinear ODE is converted into the form of a polynomial in $\exp -(-\phi(\zeta))$. We set each coefficient of the polynomial equal to zero, which results in a set of algebraic equations for $a_n, \ldots, \omega, \lambda$ and μ.

Step 5: With the help of symbolic computation software like Maple 18, the values of the constants $a_n, \ldots, \omega, \lambda, \mu$ are computed. Substituting the computed values into equation (5), results in the soliton wave solutions.

3 Solution Procedure

This section is divided into two sub sections. Both Caudrey-Dodd-Gibbon and Pochhammer-Chree equations are studied physically, then the $\exp(-\phi(\zeta))$-expansion technique is applied to obtain solitary wave solutions of these nonlinear differential equations.

3.1 Caudrey-Dodd-Gibbon Equation

The physics of the nonlinear Caudrey-Dodd-Gibbon (CDG) equation is discussed in this section. Solitary wave solutions are also constructed via the $\exp(-\phi(\zeta))$-expansion method.

The general form of the fifth order Korteweg–de Vries equation is written as:

$$\eta_t + \alpha\eta_{xxxxx} + \sigma\eta\eta_{xxx} + \gamma\eta_x\eta_{xx} + \delta\eta^2\eta_x = 0. \quad (11)$$

In equation (11) α, σ, γ and δ are arbitrary real parameters. By setting $\alpha = 1$, $\sigma = \beta$, $\gamma = \beta$ and $\delta = \frac{1}{5}\beta^2$ we obtain,

$$\eta_t + \frac{1}{5}\beta^2\eta^2\eta_x + \beta\eta_x\eta_{xx} + \beta\eta\eta_{xxx} + \eta_{xxxxx} = 0. \quad (12)$$

Which is known as the CDG equation. This equation describes the evolution of quasi one dimensional shallow water waves when it affects the surface tension; and the viscosity is negligible. Shallow water waves are produced when the depth of water is less than one half of the wavelength of the wave. Their speed is independent of their wavelength too. It depends, however, on the depth of the water. Shallow water waves show no dispersion.

Introducing a transformation as

$$\zeta = x + y + z - \omega t. \quad (13)$$

$$-\omega\eta' + \frac{1}{5}\beta^2\eta^2\eta' + \beta\eta'\eta'' + \beta\eta\eta''' + \eta^{(v)} = 0.$$

On integrating we have,

$$A - \omega\eta + \frac{1}{15}\beta^2\eta^3 + \beta\eta\eta'' + \eta^{(iv)} = 0. \quad (14)$$

Here A is the constant of integration and prime symbolizes the derivative w.r.t.ζ. To determine the value of n, we balance the highest order linear term with the highest order non-linear term of equation (14). We obtain $n = 2$. So equation (5) reduces to:

$$\eta(\zeta) = a_0 + a_1 e^{-\phi(\zeta)} + \left(e^{-\phi(\zeta)}\right)^2. \quad (15)$$

Here a_0 and a_1, a_2 are the constants which are to be calculated.

By making use of (15) into (14), we transformed the left hand side into a polynomial in $e^{-\phi(\zeta)}$. We set each coefficient of this polynomial equal to zero, and obtain a set of algebraic equations for $a_0, a_1, a_2, \lambda, \mu, A$ and ω as follows:

$$-\omega a_1 + \frac{1}{5}\beta^2 a^2 a_1 + 22a_1\mu\lambda^2 + 120a_2\mu^2\lambda + 30a_2\lambda^3\mu + 6\beta a_0 a_2\mu\lambda + \beta a_0 a_1\lambda^2 + 2\beta a a_1\mu + 2\beta a_1 a_2\mu^2 + \beta a_1^2\mu\lambda + 16a_1\mu^2 + a_1\lambda^4 = 0.$$

