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Research Article

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Analysis of projectile motion in view of conformable derivative

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Abstract: This paper presents new solutions for two-dimensional projectile motion in a free and resistive medium, obtained within the newly established conformable derivative. For free motion, we obtain analytical solutions and show that the trajectory, height, flight time, optimal angle, and maximum range depend on the order of the conformable derivative, $0 < \gamma \leq 1$. Likewise, we analyse and simulate the projectile motion in a resistive medium by assuming several scenarios. The obtained trajectories never exceed the ordinary ones, given by $\gamma = 1$, unlike results reported in other studies.

Keywords: Fractional calculus, conformable fractional derivative, projectile motion, trajectory, range, height, flight time

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1 Introduction

In the last three decades, much work has been performed on fractional calculus (FC). FC is the natural generalization of ordinary derivatives and integrals. It deals with operators having a non-integer (arbitrary) order. Today, there are excellent books and reviews on fractional calculus and its applications in science and engineering [1–7]. FC has been employed in numerous practical and theoretical problems, but, for the sake of brevity, we mention a few of them as follows: the design of optimal control systems [8]; the analysis of anomalous relaxation and diffusion processes [9], and the regularized long-wave equation [10]; the study of a bead sliding on a wire [11]; and the investigation of chaotic systems [12], heat conduction [13], nerve impulse transmission [14], chemical kinetic systems [15], and oscillating circuits [16]. However, this generalization of the ordinary calculus is still inconsistent in physical and geometric interpretation, because several definitions of fractional derivatives have been proposed. These definitions include the Riemann-Liouville, Grunwald-Letnikov, Weyl, Caputo, Marchaud and Riesz fractional derivatives [1–5]. These fractional derivatives seldom satisfy the well-known formulae, *e.g.*, the product, quotient, and chain rules. As a result, researchers have been trying to construct new definitions of fractional derivatives and integrals. As an illustrative example, helpful work on the theory of derivatives and integrals is done in [17].

Definition: Let $f : [0, \infty) \to \mathbb{R}$ be a function. Then, the γ–th order *conformable derivative* of f is defined by [17] as

$$T_\gamma(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\gamma}) - f(t)}{\epsilon}, \ \forall\, t > 0, \quad (1)$$

where $\gamma \in (0, 1]$. If f is γ-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t\to 0^+} f^\gamma(t)$ exists, then, it is defined as

$$f^\gamma(0) = \lim_{t\to 0^+} f^\gamma(t), \quad (2)$$

and the *conformable integral* of a function, $I^a_\gamma(f)(t)$, starting from $a \geq 0$, is defined as,

$$I^a_\gamma(f)(t) = \int_a^t \frac{f(x)}{x^{1-\gamma}} dx. \quad (3)$$

The integral is the usual Riemann improper integral, and $\gamma \in (0, 1]$. The most important properties of this conformable derivative are given in the following theorem.

Theorem: Let $\gamma \in (0, 1]$, and f and g be γ-differentiable at any point $t > 0$. Then,

1. $T_\gamma(af + bg) = aT_\gamma(f) + bT_\gamma(g)$ for all $a, b \in \mathbb{R}$,
2. $T_\gamma(t^p) = pt^{p-\gamma}$ for all $p \in \mathbb{R}$,

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3. $T_\gamma(\lambda) = 0$ for all constant function $f(t) = \lambda$,
4. $T_\gamma(fg) = fT_\gamma(g) + gT_\gamma(f)$,
5. $T_\gamma\left(\frac{f}{g}\right) = \frac{gT_\gamma(f) - fT_\gamma(g)}{g^2}$,
6. $T_\gamma(f)(t) = t^{1-\gamma}\frac{df}{dt}$, if f is differentiable.

This conformable derivative has attracted the interest of researchers in recent years since it seems to satisfy all the requirements of the standard derivative [18], [19]. Hence, there is a large amount of work done in this area at the present time [20–24].

