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Computing multiple ABC index and multiple GA index of some grid graphs

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Abstract: Topological indices are the atomic descriptors that portray the structures of chemical compounds and they help us to anticipate certain physico-compound properties like boiling point, enthalpy of vaporization and steadiness. The atom bond connectivity (ABC) index and geometric arithmetic (GA) index are topological indices which are defined as $ABC(G) = \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u d_v}}$ and $GA(G) = \sum_{uv\in E(G)} \frac{2\sqrt{d_u d_v}}{d_u+d_v}$, respectively, where d_u is the degree of the vertex u. The aim of this paper is to introduced the new versions of ABC index and GA index namely multiple atom bond connectivity (ABC) index and multiple geometric arithmetic (GA) index. As an application, we have computed these newly defined indices for the octagonal grid O_p^q, the hexagonal grid $H(p, q)$ and the square grid $G_{p,q}$. Also, we compared these results obtained with the ones by other indices like the ABC_4 index and the GA_5 index.

Keywords: Atom-bond-connectivity index, geometric-arithmetic index, octagonal grid, hexagonal grid, square grid

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1 Introduction

There are sure concoction exacerbates that are helpful for the survival of living things. Carbon, oxygen, hydrogen and nitrogen are the primary components that aides in the generation of cells in the living things. Carbon is a fundamental component for human life. It is helpful in the arrangements of proteins, sugars and nucleic acids. It is crucial for the development of plants as carbon dioxide. The carbon atoms can bond together in different ways, called allotropes of carbon. The outstanding structures are graphite and jewel. As of late, numerous new structures have been found including nanotubes, buckminster fullerene and sheets, precious stone cubic structure, and so forth. The utilizations of various allotropes of carbon are talked about in detail in [1].

A graph G is simply a collection of points and lines that connect the points or subset of points. The points are called vertices of G and lines are called edges of G. The vertices set and edges set of G are denoted as $V(G)$ and $E(G)$, respectively. If e is an edge of G that connects the vertices u and v, then we can write $e = uv$. A graph is called connected graph if there is a path between all pairs of vertices. The *degree of a vertex* v in the graph G is the number of edges which are incident to the vertex v and will be represented by d_v.

Let Γ be the family of finite graphs. A function T from Γ into set of real numbers having $T(G) = T(H)$ property, for isomorphic G and H, is called a topological index. Someone can clearly notice that the vertices cardinality and the edges cardinality are topological indices. The earliest known topological index is Wiener index [2] and its based on distance, it is characterized as the sum of the half of distances between every pairs of vertices in a graph.

If $u, v \in V(G)$, then the distance between the vertices u and v is given by the length of any arbitrary shortest path in G that connects u and v. Another well known and one of the earliest degree dependent index was due to Milan Randi´c [3] in 1975, characterized as the sum of the negative square root of the product of degree of the end vertices of each edge of the graph.

One can define the family of atom bond connectivity topological indices [4] consisting of elements(member) of the form $ABC(G) = \sum_{uv\in E(G)} \sqrt{\frac{J_u+J_v-2}{J_uJ_v}}$, where J_u is some number that in a uniquely way can be assigned with the vertex u of graph G. One of the element of Γ is the atom bond connectivity index introduced by Estrada et al. [5]:

$$ABC(G) = \sum_{uv\in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}. \quad (1)$$

Another well known member of Γ is the fourth version of atom bond connectivity denoted as ABC_4 topological index of a graph G, introduced by Ghorbhani *et.al.* [6]:

$$ABC_4(G) = \sum_{uv\in E(G)} \sqrt{\frac{S_u + S_v - 2}{S_u S_v}}. \quad (2)$$

where $S_u = \sum_{uv\in E(G)} d_v$, $S_v = \sum_{uv\in E(G)} d_u$.
Here, we define a new member of this family Γ, namely multiple atom bond connectivity index and it is defined as follows:

$$ABC_M(G) = \sum_{uv\in E(G)} \sqrt{\frac{M_u + M_v - 2}{M_u M_v}}. \quad (3)$$

where $M_u = \prod_{uv\in E(G)} d_v$, $M_v = \prod_{uv\in E(G)} d_u$.
A family Λ of geometric arithmetic topological indices consisting of elements(member) of the form $GA(G) = \sum_{uv\in E(G)} \frac{2\sqrt{J_uJ_v}}{J_u+J_v}$, where J_u is some number that in a uniquely way can be assigned with the vertex u of G. One of the other member of Λ is the geometric arithmetic index GA of a graph G introduced by Vukičević *et.al.* [7]:

$$GA(G) = \sum_{uv\in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v}. \quad (4)$$

Another well known member of Λ is the fifth version of geometric arithmetic index and is denoted by GA_5 topological index of a graph G, introduced by Graovoc *et.al.* [8]:

$$GA_5(G) = \sum_{uv\in E(G)} \frac{2\sqrt{S_u S_v}}{S_u + S_v}. \quad (5)$$

Here, we define a new member of Λ namely multiple geometric -arithmetic index and it is characterized as:

$$GA_M(G) = \sum_{uv\in E(G)} \frac{2\sqrt{M_u M_v}}{M_u + M_v}. \quad (6)$$

For more information and properties of topological indices, see [9–15].
Moreover this idea of computing the topological indices is helpful to discuss the concept of entropy. The entropy of Shannon, Rényi and Kolmogorov are analyzed and compared together with their main properties. The entropy of some particular antennas with a pre-fractal shape, also called fractal antennas, is studied [16]. Fractional derivative of the Riemann zeta function has been explicitly computed and the convergence of the real part and imaginary parts are studied with the help of topological indices [17, 18].
The aim of this paper is the introduction of the multiple atom bond connectivity index and multiple geometric arithmetic index. As an application we shall compute these new indices for the octagonal grid O_p^q, the hexagonal grid $H(p, q)$ and the square grid $G_{p,q}$. Also, we compared these results obtained with the ones obtained by other indices like fourth atom bond connectivity index and fifth geometric arithmetic index via their computation too. But first we shall see some examples.