$$\frac{1}{5}\beta^2 {a_0}^2 a_2 + \frac{1}{5}\beta^2 {a_1}^2 a_0 + 2\beta {a_1}^2\mu + 60a_1\mu\lambda + 232a_2\lambda^2\mu + 2\beta a_2^2\mu^2 + \beta a_1^2\lambda^2 + 7\beta a_1 a_2\mu\lambda + 8\beta a_0 a_2\mu + 3\beta a_0 a_1\lambda + 4\beta a_0 a_2\lambda^2 + 136a_2\mu^2 + 16a_2\lambda^4 - \omega a_2 + 15a_1\lambda^3 = 0.$$

$$440a_2\mu\lambda + 2\beta a_0 a_1 + 3\beta a_1^2\lambda + \frac{2}{5}\beta^2 a_0 a_1 a_2 + 10\beta a_1 a_2\mu + 6\beta a_2^2\mu\lambda + 10\beta a_0 a_2\lambda + 5\beta a_1 a_2\lambda^2 + \frac{1}{15}\beta^2 a_1^3 + 40a_1\mu + 130a_2\lambda^3 + 50a_1\lambda^2 = 0.$$

$$\frac{1}{5}\beta^2 a_0 a_2^2 + \frac{1}{5}\beta^2 a_1^2 a_2 + 8\beta a_2^2\mu + 6\beta a_0 a_2 + 4\beta a_2^2\lambda^2 + 13\beta a_1 a_2\lambda + 240a_2\mu + 330a_2\lambda^2 + 2\beta a_1^2 + 60a_1\lambda = 0.$$

$$\frac{1}{5}\beta^2 a_1 a_2^2 + 8\beta a_1 a_2 + 10\beta a_2^2\lambda + 336a_2\lambda + 24a_1 = 0.$$

$$6\beta a_2^2 + \frac{1}{15}\beta^2 a_2^3 + 120a_2 = 0.$$

$$A - \omega a_0 + \frac{1}{15}\beta^2 a_0^3 + a_1\mu\lambda^3 + 8a_1\mu^2\lambda + 14a_2\mu^2\lambda^2 + 16a_2\mu^3 + \beta a_0 a_1\mu\lambda + 2\beta a_0 a_2\mu^2 = 0.$$

With the help of symbolic computation software like Maple 18, the algebraic equations are solved. Finally, we obtain the following two solution sets:

1st Solution Set:

Consider the solution set of the form

$$\omega = \frac{1}{5}\beta^2 a_0^2 + \beta a_0\lambda^2 + 22\mu\lambda^2 + 76\mu^2 + \lambda^4 + 8\beta a_0\mu,$$

$$a_0 = a_0,\ a_1 = -\frac{30\lambda}{\beta},\ a_2 = -\frac{30}{\beta},$$

$$A = \frac{1}{15\beta}\begin{pmatrix} 2\beta^3 a_0^3 + 15\beta^2 a_0^2\lambda^2 \\ +780a_0\lambda^2\mu\beta + 2040a_0\mu^2\beta \\ +15\beta a_0\lambda^4 + 120\beta^2 a_0^2\mu \\ +450\mu\lambda^4 + 9900\mu^2\lambda^2 + 7200\mu^3 \end{pmatrix}.$$

By inserting the above values into equation (15), we obtain:

$$\eta = -\frac{-\beta a_0 + 30e^{-\phi(\zeta)}\lambda + 30e^{-2\phi(\zeta)}}{\beta}. \quad (16)$$

Where $\zeta = x + y + z - \omega t$.

Inserting the solutions of equation (1) into equation (16), we obtain five cases of soliton wave solutions for the CDG equation.

Case 6: When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, we obtain solution,

$$\eta_{11} = \frac{1}{\beta}\left(\beta a_0 - \frac{(60\mu\lambda)}{-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,(\zeta + c_1)\right) - \lambda} - \frac{(120\mu^2)}{\left(-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,(\zeta + c_1)\right) - \lambda\right)^2}\right). \quad (17)$$

Case 7: When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$, then we have,

$$\eta_{12} = \frac{1}{\beta}\left(\beta a_0 - \frac{(60\mu\lambda)}{\sqrt{-\lambda^2+4\mu}\tan\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}(\zeta+c_1)\right)-\lambda} - \frac{(120\mu^2)}{\left(\sqrt{-\lambda^2+4\mu}\tanh\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}(\zeta+c_1)\right)-\lambda\right)^2}\right). \tag{18}$$