We would like to point out that the term “fractional" is not the one used in the notation of fractional calculus [25]. However, the properties of this conformable derivative make it suitable for investigating real systems and to get new insights due to the presence of the fractional parameter $0 < \gamma \leq 1$.

In this paper, we use the conformable derivative to discuss two-dimensional projectile motion. First, we consider free two-dimensional projectile motion. Subsequently, new formulae for the trajectory, height, range, flight time, optimal angle, and maximum range are obtained. We show that these formulae depend on the conformable derivative order $0 < \gamma \leq 1$, and in the particular case of $\gamma = 1$, they become ordinary. Finally, we consider two-dimensional projectile motion in a resistive medium.

This paper is organized as follows. In Section 2, the main results of ordinary free projectile motion are reviewed. In Section 3, new analytical solutions for this motion are obtained. In Section 4, we consider motion in resistive media. Finally, the conclusions are given in Section 5.

2 Ordinary free projectile motion

We study the motion of a projectile moving in a free two-dimensional space. The projectile is treated as a particle of mass (m), under a uniform gravitational force (g), and disregarding any other resisting or external force. Assuming the particle starts from rest (*i.e.*, $x_0 = y_0 = 0$ m), with an initial velocity of modulus v_0 and angle θ with respect to the horizontal axis x, then, the classical equations of motion for this system, in the x, y plane, are,

$$m\frac{d^2x}{dt^2} = 0, \tag{4}$$

$$m\frac{d^2y}{dt^2} = -mg, \tag{5}$$

with the initial conditions,

$$x_0 = x(0) = 0, \qquad \dot{x}(0) = v_0\cos\theta, \tag{6}$$

$$y_0 = y(0) = 0, \qquad \dot{y}(0) = v_0\sin\theta. \tag{7}$$

The solutions, given in parametric form, are

$$x(t) = v_0 t\cos\theta, \qquad y(t) = v_0 t\sin\theta - \frac{1}{2}gt^2, \tag{8}$$

and the corresponding velocities are

$$v_x = v_0\cos\theta, \qquad v_y = v_0\sin\theta - gt. \tag{9}$$

By eliminating the time t in the expressions (8), we obtain the trajectory equation, given by the parabola

$$y(x) = x\tan\theta - \frac{g}{2v_0^2}\frac{x^2}{\cos^2\theta}. \tag{10}$$

Moreover, three interesting features for studying two-dimensional projectile motion are described as follows. *Range* (R) is defined as the horizontal distance traveled by the particle, from start to end. It can be obtained with (10) assuming the landing condition, $y(x = R) = 0$ m,

$$R = \frac{2v_0^2}{g}\sin\theta\cos\theta. \tag{11}$$

Maximum altitude (y_{max}) is the highest point reached by the particle. In this point, $v_y = 0$ m/s and, from (9), we have $t_{max} = v_0\sin\theta/g$. Substituting t_{max} in the second equation of (8), we have

$$y_{max} = \frac{v_0^2\sin^2\theta}{2g}. \tag{12}$$

Flight time (T_{flight}) is the amount of time that the projectile spends in the air, *i.e.*, from its shooting to its landing [26]. It is determined as

$$T_{flight} = 2t_{max} = \frac{2v_0\sin\theta}{g}. \tag{13}$$

3 Free projectile motion in view of conformable derivative

Usually, authors replace integer derivative operators with fractional ones on a purely mathematical basis [26–28]. However, this practice is *not* completely correct, from the physical and engineering point of view, and a dimensional correction in the new equation is necessary. Having this in mind, in [29] has been proposed a systematic way to construct fractional differential equations; this has been successfully applied in [30–33]. This procedure considers the introduction of parameters such as σ_t and σ_x with an appropriate dimensionality. In other words,

$$\frac{d}{dt} \to \frac{1}{\sigma_t^{1-\gamma}}\frac{d^\gamma}{dt^\gamma}, \qquad \frac{d}{dx} \to \frac{1}{\sigma_x^{1-\gamma}}\frac{d^\gamma}{dx^\gamma}, \tag{14}$$

where γ is an arbitrary parameter which represents the order of the derivative $0 < \gamma \le 1$. σ_t is a parameter representing the fractional time components in the system with dimensionality in time unit $[t]$; likewise, the parameter σ_x has dimension of length unit $[l]$, and represents the spatial fractional components [29]. These are the only conditions imposed on parameters σ_t and σ_x.