Example 1. *Let $G = K_n$ be the complete graph, then for all $u \in V(K_n)$, the $d_u = n-1$, so $M_u = (n-1)^{n-1}$. Thus*

$$\begin{aligned} ABC_M(G) &= \frac{n \times \sqrt{(n-1)^{n-1} - 1}}{\sqrt{2}(n-1)^{n-2}} \\ GA_M(G) &= \frac{n(n-1)}{2}. \end{aligned}$$

Example 2. *If $G = C_n$ be the cycle graph, then for all $u \in V(C_n)$, then $d_u = 2$, so $M_u = 4$. Thus*

$$\begin{aligned} ABC_M(G) &= \frac{n}{4}\sqrt{6} \\ GA_M(G) &= n. \end{aligned}$$

Example 3. *If $G = P_n$, $n \geq 5$ be the path graph of length n, then for all $u \in V(P_n)$, we can compute easily as:*

$$\begin{aligned} ABC_M(G) &= \frac{n-5}{4}\sqrt{6} + \frac{4}{\sqrt{2}} \\ GA_M(G) &= n - 1. \end{aligned}$$

2 Applications of topological indices

The atom bond connectivity index (ABC) is a topological descriptor that has correlated with a lot of chemical characteristics of the molecules and has been found to the parallel to computing the boiling point and Kovats constants of the molecules. Moreover, the atom bond connectivity (ABC) index provides a very good correlation for the stability of linear alkanes as well as the branched alkanes and for computing the strain energy of cyclo alkanes [19, 20]. To correlate with certain physico-chemical properties, GA index has much better predictive power than the predictive power of the Randic connectivity index [21]. The first

Zagreb index and second Zagreb index were found to occur for computation of the total π-electron energy of the molecules within specific approximate expressions [22–24]. These are among the graph invariants, who were proposed for measurement of skeleton of branching of the carbon-atom [25].

3 The octagonal grid O_p^q

In [26] and [27] Diudea *et. al.* constitute a C_4C_8 net consisting of a trivalent decoration constructed by alternating octagons and squares in two different manners. One is by alternating squares C_4 and octagons C_8 in different ways denoted by $C_4C_8(S)$ and other is by alternating rhombus and octagons in different ways denoted by $C_4C_8(R)$. We denote $C_4C_8(R)$ by O_p^q see Figure 1. In [28] they also called it as *the Octagonal grid*.

For $p, q \geq 1$ the octagonal grid O_p^q, is the grid with p horizontal octagons and q vertical octagons. Therefore, in O_p^q the number vertices and edges are $4pq + 2p + 2q$ and $6pq + p + q$, respectively. In this paper, we consider O_p^q for $p, q \geq 2$.

Table 1: Partition of edges O_p^q based on sum of degrees belonging to neighbourhood of each vertex.

(S_u, S_v)	Frequency
$(4,4)$	4
$(4,5)$	8
$(5,5)$	$2p + 2q - 8$
$(5,7)$	$4p + 4q - 8$
$(7,9)$	$2p + 2q - 4$
$(9,9)$	$6pq - 7p - 7q + 8$

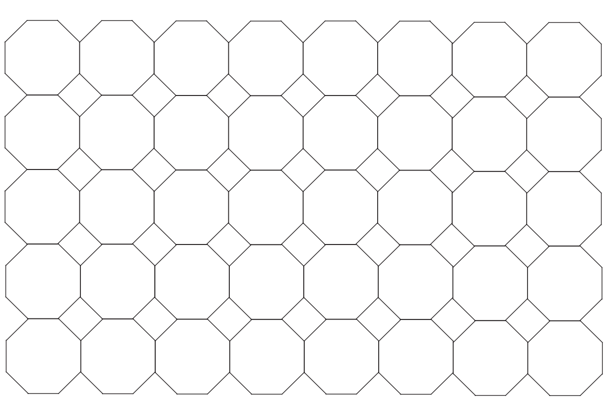

Figure 1: The octagonal grid O_8^5.

Table 2: Partition of edges O_p^q based on product of degree belonging to neighbourhood of each vertex.

(M_u, M_v)	Frequency
$(4,4)$	4
$(4,6)$	8
$(6,6)$	$2p + 2q - 8$
$(6,12)$	$4p + 4q - 8$
$(12,27)$	$2p + 2q - 4$
$(27,27)$	$6pq - 7p - 7q + 8$

3.1 Results for the octagonal grid O_p^q

Now we shall compute fourth atom bond connectivity index, fifth geometric arithmetic index, multiple atom bond connectivity index and multiple geometric arithmetic index and we shall compare the results obtained for O_p^q with for $p, q \geq 2$. For this we shall use Table 1 and Table 2. In Table 1 we have partitioned the edges of O_p^q based on the sum of degrees for each pair of vertices incident to same edge. In Table 2 we have partitioned the edges of O_p^q based on product of degrees of the neighbouring vertices to each pair of vertices incident to same edge. This will help us to develop the theorems of present section.