Case 8: When $\mu = 0$ and $\lambda \neq 0$, we obtain,

$$\eta_{13} = \frac{1}{\beta}\left(\beta a_0 - \frac{30\lambda^2}{\exp(\lambda(\zeta+c_1))-1} - \frac{30\lambda^2}{(\exp(\lambda(\zeta+c_1))-1)^2}\right). \tag{19}$$

Case 9: When $\lambda^2 - 4\mu = 0$, $\mu = 0$ and $\lambda \neq 0$,

$$\eta_{14} = -\frac{1}{\beta}\left(-\beta a_0 + \frac{30\lambda^3(\zeta+c_1)}{-4+(-2\zeta-2c_1)\lambda} + \frac{30\lambda^4(\zeta+c_1)^2}{(-4+(-2\zeta-2c_1)\lambda)^2}\right). \tag{20}$$

Case 10: When $\mu = 0$ and $\lambda = 0$, we obtain,

$$\eta_{15} = \frac{1}{\beta}\left(\beta a_0 - 30e^{-2\ln(\zeta+c_1)}\right). \tag{21}$$

Where $\zeta = x - \left(\frac{1}{5}\beta^2 a_0^2 + \beta a_0\lambda^2 + 22\mu\lambda^2 + 76\mu^2 + \lambda^4 + 8\beta a_0\mu\right)t$.

2^{nd} Solution Set:

Consider the solution set of the form:

$$\omega = \lambda^4 - 9\mu\lambda^2 + 16\mu^2, \quad a_0 = \frac{5(\lambda^2+8\mu)}{\beta}, \quad a_1 = -\frac{60\lambda}{\beta}, \quad a_2 = -\frac{60}{\beta}, \quad A = -\frac{10}{3\beta}\left(-\lambda^6 - 48\mu^2\lambda^2 + 64\mu^3 + 12\mu\lambda^4\mu\right). \tag{22}$$

Substituting values from equation (22) into equation (15), we obtain

$$\eta = -\frac{5\left(\lambda^2 + 8\mu + 12e^{-\varphi(\zeta)}\lambda + 12e^{-2\varphi(\zeta)}\right)}{\beta}. \tag{23}$$

Where $\zeta = x + y + z - \omega t$.

Inserting the solutions of equation (1) into equation (23), we get five cases of soliton wave solutions for the Caudrey-Dodd-Gibbon equation:

Case 11: When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, we obtain the solution,

$$\eta_{21} = \frac{1}{\beta}\left(-5\lambda^2 - 40\mu - \frac{(120\mu\lambda)}{-\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(\zeta+c_1)\right)-\lambda} - \frac{(240\mu^2)}{\left(-\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(\zeta+c_1)\right)\lambda\right)^2}\right). \tag{24}$$

Case 12: When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$, we obtain the solution,

$$\eta_{22} = \frac{1}{\beta}\left(-5\lambda^2 - 40\mu - \frac{(120\mu\lambda)}{\sqrt{-\lambda^2+4\mu}\tan\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}(\zeta+c_1)\right)-\lambda} - \frac{(240\mu^2)}{\left(\sqrt{-\lambda^2+4\mu}\tanh\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}(\zeta+c_1)\right)-\lambda\right)^2}\right). \tag{25}$$

Case 13: When $\mu = 0$ and $\lambda \neq 0$, we obtain the solution,

$$\eta_{23} = \frac{1}{\beta}\left(-5\lambda^2 - \frac{60\lambda^2}{\exp(\lambda(\zeta+c_1))-1} - \frac{60\lambda^2}{(\exp(\lambda(\zeta+c_1))-1)^2}\right). \tag{26}$$

