Research Article

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Integrability, exact solutions and nonlinear dynamics of a nonisospectral integral-differential system

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Abstract: The investigations of integrability, exact solutions and dynamics of nonlinear partial differential equations (PDEs) are vital issues in nonlinear mathematical physics. In this paper, we derive and solve a new Lax integrable nonisospectral integral-differential system. To be specific, we first generalize an eigenvalue problem and its adjoint equation by equipping it with a new time-varying spectral parameter. Based on the generalized eigenvalue problem and the adjoint equation, we then derive a new Lax integrable nonisospectral integral-differential system. Furthermore, we obtain exact solutions and their reduced forms of the derived system by extending the famous nonlinear Fourier analysis method–inverse scattering transform (IST). Finally, with graphical assistance we simulate a pair of reduced solutions, the dynamical evolutions of which show that the amplitudes of solutions vary with time.

Keywords: Integral-differential system; Integrability; Exact solution; Dynamical evolution; IST

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1 Introduction

Nonlinear PDEs are important mathematical models describing some nonlinear natural phenomena like those in physics, biology, chemistry and mechanics. In the field of nonlinear science, the investigation of integrability, exact solutions and dynamics of nonlinear PDEs has attracted much attention [1–15]. There is no uniform definition for the integrability of PDEs. Usually, it is necessary to indicate the type of integrability [16]. In soliton theory, the nonlinear PDEs are classified two types: the isospectral equations and the nonisospectral equations. Generally speaking, the spectral parameters of the eigenvalue problems associated with isospectral equations are time-independent. Otherwise when the spectral parameters are dependent of time, the associated equations are nonisospectral.

The purpose of this article is to study the integrability, exact solutions and nonlinear dynamics of the following new nonisospectral integral-differential system:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -q_{xx} + 2q^2r + 3q_{xx} + xq_{xxx} - 2q_x\partial^{-1}(qr) \\ -4xqrq_x - 2xq^2r_x - 8q\partial^{-1}(q_xr) \\ -2xqrq_{xx} + 2xq^2r_{xx} - 2q_x - xq_{xx} \\ +2q\partial^{-1}(qr) + 2xq^2r + q + xq_x - xq \\ \\ r_{xx} - 2qr^2 + 3r_{xx} + xr_{xxx} - 2r_x\partial^{-1}(qr) \\ -4xqrr_x - 2xr^2q_x - 8q\partial^{-1}(qr_x) \\ -2xqrr_{xx} + 2xr^2q_{xx} + 2r_x + xr_{xx} \\ -2r\partial^{-1}(qr) - 2xqr^2 + r + xr_x + xr \end{pmatrix} \qquad (1)$$

In Section 2, we derive the integral-differential system (1) and prove its Lax integrability. In Section 3, we construct exact solutions and their reduced forms of system (1) by equipping the IST method [17–19] with a new time-varying spectral parameter. In Section 4, a pair of reduced solutions are simulated to offer an insight into the dynamical evolutions of solutions.

2 Derivation and Lax Integrability

To derive system (1), firstly we equip the eigenvalue problem [20]:

$$\phi_x = M\phi, \quad M = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \qquad (2)$$

and its adjoint equation

$$\phi_t = N\phi, \quad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \qquad (3)$$

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with a new spectral parameter k satisfying

$$ik_t = \frac{1}{2}\left[\sum_{n=0}^{3}(2ik)^n\right]. \quad (4)$$

In Eqs. (2) and (3), the potential functions $q = q(x,t) \to 0$, $r = r(x,t) \to 0$ as $x \to \pm\infty$, all the derivatives of q and r have the same asymptotic properties, and $A = A(x,t,k,q,r)$, $B = B(x,t,k,q,r)$ and $C = C(x,t,k,q,r)$ are functions of the indicated variables to be determined later.

Secondly, using the compatibility condition of Eqs. (2) and (3) we have

$$A_x = qC - rB - ik_t, \quad (5)$$

$$q_t = B_x + 2ikB + 2qA, \quad (6)$$

$$r_t = C_x - 2ikC - 2rA. \quad (7)$$

In view of Eqs. (4) and (5), we suppose that

$$A = \partial^{-1}(r,q)\begin{pmatrix}-B\\ C\end{pmatrix} - \frac{1}{2}(2ik)^2 - \frac{1}{2}\left[\sum_{n=0}^{3}(2ik)^n\right]x. \quad (8)$$