Theorem 1. For every $p, q \geq 2$, consider the graph of $G \cong O_p^q$. Then the fourth atom bond connectivity index $ABC_4(G)$ is given as

$$\begin{aligned} ABC_4(G) &= \frac{8pq}{3} - \frac{28q}{9} + \frac{4\sqrt{35}}{5} + \frac{(2(2p+2q-8))\sqrt{2}}{5} \\ &+ \frac{(4p+4q-8)\sqrt{14}}{7} \\ &+ \frac{(2p+2q-4)\sqrt{2}}{3} + \frac{32}{9} + \sqrt{6} - \frac{28p}{9}. \end{aligned}$$

Proof. Let G be the graph of O_p^q. Then by using Table 1 and equation (2), the fourth atom bond connectivity index $ABC_4(G)$ is computed below.

$$\begin{aligned} ABC_4(G) &= \sum_{uv \in E(G)} \sqrt{\frac{S_u + S_v - 2}{S_u S_v}} \\ ABC_4(O_p^q) &= 4\sqrt{\frac{4+4-2}{4 \times 4}} + 8\sqrt{\frac{4+5-2}{4 \times 5}} \\ &+ (2p+2q-8)\sqrt{\frac{5+5-2}{5 \times 5}} \\ &+ (4p+4q-8)\sqrt{\frac{5+7-2}{5 \times 7}} \end{aligned}$$

$$+ \quad (2p+2q-4)\sqrt{\frac{7+9-2}{7\times 9}}$$
$$+ \quad (6pq-7p-7q+8)\sqrt{\frac{9+9-2}{9\times 9}}.$$

After some easy calculations we get:

$$\begin{aligned} ABC_4(O_p^q) &= \frac{8pq}{3} - \frac{28q}{9} + \frac{4\sqrt{35}}{5} \\ &+ \frac{(2(2p+2q-8))\sqrt{2}}{5} + \frac{(4p+4q-8)\sqrt{14}}{7} \\ &+ \frac{(2p+2q-4)\sqrt{2}}{3} + \frac{32}{9} + \sqrt{6} - \frac{28p}{9} \end{aligned}$$

□

Theorem 2. If $G \cong O_p^q$ for every $p,\ q \geq 2$, then the multiple atom bond connectivity index $ABC_M(G)$ is :

$$\begin{aligned} ABC_M(O_p^q) &= \frac{(6pq-7p-7q+8)\sqrt{52}}{27} \\ &+ \frac{(2p+2q-8)\sqrt{10}}{6} + \frac{(4p+4q-8)\sqrt{2}}{3} \\ &+ \frac{(2p+2q-4)\sqrt{37}}{18} + \sqrt{6} + \frac{8\sqrt{3}}{3}. \end{aligned}$$

Proof. Let G be the graph of O_p^q. Then by using Table 2 and equation (3), the multiple atom bond connectivity index $ABC_M(G)$ is computed as:

$$\begin{aligned} ABC_M(G) &= \sum_{uv\in E(G)} \sqrt{\frac{M_u+M_v-2}{M_uM_v}} \\ ABC_M(O_p^q) &= 4\sqrt{\frac{4+4-2}{4\times 4}} + 8\sqrt{\frac{4+6-2}{4\times 6}} \\ &+ (2p+2q-8)\sqrt{\frac{6+6-2}{6\times 6}} \\ &+ (4p+4q-8)\sqrt{\frac{6+12-2}{6\times 12}} \\ &+ (2p+2q-4)\sqrt{\frac{12+27-2}{12\times 27}} \\ &+ (6pq-7p-7q+8)\sqrt{\frac{27+27-2}{27\times 27}}. \end{aligned}$$

After some easy calculations we obtained:

$$\begin{aligned} ABC_M(O_p^q) &= \frac{(6pq-7p-7q+8)\sqrt{52}}{27} \\ &+ \frac{(2p+2q-8)\sqrt{10}}{6} + \frac{(4p+4q-8)\sqrt{2}}{3} \\ &+ \frac{(2p+2q-4)\sqrt{37}}{18} + \sqrt{6} + \frac{8\sqrt{3}}{3}. \end{aligned}$$

□

Theorem 3. If $G \cong O_p^q$ for every $p,\ q \geq 2$, then the fifth geometric arithmetic index $GA_5(G)$ is:

$$\begin{aligned} GA_5(O_p^q) &= 6pq-5p-5q + \frac{(8p+8q-16)\sqrt{35}}{12} \\ &+ \frac{(4p+4q-8)\sqrt{63}}{16} + 4 + \frac{32\sqrt{5}}{9}. \end{aligned}$$

Proof. Let G be the graph of O_p^q. Then by using Table 1 and equation (5), the fifth geometric arithmetic index $GA_5(G)$ is computed as below:

$$\begin{aligned} GA_5(G) &= \sum_{uv\in E(G)} \frac{2\sqrt{S_uS_v}}{S_u+S_v} \\ GA_5(O_p^q) &= (4)\frac{2\sqrt{4\times 4}}{4+4} + (8)\frac{2\sqrt{4\times 5}}{4+5} \\ &+ (2p+2q-8)\frac{2\sqrt{5\times 5}}{5+5} \\ &+ (4p+4q-8)\frac{2\sqrt{5\times 7}}{5+7} \\ &+ (2p+2q-4)\frac{2\sqrt{7\times 9}}{7+9} \\ &+ (6pq-7p-7q+8)\frac{2\sqrt{9\times 9}}{9+9}. \end{aligned}$$

After some easy calculations we get:

$$\begin{aligned} GA_5(O_p^q) &= 6pq-5p-5q + \frac{(8p+8q-16)\sqrt{35}}{12} \\ &+ \frac{(4p+4q-8)\sqrt{63}}{16} + 4 + \frac{32\sqrt{5}}{9}. \end{aligned}$$

□

Theorem 4. For every $p,\ q \geq 2$ consider the graph of $G \cong O_p^q$. The multiple geometric arithmetic index $GA_M(G)$ is:

$$\begin{aligned} GA_M(O_p^q) &= 6pq - \frac{41p}{13} - \frac{41q}{13} \\ &+ \frac{(8p+8q-16)\sqrt{2}}{3} + \frac{4}{13} + \frac{16\sqrt{6}}{5}. \end{aligned}$$