Case 14: When $\lambda^2 - 4\mu = 0$, $\mu = 0$ and $\lambda \neq 0$, we have the solution,

$$\eta_{24} = -\frac{1}{\beta}\left(5\left(\lambda^2 + 8\mu + \frac{12\lambda^3(\zeta+c_1)}{-4+(-2\zeta-2c_1)\lambda} + \frac{12\lambda^4(\zeta+c_1)^2}{(-4+(-2\zeta-2c_1)\lambda)^2}\right)\right). \tag{27}$$

Case 15: When $\mu = 0$ and $\lambda = 0$, we obtain the solution,

$$\eta_{25} = -\frac{60}{\beta(\zeta+c_1)^2} \tag{28}$$

Here $\zeta = x - \left(\lambda^4 - 8\mu\lambda^2 + 16\mu^2\right)t$.

3.2 Pochhammer-Chree Equation

In this section, the physics of the nonlinear Pochhammer-Chree equation is studied and then soliton wave solutions are obtained.

The generalized Pochhammer-Chree equation is given by,

$$\eta_{tt} - \eta_{ttxx} - \left(a\eta - b\eta^{n+1} - \gamma\eta^{2n+1}\right)_{xx} = 0.$$

Taking $n = 2$, we have,

$$\eta_{tt} - \eta_{ttxx} - \left(a\eta - b\eta^{3} - \gamma\eta^{5}\right)_{xx} = 0.$$

Where a, b and γ are arbitrary non-zero constants, while the exponent $n\,(> 1)$ is the power law nonlinearity parameter. This equation represents a nonlinear model of longitudinal wave propagation of elastic rods. Longitudinal waves are waves in which the displacement of the medium is in the same direction as, or in the opposite direction to, the direction of propagation of the wave. Longitudinal waves include sound waves and particle velocity propagated in an elastic medium.

Considering $\gamma = 0$, the above equation becomes,

$$\eta_{tt} - \eta_{ttxx} - \left(a\eta - b\eta^{3}\right)_{xx} = 0. \tag{29}$$

Introducing the transformation $\zeta = x+y+z-\omega t$, equation (29) can be converted to an ordinary differential equation.

$$\omega^2\eta'' - \omega^2\eta'''' - a\eta'' + 6b\eta(\eta')^2 + 3b\eta^2\eta'' = 0$$

Integrating twice, we have,

$$A + B\xi + \omega^2\eta - a\eta + b\eta^3 - \omega^2\eta'' = 0 \tag{30}$$

Here A and B are the constants of integration and prime symbolizes the derivative w.r.t.ζ. Now for the value of n, we balance the highest order linear term with the highest order non-linear term of equation (30). We obtain $n = 1$. So equation (5) reduces to,

$$\eta(\zeta) = a_0 a_1 e^{-\varphi(\zeta)}. \tag{31}$$

Here a_0 and a_1 are the constants which are to be calculated.

By substituting equation (31) into equation (30), we transform the left hand side into a polynomial in $e^{-\phi(\zeta)}$. We set each coefficient of this polynomial equal to zero, and obtain a set of algebraic equations for $a_0, a_1, a_2, \lambda, \mu, A, B$ and ω as follows:

$$a_1\omega^2 - aa_1 + 3ba_0^2a_1 - 2a_1\omega^2\mu - a_1\omega^2\lambda^2 = 0.$$
$$3ba_1^2\alpha_0 - 3a_1\omega^2\lambda = 0.$$
$$ba_1^3 - 2a_1\omega^2 = 0.$$
$$A + B\zeta + a_0\omega^2 - aa_0 + ba_0^3 - a_1\omega^2\mu\lambda = 0.$$

After solving these algebraic equations with the help of computer software like Maple 18, we obtain the solution set:

$$\omega = -\frac{\sqrt{-(-4-2\lambda^2+8\mu)\mathrm{a}}}{-2-\lambda^2+4\mu}, \quad A = -B\zeta, \quad B = B, \tag{32}$$

$$a_0 = -\frac{\mathrm{a}\lambda}{\sqrt{-b(-2-\lambda^2+4\mu)}}, \quad a_1 = \frac{2\sqrt{-b(-2-\lambda^2+4\mu)\mathrm{a}}}{b(-2-\lambda^2+4\mu)}$$

By substituting equation (32) into equation (31), we obtain,

$$\eta = -\frac{\left(\lambda + 2e^{-\varphi(\zeta)}\right)\mathrm{a}}{\sqrt{-b(-2-\lambda^2+4\mu)\mathrm{a}}}. \tag{33}$$

Where $\zeta = x+y+z-\omega t$.