Then Eqs. (5)-(7) can be simplified as

$$\begin{pmatrix}q\\ r\end{pmatrix}_t = L\begin{pmatrix}-B\\ C\end{pmatrix} - 2ik\begin{pmatrix}-B\\ C\end{pmatrix} + (2ik)^2\begin{pmatrix}-q\\ r\end{pmatrix} + \sum_{m=0}^{3}(2ik)^m\begin{pmatrix}-xq\\ xr\end{pmatrix}, \quad (9)$$

by the usage of the operator

$$L = \begin{pmatrix}-\partial & 0\\ 0 & \partial\end{pmatrix} + 2\begin{pmatrix}q\\ -r\end{pmatrix}\partial^{-1}(r,q), \quad \partial = \frac{\partial}{\partial x}, \quad (10)$$

$$\partial^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{+\infty}\right)\mathrm{d}x.$$

Thirdly, we let

$$\begin{pmatrix}-B\\ C\end{pmatrix} = \sum_{m=1}^{3}\begin{pmatrix}-b_m\\ c_m\end{pmatrix}(2ik)^{3-m}, \quad (11)$$

then the coefficients of $2ik$ in Eq. (9) give

$$(2ik)^0: \quad \begin{pmatrix}q\\ r\end{pmatrix}_t = L\begin{pmatrix}-b_3\\ c_3\end{pmatrix} + \begin{pmatrix}-xq\\ xr\end{pmatrix}, \quad (12)$$

$$(2ik)^1: \quad \begin{pmatrix}-b_3\\ c_3\end{pmatrix} = L\begin{pmatrix}-b_2\\ c_2\end{pmatrix} + \begin{pmatrix}-xq\\ xr\end{pmatrix}, \quad (13)$$

$$(2ik)^2: \quad \begin{pmatrix}-b_2\\ c_2\end{pmatrix} = L\begin{pmatrix}-b_1\\ c_1\end{pmatrix} + \begin{pmatrix}-q\\ r\end{pmatrix} + \begin{pmatrix}-xq\\ xr\end{pmatrix}, \quad (14)$$

$$(2ik)^3: \quad \begin{pmatrix}-b_1\\ c_1\end{pmatrix} = \begin{pmatrix}-xq\\ xr\end{pmatrix}. \quad (15)$$

Finally, substituting Eqs. (13)-(15) into Eq. (12) yields system (1). This shows that system (1) is Lax integrable.

3 Exact solutions and their reductions

This section has three aspects: the first aspect is to determine the scattering data for the generalized eigenvalue problem (2); the second aspect is to construct exact solutions of system (1); and the last aspect is to reduce the obtained exact solutions.

3.1 Scattering data

Theorem 1. *The generalized eigenvalue problem (2) has the following scattering data:*

$$\kappa_{jt}(t) = -\frac{i}{2}\sum_{n=0}^{3}\left[2i\kappa_j(t)\right]^n \quad (16)$$
$$c_j^2(t) = c_j^2(0)e^{\int_0^t\left[\sum_{s=1}^{3}s(2i\kappa_j(w))^{s-1}-4\kappa_j^2(w)\right]dw},$$
$$R(t,k) = R(0,k)e^{-4\int_0^t k^2(\tau)d\tau}$$

$$\overline{\kappa}_{mt}(t) = -\frac{i}{2}\sum_{n=0}^{3}\left[2i\overline{\kappa}_m(t)\right]^n \quad (17)$$
$$\overline{c}_m^2(t) = \overline{c}_m^2(0)e^{-\int_0^t\left[\sum_{s=1}^{3}s(2i\overline{\kappa}_m(w))^{s-1}-4\overline{\kappa}_m^2(w)\right]dw},$$
$$\overline{R}(t,k) = \overline{R}(0,k)e^{4\int_0^t \overline{k}^2(\tau)d\tau}$$

for $j = 1, 2, \cdots, n$ and $m = 1, 2, \cdots, \overline{n}$, here $c_j(0)$, $\overline{c}_m(0)$, $R(0,k)$ and $\overline{R}(0,k)$ are the scattering data of the generalized eigenvalue problem (2) in the case of $q(0,x)$ and $r(0,x)$.

Proof. When $\phi(x,k)$ satisfies Eq. (2), $\phi_t(x,k) - N\phi(x,k)$ also satisfies Eq. (2), then one has

$$\phi_t(x,k) - N\phi(x,k) = \rho(t,k)\phi(x,k) + \theta(t,k)\tilde{\phi}(x,k). \quad (18)$$

where $\phi(x,k)$ and $\tilde{\phi}(x,k)$ are a pair of fundamental solutions of Eq. (2), $\rho(t,k)$ and $\theta(t,k)$ are undetermined coefficient functions.