Proof. Let G be the graph of O_p^q. Then by using Table 2 and equation (6). the multiple geometric arithmetic index $GA_M(G)$ is computed below:

$$\begin{aligned} GA_M(G) &= \sum_{uv\in E(G)} \frac{2\sqrt{M_uM_v}}{M_u+M_v} \\ GA_M(O_p^q) &= (4)\frac{2\sqrt{4\times 4}}{4+4} + (8)\frac{2\sqrt{4\times 6}}{4+6} \\ &+ (2p+2q-8)\frac{2\sqrt{6\times 6}}{6+6} \\ &+ (4p+4q-8)\frac{2\sqrt{6\times 12}}{6+12} \end{aligned}$$

$$+ \quad (2p+2q-4)\frac{2\sqrt{12\times 27}}{12+27}$$
$$+ \quad (6pq-7p-7q+8)\frac{2\sqrt{27\times 27}}{27+27}.$$

After some easy calculations we get:

$$GA_M(O_p^q) = 6pq - \frac{41p}{13} - \frac{41q}{13} + \frac{(8p+8q-16)\sqrt{2}}{3} + \frac{4}{13} + \frac{16\sqrt{6}}{5}.$$

□

4 The hexagonal grid $H(p,q)$

In this section we shall compute fourth atom bond connectivity index, fifth geometric arithmetic index, multiple atom bond connectivity index and multiple geometric arithmetic index for hexagonal grid $H(p,q)$. Also we shall compare these results obtained in the last section. For $p,q \geq 1$ the hexagonal grid $H(p,q)$ consists of p octagons in a row (horizontal) and q represents the number of rows see Figure 2. One can easily see that in $H(p,q)$ the number vertices and edges are $2pq+2p+2q$ and $3pq+2p+2q-1$, respectively. In this paper the we consider $H(p,q)$ for $p,q \geq 2$.

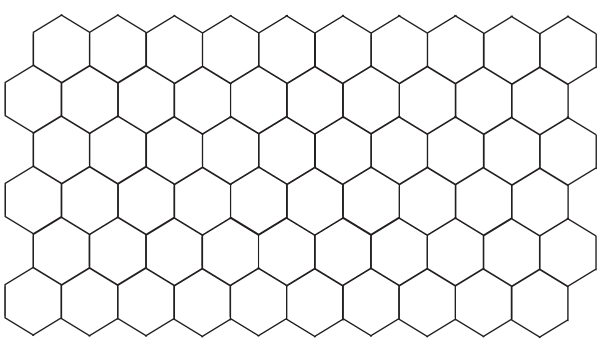

Figure 2: The hexagonal grid $H(10,6)$.

For this we shall use Table 3 and Table 4. In Table 3 we have partitioned the edges of $H(p,q)$ based on the sum of degrees for each pair of vertices incident to same edge. In Table 4 we have partitioned the edges of $H(p,q)$ based on product of degrees of the neighbouring vertices to each pair of vertices incident to same edge. This will help us to develop the theorems of present section.

Table 3: Partition of edges $H(p,q)$ based on sum of degrees belonging to neighbourhood of each vertex.

(S_u, S_v)	Frequency
$(4,5)$	4
$(5,5)$	q
$(5,7)$	8
$(5,8)$	$2q-4$
$(6,7)$	$4p-8$
$(7,9)$	$2p$
$(8,8)$	$q-2$
$(8,9)$	$2q-4$
$(9,9)$	$3pq-4p-4q+5$

Table 4: Partition of edges $H(p,q)$ based on product of degree belonging to neighbourhood of each vertex.

(M_u, M_v)	Frequency
$(4,6)$	4
$(6,6)$	q
$(6,12)$	8
$(6,18)$	$2q-4$
$(9,12)$	$4p-8$
$(12,27)$	$2p$
$(18,18)$	$q-2$
$(18,27)$	$2q-4$
$(27,27)$	$3pq-4p-4q+5$

4.1 Results for the hexagonal grid $H(p,q)$

Now we shall compute fourth atom bond connectivity index, fifth geometric arithmetic index, multiple atom bond connectivity index and multiple geometric arithmetic index and we shall compare the results obtained for $H(p,q)$ with for $p,q \geq 2$. For this we shall use Table 3 and Table 4. In Table 3 we have partitioned the edges of $H(p,q)$ based on the sum of degrees for each pair of vertices incident to same edge. In Table 4 we have partitioned the edges of $H(p,q)$ based on product of degrees of the neighbouring vertices to each pair of vertices incident to same edge. This will help us to develop the theorems of present section.

Theorem 5. For every $p,\ q \geq 2$ consider the graph of $G \cong H(p,q)$. The fourth atom bond connectivity index $ABC_4(G)$ of $H(p,q)$ is given as

$$ABC_4(H(p,q)) = \frac{4pq}{3} - \frac{16q}{9} - \frac{16p}{9} + \frac{(2q-4)\sqrt{110}}{20} + \frac{(4p-8)\sqrt{462}}{42} + \frac{2\sqrt{2}p}{3}$$

$$+ \frac{(q-2)\sqrt{14}}{8} + \frac{(2q-4)\sqrt{30}}{12} + \frac{2\sqrt{35}}{5} + \frac{2\sqrt{2}q}{5} + \frac{8\sqrt{14}}{7} + \frac{20}{9}$$

Proof. Let G be the graph of $H(p,q)$. Then as in Theorem 1, by using Table 3, equation (1) and following computations, the result follows:

$$\begin{aligned}
ABC_4(G) &= \sum_{uv\in E(G)} \sqrt{\frac{S_u+S_v-2}{S_uS_v}}\\
ABC_4(H) &= (4)\sqrt{\frac{4+5-2}{4\times5}} + (q)\sqrt{\frac{5+5-2}{5\times5}}\\
&+ (8)\sqrt{\frac{5+7-2}{5\times7}} + (2q-4)\sqrt{\frac{5+8-2}{5\times8}}\\
&+ (4p-8)\sqrt{\frac{6+7-2}{6\times7}} + (2p)\sqrt{\frac{7+9-2}{7\times9}}\\
&+ (q-2)\sqrt{\frac{8+8-2}{8\times8}}\\
&+ (2q-4)\sqrt{\frac{8+9-2}{8\times9}}\\
&+ (3pq-4q-4p+5)\sqrt{\frac{9+9-2}{9\times9}}
\end{aligned}$$