Inserting the solutions of equation (1) into equation (33), we get five cases of travelling wave solutions for the PC equation (29).

Case 16: When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, we obtain the soliton wave solution,

$$\eta_1 = -\frac{1}{\sqrt{-b(-2-\lambda^2+4\mu)\mathrm{a}}}\left(\left(\lambda + \frac{4\mu}{-\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(\zeta+c_1)\right)-\lambda}\right)\mathrm{a}\right). \tag{34}$$

Case 17: When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$, we obtain,

$$\eta_2 = -\frac{1}{\sqrt{-b(-2-\lambda^2+4\mu)\mathrm{a}}}\left(\left(\lambda + \frac{4\mu}{\sqrt{\lambda^2-4\mu}\tan\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\,(\zeta+c_1)\right)-\lambda}\right)\mathrm{a}\right). \tag{35}$$

Case 18: When $\mu = 0$ and $\lambda \neq 0$, we obtain,

$$\eta_3 = -\frac{\mathrm{a}}{\sqrt{b(2+\lambda^2)\mathrm{a}}}\left(\lambda + \frac{2\lambda}{\exp\left(\lambda(\zeta+c_1)\right)-1}\right). \tag{36}$$

Case 19: When $\lambda^2 - 4\mu = 0$, $\mu = 0$ and $\lambda \neq 0$, we obtain,

$$\eta_4 = -\frac{2\lambda\mathrm{a}}{\sqrt{-b(-2-\lambda^2+4\mu)\mathrm{a}}\,(\lambda\,(\zeta+c_1))+2}. \tag{37}$$

Case 20: When $\mu = 0$ and $\lambda = 0$, we obtain the travelling wave solution,

$$\eta_5 = -\frac{\mathrm{a}\sqrt{2}}{(\zeta+c_1)\sqrt{b\mathrm{a}}}. \tag{38}$$

Here $\zeta = x + \frac{\sqrt{-(-4-2\lambda^2+8\mu)\mathrm{a}}}{-2-\lambda^2+4\mu}t$.

4 Results and Discussion

The construction of soliton waves by solving nonlinear Caudrey-Dodd-Gibbon (CDG) and Pochhammer-Chree (PC)

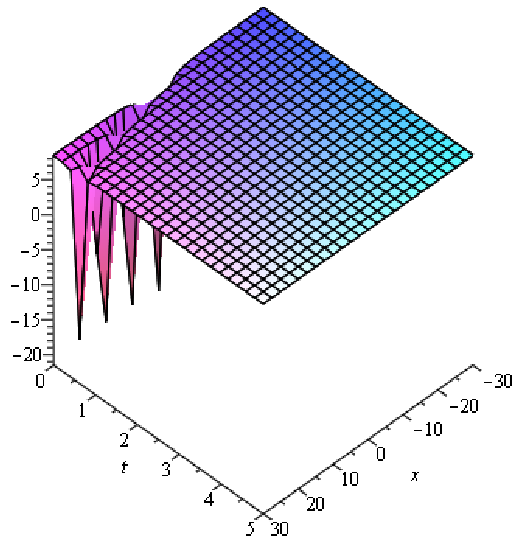

Figure 1: 3D plot for $\lambda = 0.2,\ \mu = 1,\ c_1 = 2,\ a_0 = 1, -30 \le x \le 30,\ 0 \le t \le 5$.