Firstly, we begin with the spectral parameter $k = \kappa_j(t)(\mathrm{Im}\kappa_j > 0)$ of the discrete case. Then we must have $\theta(t,k) = 0$ because $\phi(x,\kappa_j(t))$ and $\tilde{\phi}(x,k)$ have opposite exponentially asymptotic properties as $x \to +\infty$. Thus Eq. (18) becomes

$$\phi_t(x,\kappa_j(t)) - N\phi(x,\kappa_j(t)) = \rho(t,\kappa_j(t))\phi(x,\kappa_j(t)), \quad (19)$$

which can be further written as [6]

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\phi_1(x,\kappa_j(t))\phi_2(x,\kappa_j(t)) \\ &- \left[C\phi_1^2(x,\kappa_j(t)) + B\phi_2^2(x,\kappa_j(t))\right] \\ &= 2\rho(t,\kappa_j(t))\phi_1(x,\kappa_j(t))\phi_2(x,\kappa_j(t))\end{aligned} \quad (20)$$

Supposing $\phi(x,\kappa_j(t))$ and $c_j^2(t)$ are the normalization eigenfunction and the normalization constant respectively, then we have [6]

$$\rho(t,\kappa_j(t)) = \frac{1}{2}\sum_{s=1}^{3} s(2i\kappa_j(t))^{s-1}. \quad (21)$$

In view of Eqs. (3), (19), (21) and

$$\begin{aligned}&A \to -\frac{1}{2}\left[\sum_{m=0}^{3}(2i\kappa_j(t))^m\right]x - \frac{1}{2}(2i\kappa_j(t))^2, \\ &B \to 0, \quad C \to 0,\end{aligned} \quad (22)$$

$$\phi(x,\kappa_j(t)) \to c_j(t)\begin{pmatrix}0\\1\end{pmatrix}e^{i\kappa_j(t)x}, \quad (23)$$

as $x \to +\infty$, we obtain

$$\begin{aligned}&\kappa_{jt}(t) = -\frac{i}{2}\sum_{n=0}^{3}(2i\kappa_j(t))^n \\ &c_{jt}(t) - c_j(t)\left[\frac{1}{2}\sum_{s=1}^{3}s(2i\kappa_j(t))^{s-1} + \frac{1}{2}(2i\kappa_j(t))^2\right] = 0.\end{aligned} \quad (24)$$

Similarly, we obtain

$$\begin{aligned}&\overline{\kappa}_{mt}(t) = -\frac{i}{2}\sum_{n=0}^{3}(2i\overline{\kappa}_m(t))^n, \\ &\overline{c}_{mt} + \overline{c}_m\left[\frac{1}{2}\sum_{s=1}^{3}s(2i\overline{\kappa}_m)^{s-1} + \frac{1}{2}(2i\overline{\kappa}_m)^2\right] = 0.\end{aligned} \quad (25)$$

Secondly, we consider the case when k is a real continuous spectral parameter. Since

$$\varphi_t(x,k) - N\varphi(x,k) = \varepsilon(t,k)\varphi(x,k) + \delta(t,k)\overline{\varphi}(x,k), \quad (26)$$

where $\varphi(x,k)$ and $\overline{\varphi}(x,k)$ are another pair of fundamental solutions of Eq. (2), $\varepsilon(t,k)$ and $\delta(t,k)$ are undetermined coefficient functions.

Letting $x \to -\infty$ and using

$$\varphi(x,k) \to \begin{pmatrix}1\\0\end{pmatrix}\mathrm{e}^{-ikx}, \quad \overline{\varphi}(x,k) \to \begin{pmatrix}0\\-1\end{pmatrix}\mathrm{e}^{ikx}, \quad (27)$$

from Eq. (26) we have

$$\varepsilon(t,k) = -4k^2, \quad \delta(t,k) = 0. \quad (28)$$

Taking the following Jost relationship with two undetermined coefficient functions $\lambda(t,k)$ and $\mu(t,k)$:

$$\varphi(x,k) = \lambda(t,k)\overline{\phi}(x,k) + \mu(t,k)\phi(x,k), \quad (29)$$

and substituting Eq. (29) into Eq. (26), we have

$$\begin{aligned}&\left(\lambda(t,k)\overline{\phi}(x,k) + \mu(t,k)\phi(x,k)\right)_t - N(\lambda(t,k)\overline{\phi}(x,k) \\ &+ \mu(t,k)\phi(x,k)) = -4k^2\phi(x,k).\end{aligned} \quad (30)$$