Therefore, we fractionalize the equations (4) and (5), considering the first relation in (14), and using the conformable derivative, in particular the property 6 from the previous theorem

$$\frac{d^\gamma}{dt^\gamma} f(t) = t^{1-\gamma} \frac{d}{dt} f(t), \tag{15}$$

and arrive at [34]

$$\frac{d}{dt} \to \frac{t^{1-\gamma}}{\sigma^{1-\gamma}} \frac{d}{dt}, \qquad 0 < \gamma \le 1, \tag{16}$$

where we have omitted the subscript t in σ. Substituting into (9), we obtain the corresponding conformable differential equations

$$\frac{dx}{dt} = v_0 \sigma^{1-\gamma} t^{\gamma-1} \cos\theta, \tag{17}$$

$$\frac{dy}{dt} = v_0 \sigma^{1-\gamma} t^{\gamma-1} \sin\theta - g\sigma^{1-\gamma} t^{\gamma}. \tag{18}$$

We can integrate them directly

$$x(t;\gamma) = \frac{\sigma^{1-\gamma}}{\gamma} t^{\gamma} v_0 \cos\theta, \tag{19}$$

$$y(t;\gamma) = -\frac{g\sigma^{1-\gamma}}{\gamma+1} t^{\gamma+1} + \frac{\sigma^{1-\gamma}}{\gamma} t^{\gamma} v_0 \sin\theta. \tag{20}$$

We also determine the components of the conformable velocity of the system by differentiating with respect to time as

$$v_x(t;\gamma) = v_0 \sigma^{1-\gamma} t^{\gamma-1} \cos\theta, \tag{21}$$

$$v_y(t;\gamma) = v_0 \sigma^{1-\gamma} t^{\gamma-1} \sin\theta - g\sigma^{1-\gamma} t^{\gamma}. \tag{22}$$

Unlike expressions (8) and (9), the coordinates and velocities in (19)–(22) depend on the fractional parameter $0 < \gamma \le 1$. In the particular case $\gamma = 1$, those are reduced to (8) and (9). We want to emphasize that the solutions given by (19) and (20) are qualitatively different from the solutions previously obtained in [26], [27] and [32]. Therefore, in the projectile motion under the conformable derivative sense, we can achieve new formulae for the trajectory, maximum height, range, flight time, maximum range, and optimal angle, as will be shown:

1. *Trajectory* ($y = y(x)$)*:* This is determined by eliminating t from (19) and replacing it in (20), for an arbitrary value of $\gamma \in (0, 1]$), giving

$$y(x;\gamma) = x\tan\theta - \frac{g\sigma^{1-\gamma}}{\gamma+1}\left(\frac{\gamma x}{\sigma^{1-\gamma} v_0 \cos\theta}\right)^{\frac{1}{\gamma}+1}. \tag{23}$$

   Note that (23) becomes (10), when $\gamma = 1$.
2. *Maximum height* (H)*:* This is obtained by using the condition $v_y(t;\gamma) = 0$ m/s, and employing (22), to find

$$t_{max} = \frac{v_0 \sin\theta}{g}, \tag{24}$$

   since t_{max} indicates the time when the particle has reached its maximum height; then, substituting this time in (20), we obtain the maximum height H, as

$$H = y_{max}(\theta;\gamma) = \frac{\sigma^{1-\gamma}(v_0 \sin\theta)^{\gamma+1}}{(\gamma+1) g^{\gamma}}. \tag{25}$$