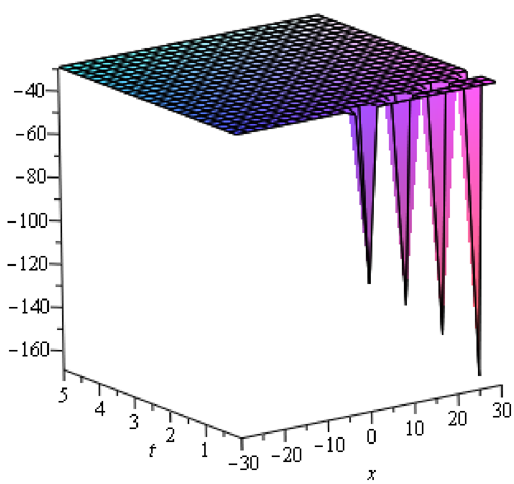

Figure 2: 3D plot for $\lambda = 0.2,\ \mu = 1,\ c_1 = 2,\ a_0 = 1, -30 \le x \le 30,\ 0 \le t \le 5$.

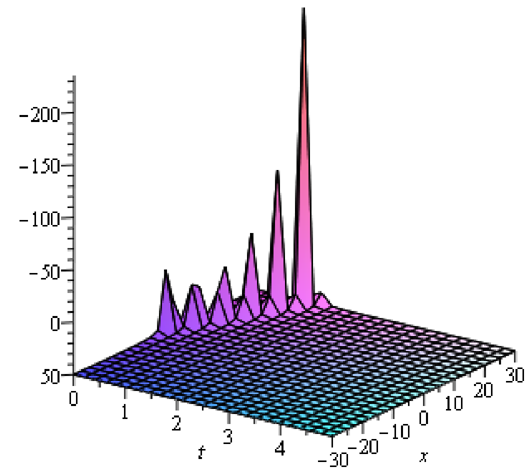

Figure 3: 3D plot for $\lambda = 1,\ \mu = -1,\ c_1 = 0.1,\ a_0 = 2, -30 \le x \le 30,\ 0 \le t \le 4$.

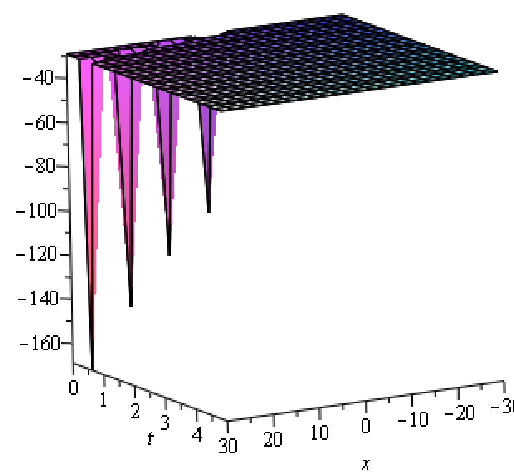

Figure 4: 3D plot for $\lambda = 1,\ \mu = -1,\ c_1 = 0.1,\ a_0 = 2, -30 \le x \le 30,\ 0 \le t \le 4$.

equations have been examined via an analytical technique, the $\exp(-\phi(\zeta))$-expansion method. The findings are summarised and discussed in this section.

Solitary waves arise due to an indirect balance of nonlinear effects with dispersive effects. From the above graphs, we are able to judge that a soliton is a wave which preserves its shape after it strikes another wave of a similar kind. The waves produced by linear description tend to experience dispersion and create a localized disturbance in the spreading. The solitary waves can intuitively be anticipated to be the outcome of two effects of steepening and spreading with a marginally nonlinear amplitude. The waves of various type of wave-number, being generated without changing their actual shapes. The wave speed and amplitude depends upon the process of dispersion. We have observed that some media undergo strong dispersion and generate high amplitude waves while weak dispersive media generate waves of small amplitude.