One can easily calculate that

$$\begin{aligned}
ABC_4(H(p,q)) &= \frac{4pq}{3} - \frac{16q}{9} - \frac{16p}{9} + \frac{(2q-4)\sqrt{110}}{20}\\
&+ \frac{(4p-8)\sqrt{462}}{42} + \frac{2\sqrt{2}p}{3}\\
&+ \frac{(q-2)\sqrt{14}}{8} + \frac{(2q-4)\sqrt{30}}{12} + \frac{2\sqrt{35}}{5}\\
&+ \frac{2\sqrt{2}q}{5} + \frac{8\sqrt{14}}{7} + \frac{20}{9}
\end{aligned}$$

□

Theorem 6. For every $p,\ q \geq 2$ consider the graph of $G \cong H(p,q)$. The multiple atom bond connectivity index $ABC_M(G)$ of $H(p,q)$ is given as

$$\begin{aligned}
ABC_M(H(p,q)) &= \frac{4\sqrt{3}}{3} + \frac{\sqrt{10}q}{6} + \frac{8\sqrt{2}}{3} + \frac{(2q-4)\sqrt{66}}{18}\\
&+ \frac{(4p-8)\sqrt{57}}{18} + \frac{\sqrt{37}p}{9}\\
&+ \frac{(q-2)\sqrt{34}}{18} + \frac{(2q-4)\sqrt{258}}{54}\\
&+ \frac{(3pq-4q-4p+5)\sqrt{52}}{27}
\end{aligned}$$

Proof. Let G be the graph of $H(p,q)$. Then as in Theorem 2, by using Table 4, equation (2) and following computations, the result follows.

$$\begin{aligned}
ABC_M(H(p,q)) &= (4)\sqrt{\frac{4+6-2}{4\times6}} + (q)\sqrt{\frac{6+6-2}{6\times6}}\\
&+ (8)\sqrt{\frac{6+12-2}{6\times12}} + (2q-4)\sqrt{\frac{6+18-2}{6\times18}}\\
&+ (4p-8)\sqrt{\frac{9+12-2}{9\times12}} + (2p)\sqrt{\frac{12+27-2}{12\times27}}\\
&+ (q-2)\sqrt{\frac{18+18-2}{18\times18}}\\
&+ (2q-4)\sqrt{\frac{18+27-2}{18\times27}}\\
&+ (3pq-4q-4p+5)\sqrt{\frac{27+27-2}{27\times27}}
\end{aligned}$$

One can easily calculate that

$$\begin{aligned}
ABC_M(H(p,q)) &= \frac{4\sqrt{3}}{3} + \frac{\sqrt{10}q}{6} + \frac{8\sqrt{2}}{3} + \frac{(2q-4)\sqrt{66}}{18}\\
&+ \frac{(4p-8)\sqrt{57}}{18} + \frac{\sqrt{37}p}{9}\\
&+ \frac{(q-2)\sqrt{34}}{18} + \frac{(2q-4)\sqrt{258}}{54}\\
&+ \frac{(3pq-4q-4p+5)\sqrt{52}}{27}
\end{aligned}$$

□

Theorem 7. For every $p,\ q \geq 2$ consider the graph of $G \cong H(p,q)$. The fifth geometric arithmetic index GA_5 of $H(p,q)$ is given as

$$\begin{aligned}
GA_5(H(p,q)) &= \frac{16\sqrt{5}}{9} - 2q + \frac{4\sqrt{35}}{3} + \frac{(4q-8)\sqrt{40}}{13}\\
&+ \frac{(8p-16)\sqrt{42}}{13} + \frac{3p\sqrt{7}}{4}\\
&+ 3 + \frac{(4q-8)\sqrt{72}}{17} + 3pq - 4p.
\end{aligned}$$

Proof. Let G be the graph of $H(p,q)$. Then as in Theorem 3, by using Table 3, equation (3) and the following computations, the result follows.

$$\begin{aligned}
GA_5(H(p,q)) &= (4)\frac{2\sqrt{4\times5}}{4+5} + (q)\frac{2\sqrt{5\times5}}{5+5}\\
&+ (8)\frac{2\sqrt{5\times7}}{5+7} + (2q-4)\frac{2\sqrt{5\times8}}{5+8}\\
&+ (4p-8)\frac{2\sqrt{6\times7}}{6+7} + (2p)\frac{2\sqrt{7\times9}}{7+9}\\
&+ (q-2)\frac{2\sqrt{8\times8}}{8+8} + (2q-4)\frac{2\sqrt{8\times9}}{8+9}\\
&+ (3pq-4p-4q+5)\frac{2\sqrt{9\times9}}{9+9}.
\end{aligned}$$

After some easy calculations we get

$$\begin{aligned} GA_5(H(p,q)) &= \frac{16\sqrt{5}}{9} - 2q + \frac{4\sqrt{35}}{3} + \frac{(4q-8)\sqrt{40}}{13} \\ &+ \frac{(8p-16)\sqrt{42}}{13} + \frac{3p\sqrt{7}}{4} \\ &+ 3 + \frac{(4q-8)\sqrt{72}}{17} + 3pq - 4p. \end{aligned}$$