Using Eq. (30) and the asymptotical properties

$$\begin{aligned}&\phi(x,k) \to \begin{pmatrix}0\\1\end{pmatrix}\mathrm{e}^{ikx}, \quad \overline{\phi}(x,k) \to \begin{pmatrix}1\\0\end{pmatrix}\mathrm{e}^{-ikx}, \\ &x \to -\infty,\end{aligned} \quad (31)$$

yields

$$\frac{\mathrm{d}\lambda(t,k)}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\mu(t,k)}{\mathrm{d}t} = -4k^2\mu(t,k). \quad (32)$$

With a similar process, we obtain

$$\frac{\mathrm{d}\overline{\lambda}(t,k)}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\overline{\mu}(t,k)}{\mathrm{d}t} = 4k^2\overline{\mu}(t,k). \quad (33)$$

Directly solving Eqs. (24), (25), (32) and (33), we arrive at Eqs. (16) and (17). Thus, we finish the proof of Theorem 1.

3.2 Exact solutions

Theorem 2. *With the help of scattering data $\kappa_j(t)$, $c_j(t)$, $R(t,k)$, $\overline{\kappa}_m(t)$, $\overline{c}_m(t)$, $\overline{R}(t,k)$ in Eqs. (16) and (17), we can determine the following exact solutions of system (1):*

$$q(x,t) = -2K_1(t,x,x), \quad (34)$$

$$r(x,t) = \frac{K_{2x}(t,x,x)}{K_1(t,x,x)}, \quad (35)$$

where $K_1(t,x,y)$ and $K_2(t,x,y)$ satisfy:

$$\begin{aligned}&K_1(t,x,y) - \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{R}(t,k)\mathrm{e}^{-ik(x+y)}\mathrm{d}k \\ &+ \sum_{j=1}^{\overline{n}}\overline{c}_j^2(t)\mathrm{e}^{-i\overline{\kappa}_j(t)(x+y)} \\ &- \int_x^{\infty}K_1(t,x,s)\sum_{j=1}^{n}\sum_{m=1}^{\overline{n}}\frac{ic_j^2(t)\overline{c}_m^2(t)}{\kappa_j(t) - \overline{\kappa}_m(t)}\mathrm{e}^{i\kappa_j(t)(x+s) - i\overline{\kappa}_m(t)(x+y)}\mathrm{d}s \\ &= 0\end{aligned} \quad (36)$$

$$K_2(t,x,y)-\sum_{j=1}^{n}\sum_{m=1}^{\overline{n}}\frac{ic_j^2(t)\overline{c}_m^2(t)}{\kappa_j(t)-\overline{\kappa}_m(t)}e^{2i\kappa_j(t)x-i\overline{\kappa}_m(t)(x+y)} - \int_x^{\infty}K_2(t,x,s)\sum_{j=1}^{n}\sum_{m=1}^{\overline{n}}\frac{ic_j^2(t)\overline{c}_m^2(t)}{\kappa_j(t)-\overline{\kappa}_m(t)}e^{i\kappa_j(t)(x+s)-i\overline{\kappa}_m(t)(x+y)}ds = 0 \tag{37}$$

Proof. According to Theorem 1 and the results in [6], we can easily prove Theorem 2. Here the process of the proof is omitted for simplification.

3.3 Reduction of exact solutions

Theorem 3. *System (1) has the following reduced solutions:*

$$q(x,t)=2\mathrm{tr}(P^{-1}(x,t)\overline{\Lambda}\overline{\Lambda}^T), \tag{38}$$

$$r(x,t)=-\frac{\frac{d}{dx}\mathrm{tr}(P^{-1}(x,t)Q(x,t)\frac{d}{dx}Q^T(x,t))}{\mathrm{tr}(P^{-1}(x,t)\overline{\Lambda}\overline{\Lambda}^T)}, \tag{39}$$

determined by

$$P(x,t)=E+Q(x,t)Q^T(x,t), \tag{40}$$

$$Q(x,t)=\left(\frac{c_j(t)\overline{c}_m(t)}{\kappa_j(t)-\overline{\kappa}_m(t)}e^{i(\kappa_j(t)-\overline{\kappa}_m(t))x}\right)_{\overline{n}\times n}, \tag{41}$$