   We have the ordinary maximum height (12), when $\gamma = 1$.
3. *Range* (R)*:* The maximum displacement is calculated from the trajectory equation, and it occurs when $y(x = R) = 0$ m. That is,

$$\begin{aligned} R &= x_{max}(\theta;\gamma) \\ &= \frac{\sigma^{1-\gamma}}{\gamma}\left(\frac{1+\gamma}{\gamma g}\right)^{\gamma} v_0^{\gamma+1} \cos\theta (\sin\theta)^{\gamma}. \end{aligned} \tag{26}$$

   Note again, if $\gamma = 1$, we have (11).
4. *Flight time* (T_{flight})*:* This is defined as the time value T at which the projectile hits the ground, $y(t = T) = 0$ m, as

$$T_{flight}(\theta;\gamma) = \left(\frac{\gamma+1}{\gamma}\right)\frac{v_0 \sin\theta}{g}. \tag{27}$$

   In the case $\gamma = 1$, we have (13).
5. *Maximum range* (R_{max})*:* This feature is considerably interesting for both practical and theoretical studies. It can be determined by finding the optimal projection angle θ_{op}, and evaluating it in $R_{max} = R(\theta_{op})$. The necessary condition to maximize the range, $R \to R_{max}$ is given by $\left[dR/d\theta\right]_{\theta_{op}} = 0$. Thus, from (26) we have

$$\begin{aligned} \left.\frac{dR}{d\theta}\right|_{\theta_{op}} &= \frac{d}{d\theta}\left[\cos\theta(\sin\theta)^{\gamma}\right]_{\theta_{op}} \\ &= -(\sin\theta_{op})^{\gamma+1} + \gamma(\cos\theta_{op})^2(\sin\theta_{op})^{\gamma-1} \\ &= 0. \end{aligned} \tag{28}$$

   By solving this equation with respect to θ_{op}, we obtain

$$\theta_{op} = \tan^{-1}\sqrt{\gamma}, \qquad 0 < \gamma \le 1. \tag{29}$$

In the case $\gamma = 1$, the optimal angle is $\theta_{op} = \pi/4$ (as expected). Hence, using trigonometric formulae,

$$\sin\theta = \frac{\tan\theta}{\sqrt{1 + (\tan\theta)^2}} = \frac{\sqrt{\gamma}}{\sqrt{1+\gamma}}, \tag{30}$$

$$\cos\theta = \frac{1}{\sqrt{1 + (\tan\theta)^2}} = \frac{1}{\sqrt{1+\gamma}}, \tag{31}$$

and, by replacing this expression in (26), we obtain the maximum projectile range R_{max} as

$$R_{max} = \frac{\sigma^{1-\gamma}}{\gamma}\left(\frac{1+\gamma}{\gamma g}\right)^{\gamma}\left(\frac{\gamma}{1+\gamma}\right)^{\gamma/2}\frac{v_0^{\gamma+1}}{\sqrt{1+\gamma}}. \tag{32}$$

From this expression, in the case $\gamma = 1$, we get the classical maximum range $R_{cl,maximum} = v_0^2/g$.

These new formulae have the same dimensionality as in the ordinary case, because σ has dimension of seconds. Furthermore, for demonstrative purposes, Figure 1 shows the trajectory (Figure 1a), the maximum height (Figure 1b), and the range (Figure 1c) of a particle in projectile motion with different values of γ.