□

Theorem 8. For every $p,\ q \geq 2$ consider the graph of $G \cong H(p,q)$. The multiple geometric arithmetic index GA_M of $H(p,q)$ is given as

$$\begin{aligned} GA_M(H(p,q)) &= \frac{8\sqrt{6}}{5} - 2q + \frac{16\sqrt{2}}{3} \\ &+ \frac{(4q-8)\sqrt{3}}{4} + \frac{(8p-16)\sqrt{12}}{7} \\ &- \frac{28p}{13} + 3 \\ &+ \frac{(4q-8)\sqrt{6}}{5} + 3pq. \end{aligned}$$

Proof. Let G be the graph of $H(p,q)$. Then as in Theorem 4, by using Table 4, equation (4) and the calculations below, the result follows.

$$\begin{aligned} GA_M(H(p,q)) &= (4)\frac{2\sqrt{4\times 6}}{4+6} + (q)\frac{2\sqrt{6\times 6}}{6+6} \\ &+ (8)\frac{2\sqrt{6\times 12}}{6+12} + (2q-4)\frac{2\sqrt{6\times 18}}{6+18} \\ &+ (4p-8)\frac{2\sqrt{9\times 12}}{9+12} + (2p)\frac{2\sqrt{12\times 27}}{12+27} \\ &+ (q-2)\frac{2\sqrt{18\times 18}}{18+18} \\ &+ (2q-4)\frac{2\sqrt{18\times 27}}{18+27} \\ &+ (3pq-4p-4q+5)\frac{2\sqrt{27\times 27}}{27+27}. \end{aligned}$$

After some computation, we obtained the following result

$$\begin{aligned} GA_M(H(p,q)) &= \frac{8\sqrt{6}}{5} - 2q + \frac{16\sqrt{2}}{3} \\ &+ \frac{(4q-8)\sqrt{3}}{4} + \frac{(8p-16)\sqrt{12}}{7} \\ &- \frac{28p}{13} + 3 \\ &+ \frac{(4q-8)\sqrt{6}}{5} + 3pq. \end{aligned}$$

□

5 The square grid $G_{p,q}$

In this section we shall compute fourth atom bond connectivity index, fifth geometric arithmetic index, multiple atom bond connectivity index and multiple geometric arithmetic index for square grid $G_{p,q}$ and we shall compare the results obtained in the last section. For $p,q \geq 1$ the square grid $G_{p,q}$ consists of p horizontal squares and q vertical squares, see Figure 3. One can easily see that in $G_{p,q}$ the number vertices and edges are $pq + p + q + 1$ and $2pq + p + q$, respectively. In this paper the we consider $G_{p,q}$ for $p,q \geq 4$. For this we shall use Table 5 and Table 6 . In Table 5 we have partitioned the edges of $G_{p,q}$ based on the sum of degrees for each pair of vertices incident to same edge. In Table 6 we have partitioned the edges of $G_{p,q}$ based on product of degrees of the neighbouring vertices to each pair of vertices incident to same edge. This will help us to develop the theorems of present section.

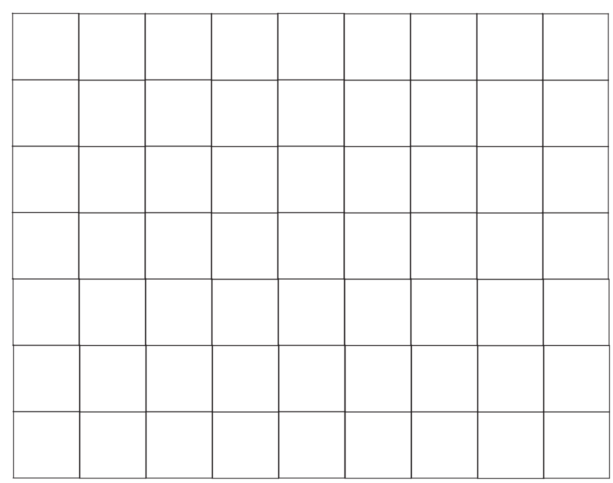

Figure 3: The square grid $G_{9,7}$.

5.1 Results for the square grid $G_{p,q}$

Now we shall compute fourth atom bond connectivity index, fifth geometric arithmetic index, multiple atom bond connectivity index and multiple geometric arithmetic index and we shall compare the results obtained for $G_{p,q}$ with for $p,q \geq 4$. For this we shall use Table 5 and Table 6. In Table 5 we have partitioned the edges of $G_{p,q}$ based on the sum of degrees for each pair of vertices incident to same edge. In Table 6 we have partitioned the edges of $G_{p,q}$ based on product of degrees of the neighbouring vertices

Table 5: Partition of edges $G_{p,q}$ based on sum of degrees belonging to neighbourhood of each vertex.

(S_u, S_v)	Frequency
$(6, 9)$	8
$(9, 10)$	8
$(10, 10)$	$2p + 2q - 16$
$(9, 14)$	8
$(10, 15)$	$2p + 2q - 12$
$(14, 15)$	8
$(15, 15)$	$2p + 2q - 16$
$(15, 16)$	$2p + 2q - 12$
$(16, 16)$	$2pq - 7p - 7q + 24$

Table 6: Partition of edges $G_{p,q}$ based on product of degree belonging to neighbourhood of each vertex.

(M_u, M_v)	Frequency
$(4, 6)$	4
$(6, 6)$	q
$(6, 12)$	8
$(6, 18)$	$2q - 4$
$(9, 12)$	$4p - 8$
$(12, 27)$	$2p$
$(18, 18)$	$q - 2$
$(18, 27)$	$2q - 4$
$(27, 27)$	$3pq - 4p - 4q + 5$

to each pair of vertices incident to same edge. This will help us to develop the theorems of present section.