$$\Lambda=(c_1(t)e^{i\kappa_1(t)x},c_2(t)e^{i\kappa_2(t)x},\cdots,c_n(t)e^{i\kappa_n(t)x})^T, \tag{42}$$

$$\overline{\Lambda}=(\overline{c}_1(t)e^{-i\overline{\kappa}_1(t)x},\overline{c}_2(t)e^{-i\overline{\kappa}_2(t)x},\cdots,\overline{c}_n(t)e^{-i\overline{\kappa}_{\overline{n}}(t)x})^T, \tag{43}$$

where E is unit matrix, $\kappa_j(t)$, $c_j(t)$, $\overline{\kappa}_m(t)$ and $\overline{c}_m(t)$ are determined in Eqs. (16) and (17).

Proof. In order to reduce soliton solutions from the exact solutions (38) and (39), we set $R(t,k)=\overline{R}(t,k)=0$ and suppose that

$$K_1(x,y,t)=\sum_{p=1}^{\overline{n}}\overline{c}_p(t)g_p(t,x)e^{-i\overline{\kappa}_p(t)y}, \tag{44}$$

$$K_2(x,y,t)=\sum_{p=1}^{\overline{n}}\overline{c}_p(t)h_p(t,x)e^{-i\overline{\kappa}_p(t)y}. \tag{45}$$

Substituting Eqs. (44) and (45) into Eqs. (36) and (37), we obtain

$$K_1(x,y,t)=-\mathrm{tr}(P^{-1}(t,x)\overline{\Lambda}\overline{\Lambda}^T), \tag{46}$$

$$K_2(x,y,t)=i\mathrm{tr}(P^{-1}(t,x)Q(t,x)\Lambda\overline{\Lambda}^T), \tag{47}$$

and hence reach the reduced solutions (38) and (39), here $P^{-1}(t,x)$ is supposed to exist. Thus, we finish the proof Theorem 3.

As a particular case, when $n=\overline{n}=1$, Eqs. (38) and (39) give:

$$q=\frac{2\overline{c}_1^2(0)e^{-2i\overline{\kappa}_1(t)x-\int_0^t\left[\sum_{s=1}^{3}s(2i\overline{\kappa}_1(w))^{s-1}-4\overline{\kappa}_1^2(w)\right]dw}}{1+\frac{c_1^2(0)\overline{c}_1^2(0)}{(\kappa_1(t)-\overline{\kappa}_1(t))^2}e^A} \tag{48}$$

where

$$A=2i(\kappa_1(t)-\overline{\kappa}_1(t))x+\int_0^t\left\{\sum_{s=1}^{3}s\left[(2i\kappa_1(w))^{s-1}-(2i\overline{\kappa}_1(w))^{s-1}\right]-\kappa_1^2(w)+4\overline{\kappa}_1^2(w)\right\}dw$$

$$r=\frac{2c_1^2(0)e^{2i\kappa_1x+\int_0^t\left[\sum_{s=1}^{3}s(2i\kappa_1(w))^{s-1}-4\kappa_1^2(w)\right]dw}}{1+\frac{c_1^2(0)\overline{c}_1^2(0)}{(\kappa_1(t)-\overline{\kappa}_1(t))^2}e^B} \tag{49}$$

where

$$B=2i(\kappa_1(t)-\overline{\kappa}_1(t))x+\int_0^t\left\{\sum_{s=1}^{3}s\left[(2i\kappa_1(w))^{s-1}-(2i\overline{\kappa}_1(w))^{s-1}\right]-4\kappa_1^2(w)+4\overline{\kappa}_1^2(w)\right\}dw$$

where $\kappa_1(t)$ and $\overline{\kappa}_1(t)$ are respectively determined by

$$\kappa_{1t}(t)=-\frac{i}{2}+\kappa_1(t)+2i\kappa_1^2(t)-4\kappa_1^3(t), \tag{50}$$

$$\overline{\kappa}_{1t}(t)=-\frac{i}{2}+\overline{\kappa}_1(t)+2i\overline{\kappa}_1^2(t)-4\overline{\kappa}_1^3(t). \tag{51}$$