We obtained the desired solution through rational functions. The hyperbolic and trigonometric function travelling wave solutions of the nonlinear Caudrey-Dodd-Gibbon equation are shown in Figure 1 and Figure 2 respectively for different values of physical parameters. Figure 3 and Figure 4 show soliton solutions for various physical parameters through exponential and rational functions. Figure 5 shows rational function solution for different values of λ, μ and c_1. In these graphical results, $\beta = 30$. By changing the values of physical and additional free parameters, the velocity and amplitude of solitary waves are controlled. It is observed that the displacement potential η be-

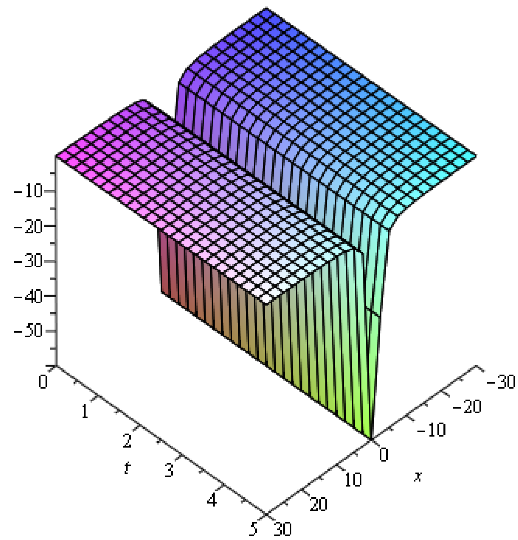

Figure 5: 3D plot for $\lambda = -1,\ \mu = 2,\ c_1 = 1, -30 \le x \le 30,\ 0 \le t \le 5$.

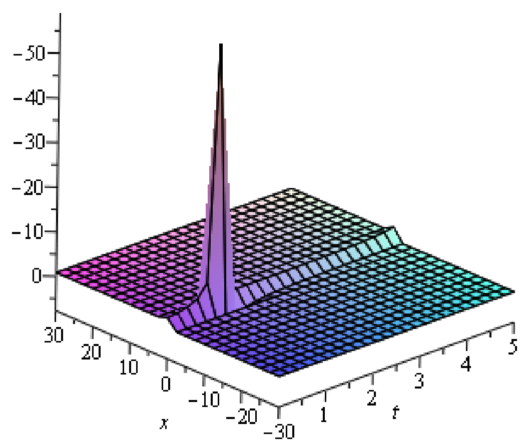

Figure 6: 3D plot for $\lambda = -1,\ \mu = 2,\ c_1 = 0.1,\ a = 3, b = 1, -30 \le x \le 30,\ 0 \le t \le 5$.

comes sharp at leading and trailing edges. The amplitude is proportional to the velocity of propagation and taller solitary waves are thinner and move faster.

Figure 6 and 7 shows hyperbolic and trigonometric function travelling wave solutions of the Pochhammer-Chree equation respectively by setting suitable values of physical parameters which control the solitary wave amplitude. Figure 8 and Figure 9 show the exponential and rational function solutions respectively. Figure 10 shows the rational function solution for different values of a, c_1 and b. The solitary wave moves towards right if the velocity is positive and towards the left if the velocity is negative. The amplitudes and velocities are controlled by various physical parameters. Solitary waves show more complicated behaviours which are controlled by various physical and additional free parameters. Figures indicate graphical solu-

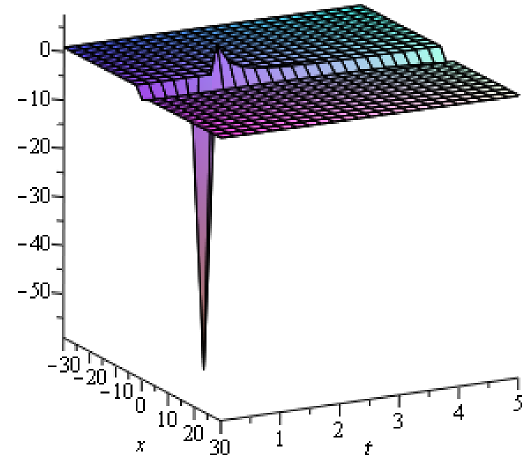

Figure 7: 3D plot for $\lambda = -1,\ \mu = 2,\ c_1 = 0.1,\ a = 3, b = 1, -30 \le x \le 30,\ 0 \le t \le 5$.