Theorem 9. For every $p,\ q \geq 4$ consider the graph of $G \cong G_{p,q}$. Then the fourth atom bond connectivity index $ABC_4(G)$ of $G_{p,q}$ is given as

$$\begin{aligned} ABC_4(G_{p,q}) &= \frac{4\sqrt{78}}{9} + \frac{4\sqrt{170}}{15} + \frac{(2p+2q-16)\sqrt{18}}{10} \\ &+ \frac{4\sqrt{6}}{3} + \frac{(2p+2q-12)\sqrt{138}}{30} + \frac{12\sqrt{70}}{35} \\ &+ \frac{(2p+2q-16)\sqrt{28}}{15} \\ &+ \frac{(2p+2q-12)\sqrt{435}}{60} \\ &+ \frac{(2pq-7p-7q+24)\sqrt{30}}{16} \end{aligned}$$

Proof. Let G be the graph of $G_{p,q}$. Then as in Theorem 1, by using Table 5, equation (1) and the computations below, the result follows.

$$\begin{aligned} ABC_4(G_{p,q}) &= (8)\sqrt{\frac{6+9-2}{6\times 9}} + (8)\sqrt{\frac{9+10-2}{9\times 10}} \\ &+ (2p+2q-16)\sqrt{\frac{10+10-2}{10\times 10}} \\ &+ (8)\sqrt{\frac{9+14-2}{9\times 14}} \\ &+ (2p+2q-12)\sqrt{\frac{10+15-2}{10\times 15}} \\ &+ (8)\sqrt{\frac{14+15-2}{14\times 15}} \\ &+ (2p+2q-16)\sqrt{\frac{15+15-2}{15\times 15}} \\ &+ (2p+2q-12)\sqrt{\frac{15+16-2}{15\times 16}} \\ &+ (2pq-7q-7p+24)\sqrt{\frac{16+16-2}{16\times 16}} \end{aligned}$$

After simplification, we obtained required result:

$$\begin{aligned} ABC_4(G_{p,q}) &= \frac{4\sqrt{78}}{9} + \frac{4\sqrt{170}}{15} + \frac{(2p+2q-16)\sqrt{18}}{10} \\ &+ \frac{4\sqrt{6}}{3} + \frac{(2p+2q-12)\sqrt{138}}{30} + \frac{12\sqrt{70}}{35} \\ &+ \frac{(2p+2q-16)\sqrt{28}}{15} \\ &+ \frac{(2p+2q-12)\sqrt{435}}{60} \\ &+ \frac{(2pq-7p-7q+24)\sqrt{30}}{16} \end{aligned}$$

□

Theorem 10. For every $p,\ q \geq 4$ consider the graph of $G \cong G_{p,q}$. Then the multiple atom bond connectivity index $ABC_M(G)$ of $G_{p,q}$ is given as

$$\begin{aligned} ABC_M(G_{p,q}) &= \frac{2\sqrt{186}}{9} + \frac{2\sqrt{87}}{9} \\ &+ \frac{(2p+2q-16)\sqrt{70}}{36} + \frac{\sqrt{249}}{9} \\ &+ \frac{(2p+2q-12)\sqrt{678}}{144} \\ &+ \frac{\sqrt{1002}}{36} + \frac{(2p+2q-16)\sqrt{382}}{192} \\ &+ \frac{(2p+2q-12)\sqrt{1338}}{384} \\ &+ \frac{(2pq-7p-7q+24)\sqrt{510}}{256} \end{aligned}$$

Proof. Let G be the graph of $G_{p,q}$. Then as in Theorem 2, by using Table 6, equation (2) and the computations below, the result follows.

$$\begin{aligned} ABC_M(G_{p,q}) &= (8)\sqrt{\frac{9+24-2}{9\times 24}} + (8)\sqrt{\frac{24+36-2}{24\times 36}} \\ &+ (2p+2q-16)\sqrt{\frac{36+36-2}{36\times 36}} \end{aligned}$$

$$
\begin{aligned}
&+ (8)\sqrt{\frac{24+144-2}{24\times 144}}\\
&+ (2p+2q-12)\sqrt{\frac{36+192-2}{36\times 192}}\\
&+ (8)\sqrt{\frac{144+192-2}{144\times 192}}\\
&+ (2p+2q-16)\sqrt{\frac{192+192-2}{192\times 192}}\\
&+ (2p+2q-12)\sqrt{\frac{192+256-2}{192\times 256}}\\
&+ (2pq-7q-7p+24)\sqrt{\frac{256+256-2}{256\times 256}}
\end{aligned}
$$

After some calculation, we get:

$$
\begin{aligned}
ABC_M(G_{p,q}) &= \frac{2\sqrt{186}}{9}+\frac{2\sqrt{87}}{9}\\
&+ \frac{(2p+2q-16)\sqrt{70}}{36}+\frac{\sqrt{249}}{9}\\
&+ \frac{(2p+2q-12)\sqrt{678}}{144}\\
&+ \frac{\sqrt{1002}}{36}+\frac{(2p+2q-16)\sqrt{382}}{192}\\
&+ \frac{(2p+2q-12)\sqrt{1338}}{384}\\
&+ \frac{(2pq-7p-7q+24)\sqrt{510}}{256}
\end{aligned}
$$

□

Theorem 11. For every $p,\ q \geq 4$ consider the graph of $G \cong G_{p,q}$. Then the fifth geometric arithmetic index GA_5 of $G_{p,q}$ is given as

$$
\begin{aligned}
GA_5(G_{p,q}) &= \frac{16\sqrt{6}}{5}+\frac{48\sqrt{10}}{19}-3p-3q-8+\frac{48\sqrt{14}}{23}\\
&+ \frac{(4p+4q-24)\sqrt{6}}{5}+\frac{16\sqrt{210}}{29}\\
&+ \frac{(4p+4q-24)\sqrt{120}}{31}+2pq.
\end{aligned}
$$

Proof. Let G be the graph of $G_{p,q}$. Then as in Theorem 3, by using Table 5, equation (3) and the computations below, the result follows.