4 Nonlinear dynamics

We have gained an insight into the nonlinear dynamics of system (1) by means of solutions (50) and (51). For such purpose, with the help of Mathematica 8 we determine $\kappa_1(t)$ and $\overline{\kappa}_1(t)$ of Eqs. (50) and (51) as follows:

$$\kappa_1(t)=\mathrm{InverseFunction}\left[-\frac{i}{4}\mathrm{ArcTan}\,[2\#1]+\left(\frac{1}{16}+\frac{i}{16}\right)\log\left[(1-2\#1)^2\right]+\left(\frac{1}{16}-\frac{i}{16}\right)\log\left[(1+2\#1)^2\right]-\frac{1}{8}\log\left[1+4\#1^2\right]\&\right]\left[-\frac{t}{2}+C[1]\right], \tag{52}$$

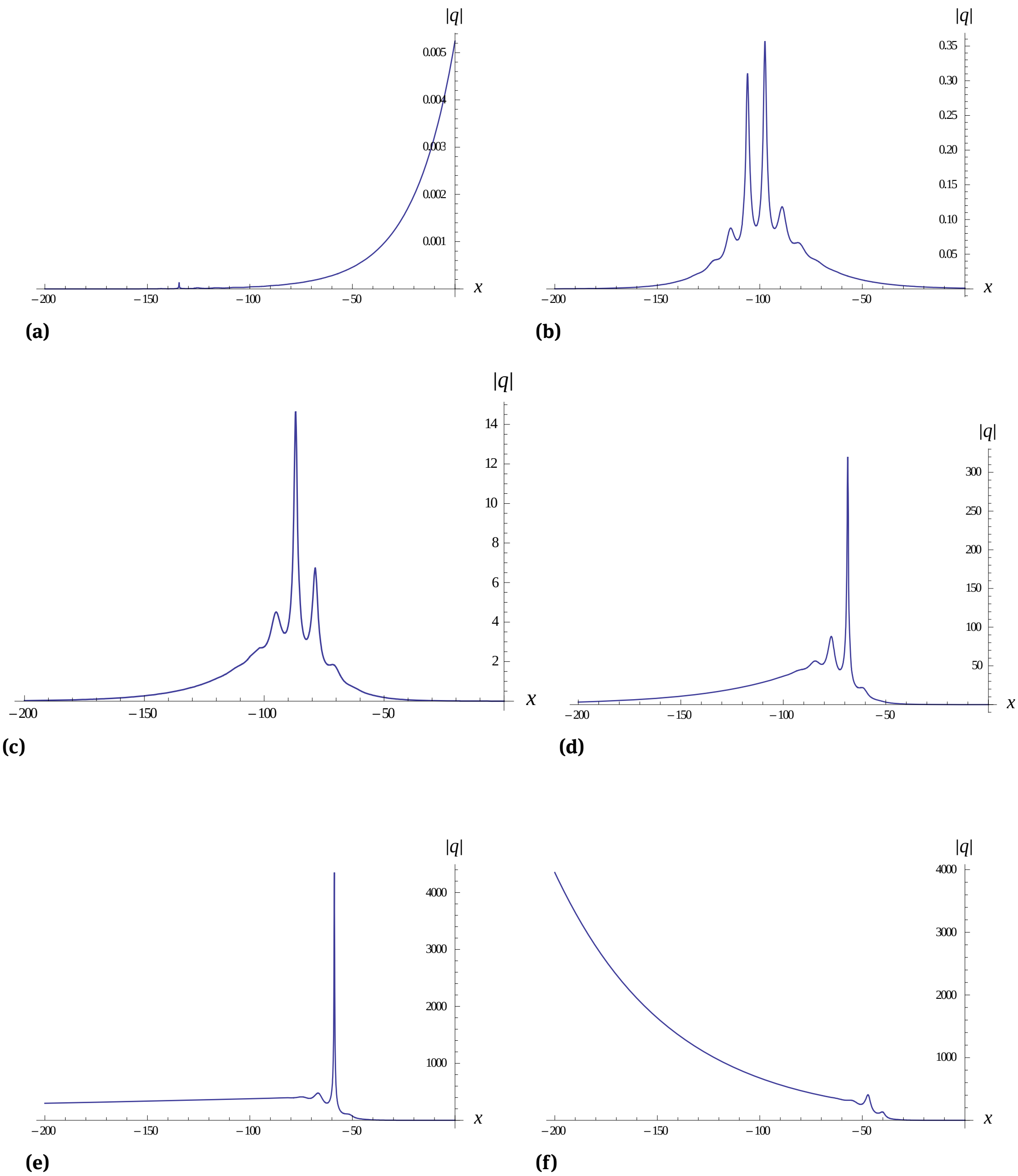


Figure 1: Dynamical evolutions of solution (48) determined by Eqs. (52)-(55) at the times: (a) $t = -0.05$, (b) $t = 0.05$, (c) $t = 0.1$, (d) $t = 0.15$, (e) $t = 0.2$, (f) $t = 0.25$