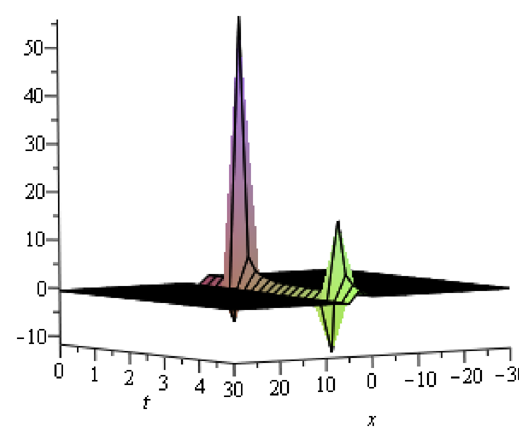

Figure 8: 3D plot for $\lambda = 1,\ \mu = 2,\ a = 0.1,\ c_1 = 1, b = 3, -30 \le x \le 30,\ 0 \le t \le 4$.

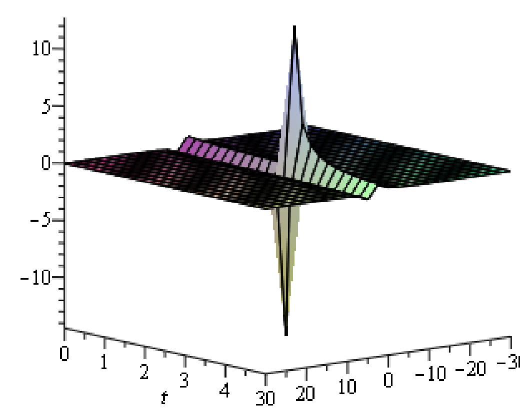

Figure 9: 3D plot for $\lambda = 1,\ \mu = 2, a = 0.1,\ \ c_1 = 1, b = 3, -30 \le x \le 30,\ 0 \le t \le 4$.

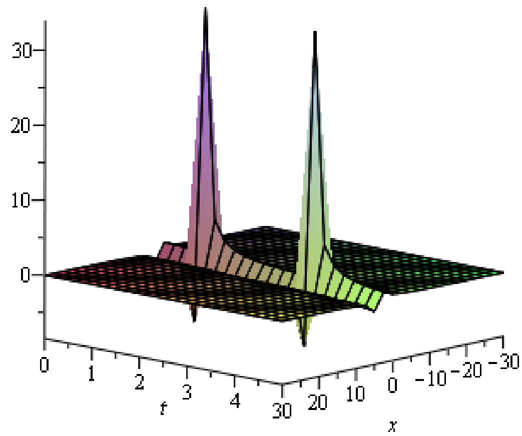

Figure 10: 3D plot for $a = 0.1,\ b = 1,\ c_1 = 2, -30 \le x \le 30,\ 0 \le t \le 4$.

tions for altered values of physical parameters. Arbitrary functions can be affected by the solitary wave solution. It is concluded that various constraints can be selected as an input to our simulation. Solitary waves for various values of physical and additional free parameters are highlighted by the figures. From the above discussed graphical cases it has been observed that the solution of soliton waves does not rely completely upon the additional free parameters. The soliton waves of different types are clearly described by the graphical outcomes.

5 Conclusion

In this paper, the main focus was to find, test and analyze the new travelling wave solutions and physical properties of nonlinear Caudrey-Dodd-Gibbon and Pochhammer-Chree equations by applying a reliable mathematical technique. It is noted that these nonlinear differential equations exhibits soliton type solutions. The applied algorithm is helpful to verify the results that are acquired by the exact solution. The obtained results from this method reveals that for solving nonlinear differential equations, it work as a powerful method. CDG and PC equations have a very important part in the theory of solitary waves as they open up a variety of aspects of the solitary wave solution. To get a grip on these equations we use an established expansion technique as a tool. The method has been applied directly without the need for linearization, discretization, or perturbation. The obtained results demonstrate the reliability of the algorithm and give it a wider applicability to nonlinear differential equations. Using the Maple software permits us to determine more solution sets, with less computational work and less computer memory.

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