$$
\begin{aligned}
GA_5(G_{p,q}) &= (8)\frac{2\sqrt{6\times 9}}{6+9}+(8)\frac{2\sqrt{9\times 10}}{9+10}\\
&+ (2p+2q-16)\frac{2\sqrt{10\times 10}}{10+10}+(8)\frac{2\sqrt{9\times 14}}{9+14}\\
&+ (2p+2q-12)\frac{2\sqrt{10\times 15}}{10+15}+(8)\frac{2\sqrt{14\times 15}}{14+15}\\
&+ (2p+2q-16)\frac{2\sqrt{15\times 15}}{15+15}\\
&+ (2p+2q-12)\frac{2\sqrt{15\times 16}}{15+16}\\
&+ (2pq-7p-7q+24)\frac{2\sqrt{16\times 16}}{16+16}.
\end{aligned}
$$

Simplification provide our required result as follows:

$$
\begin{aligned}
GA_5(G_{p,q}) &= \frac{16\sqrt{6}}{5}+\frac{48\sqrt{10}}{19}-3p-3q-8+\frac{48\sqrt{14}}{23}\\
&+ \frac{(4p+4q-24)\sqrt{6}}{5}+\frac{16\sqrt{210}}{29}\\
&+ \frac{(4p+4q-24)\sqrt{120}}{31}+2pq.
\end{aligned}
$$

□

Theorem 12. For every $p,\ q \geq 4$ consider the graph of $G \cong G_{p,q}$. Then the multiple geometric arithmetic index GA_M of $G_{p,q}$ is given as

$$
\begin{aligned}
GA_M(G_{p,q}) &= \frac{3232\sqrt{6}}{385}-3p-3q-8\\
&+ \frac{(66(4p+4q-24))\sqrt{3}}{133}+\frac{32\sqrt{3}}{7}+2pq.
\end{aligned}
$$

Proof. Let G be the graph of $G_{p,q}$. Then as in Theorem 4, by using Table 6, equation (4) and the computations below, the result follows.

$$
\begin{aligned}
GA_M(G_{p,q}) &= (8)\frac{2\sqrt{9\times 24}}{9+24}+(8)\frac{2\sqrt{24\times 36}}{24+36}\\
&+ (2p+2q-16)\frac{2\sqrt{36\times 36}}{36+36}+(8)\frac{2\sqrt{24\times 144}}{24+144}\\
&+ (2p+2q-12)\frac{2\sqrt{36\times 192}}{36+192}+(8)\frac{2\sqrt{144\times 192}}{144+192}\\
&+ (2p+2q-16)\frac{2\sqrt{192\times 192}}{192+192}\\
&+ (2p+2q-12)\frac{2\sqrt{192\times 256}}{192+256}\\
&+ (2pq-7p-7q+24)\frac{2\sqrt{256\times 256}}{256+256}.
\end{aligned}
$$

After some calculation we obtained our required result:

$$
\begin{aligned}
GA_M(G_{p,q}) &= \frac{3232\sqrt{6}}{385}-3p-3q-8\\
&+ \frac{(66(4p+4q-24))\sqrt{3}}{133}+\frac{32\sqrt{3}}{7}+2pq.
\end{aligned}
$$

□

6 Comparison and conclusion

In this paper, we have introduced the multiple atom bond connectivity index and multiple geometric arithmetic index. As their application we have computed these new indices for octagonal grid O_p^q, hexagonal grid $H(p,q)$ and square grid $G_{p,q}$. We then give comparisons of the results obtained by these indices with the ones obtained by other indices like the fourth atom bond connectivity index and the fifth geometric arithmetic index via their computation too for the octagonal grid, hexagonal grid and square grid graphs. Table 7, Table 8 and Table 9 show the comparisons between ABC_4, ABC_M, GA_5 and GA_M for O_p^q, $H(p,q)$ and $G_{p,q}$, respectively.

Table 7: Comparison of ABC_4, ABC_M, GA_5 and GA_M for O_p^q.

$[p,\ q]$	ABC_4	ABC_M	GA_5	GA_M
[2,2]	15.123	13.260	27.808	15.073
[3,3]	30.657	24.764	59.664	58.308
[4,4]	51.526	39.474	103.52	101.54
[5,5]	77.728	57.388	159.38	156.78
[6,6]	109.26	78.506	227.23	224.01
[7,7]	146.13	102.83	307.09	303.24
[8,8]	188.33	130.36	398.95	394.49
[9,9]	235.88	161.09	502.81	497.72
[10,10]	288.74	195.04	618.66	612.95

Table 8: Comparison of ABC_4, ABC_M, GA_5 and GA_M for $H(p,q)$.

$[p,\ q]$	ABC_4	ABC_M	GA_5	GA_M
[2,2]	7.4375	8.7535	18.832	18.154
[3,3]	19.201	15.326	37.747	36.650
[4,4]	30.963	23.5	62.661	61.147
[5,5]	45.392	33.276	93.578	91.644
[6,6]	62.487	44.656	130.50	128.14
[7,7]	82.250	57.638	173.41	170.64
[8,8]	104.68	72.221	222.34	219.14
[9,9]	129.78	88.408	277.25	273.64
[10,10]	157.54	106.20	338.15	334.13

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Table 9: Comparison of ABC_4, ABC_M, GA_5 and GA_M for $G_{p,q}$.

$[p,\ q]$	ABC_4	ABC_M	GA_5	GA_M
[4,4]	16.494	8.8402	39.549	35.358
[5,5]	23.928	11.634	59.466	54.233
[6,6]	32.731	14.781	83.383	77.110
[7,7]	42.904	18.280	111.30	103.99
[8,8]	54.445	22.133	143.21	134.86
[9,9]	67.359	26.339	179.13	169.74
[10,10]	81.638	30.896	219.05	208.61

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