$$\overline{\kappa}_1(t) = \text{InverseFunction}\left[-\frac{i}{4}\text{ArcTan}\left[2\#1\right] + \left(\frac{1}{16}+\frac{i}{16}\right)\log\left[(1-2\#1)^2\right] + \left(\frac{1}{16}-\frac{i}{16}\right)\log\left[(1+2\#1)^2\right] - \frac{1}{8}\log\left[1+4\#1^2\right]\&\right]\left[-\frac{t}{2}+C[2]\right], \tag{53}$$

where InverseFunction[·] is a built-in function of Mathematica 8, and

$$C[1] = -\frac{i}{4}\text{ArcTan}[2\kappa_1[0]] + \left(\frac{1}{16}+\frac{i}{16}\right)\log[(1-2\kappa_1[0])^2] + \left(\frac{1}{16}-\frac{i}{16}\right)\log[(1-2\kappa_1[0])^2] - \frac{1}{8}\log[1+4\kappa_1^2[0]], \tag{54}$$

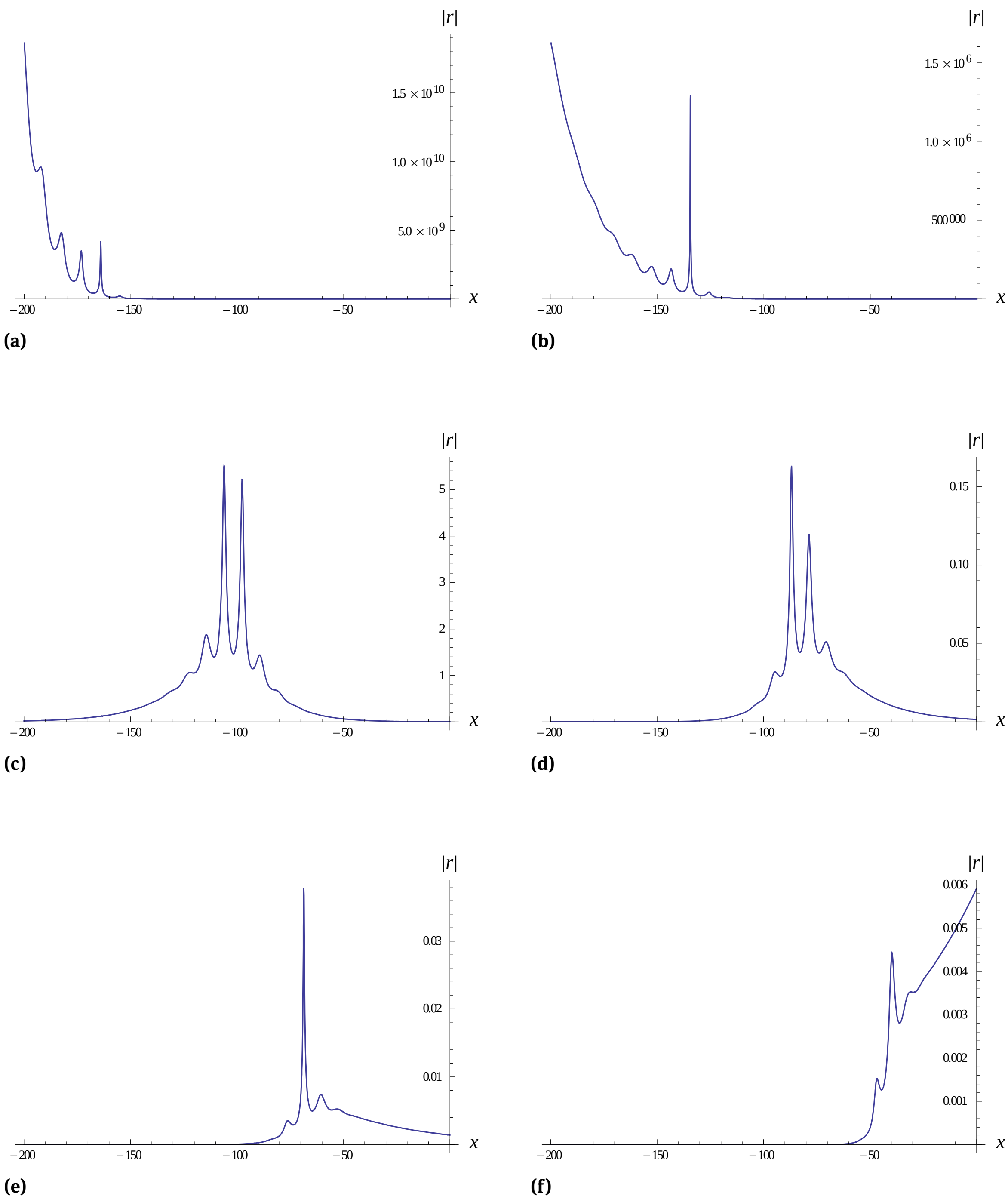
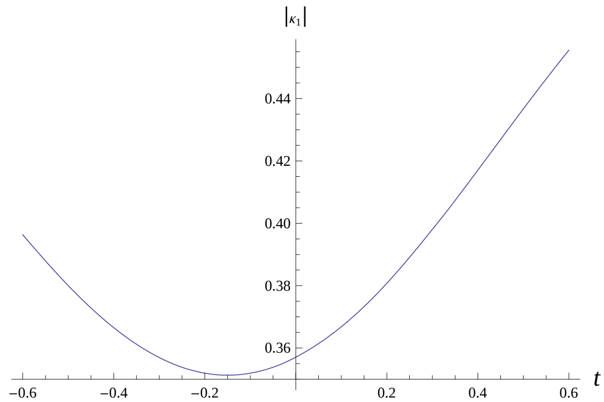


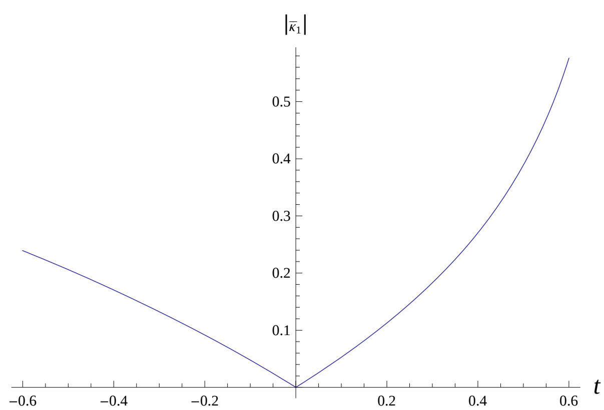
Figure 2: Dynamical evolutions of solution (49) determined by Eqs. (52)-(55) at the times: (a) $t = -0.1$, (b) $t = -0.05$, (c) $t = 0.05$, (d) $t = 0.15$, (e) $t = 0.2$, (f) $t = 0.25$

$$C[2] = -\frac{i}{4}\text{ArcTan}[2\overline{\kappa}_1[0]] + \left(\frac{1}{16} + \frac{i}{16}\right)\log[(1 - 2\overline{\kappa}_1[0])^2] + \left(\frac{1}{16} - \frac{i}{16}\right)\log[(1 - 2\overline{\kappa}_1[0])^2] - \frac{1}{8}\log[1 + 4\overline{\kappa}_1^2[0]], \quad (55)$$

In Figures 1 and 2, the dynamical evolutions of solutions (48) and (49) determined by Eqs. (52)-(55) are shown by selecting the parameters as $\kappa_1(0) = 1$, $\overline{\kappa}_1(0) = 0$, $c_1(0) = -0.2$ and , respectively. We can see from Figures 1



(a)



(b)

Figure 3: Graphs of the spectral parameters (52) and (53) with parameters: (a) $\kappa_1(0) = 1$, (b) $\overline{\kappa}_1(0) = 0$

and 2 that the amplitudes of solutions (48) and (49) vary with time. In Figure 3, we simulate the graphs of the spectral parameters (52) and (53) with parameters $\kappa_1(0) = 1$ and $\overline{\kappa}_1(0) = 0$.

5 Conclusion

In summary, we have derived and solved a new nonlinear integral-differential system (1) that is nonisospectral and Lax integrable. This is due to the new spectral parameter (4) embedded into the eigenvalue problem (2) and its adjoint equation (3). By extending the IST method with the spectral parameter (4), we obtain exact solutions (38) and (39) and their reduced forms (48) and (49) of the system (1). To gain further insight into the reduced solutions (48) and (49), with the help of the dynamical evolutions we show that the amplitudes of solutions vary with time. Though there are some generalizations [21–46] of the IST method, we conclude that constructing new nonlinear systems and their exact solutions and researching the related issues [47–51] by analytical methods are worthy of study.

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