4 Resistive projectile motion in view of conformable derivative

The motion equations of a projectile going through an isotropic medium in two-dimensional space with a resistive force proportional to its velocity, have the form [35]:

$$\frac{dv_x}{dt} + kv_x = 0, \tag{33}$$

$$\frac{dv_y}{dt} + kv_y = -g, \tag{34}$$

where k is a positive constant and its dimensionality is the inverse of seconds $[k] = s^{-1}$. The initial conditions are given by

$$x_0 = x(0) = 0\,\text{m}, \qquad \dot{x} = v_0\cos\theta, \tag{35}$$

$$y_0 = y(0) = 0\,\text{m}, \qquad \dot{y} = v_0\sin\theta, \tag{36}$$

namely, the projectile starts from rest, with an initial velocity of module v_0 and an angle θ with respect the x-axis. The solutions of these equations (33) and (34) and satisfying the initial conditions (35) and (36) are given in [35]

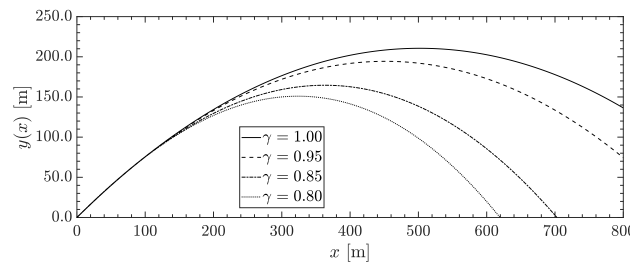
$$v_x(t) = v_0\cos\theta\, e^{-kt}, \tag{37}$$

$$v_y(t) = -\frac{g}{k} + \left(v_0\sin\theta + \frac{g}{k}\right) e^{-kt}, \tag{38}$$

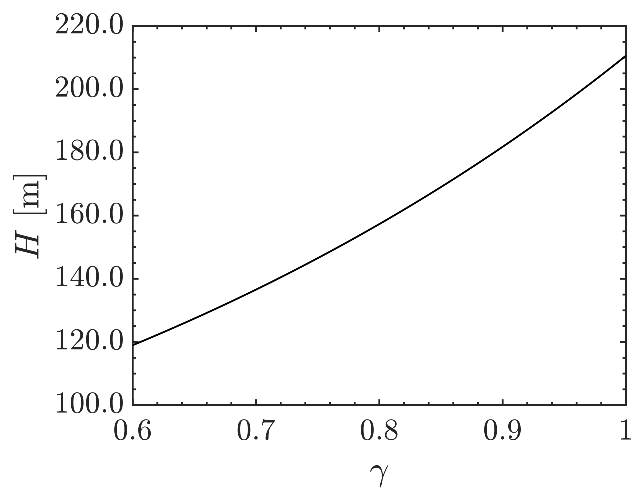
and

$$x(t) = \frac{v_0\cos\theta}{k}\left(1 - e^{-kt}\right), \tag{39}$$

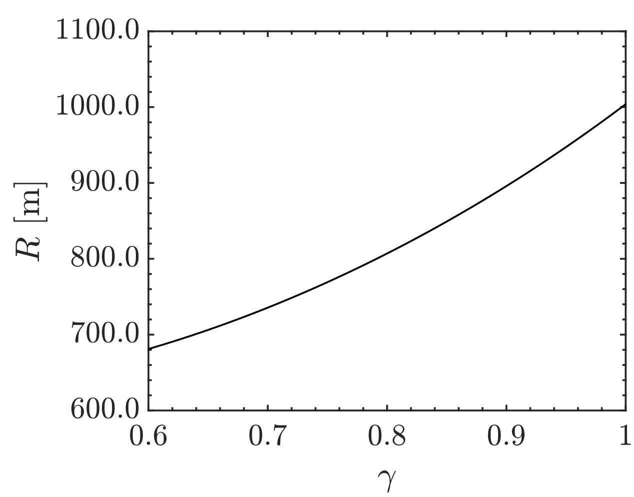
$$y(t) = -\frac{g}{k}t + \frac{1}{k}\left(v_0\sin\theta + \frac{g}{k}\right)\left(1 - e^{-kt}\right). \tag{40}$$



(a) Trajectory from (23)



(b) Height from (25)



(c) Range from (26)

Figure 1: Plots of trajectory, maximum height and range for different values of γ, with $v_0 = 100\,\text{m/s}$, $g = 9.81\,\text{m/s}^2$, and $\theta = 40°$

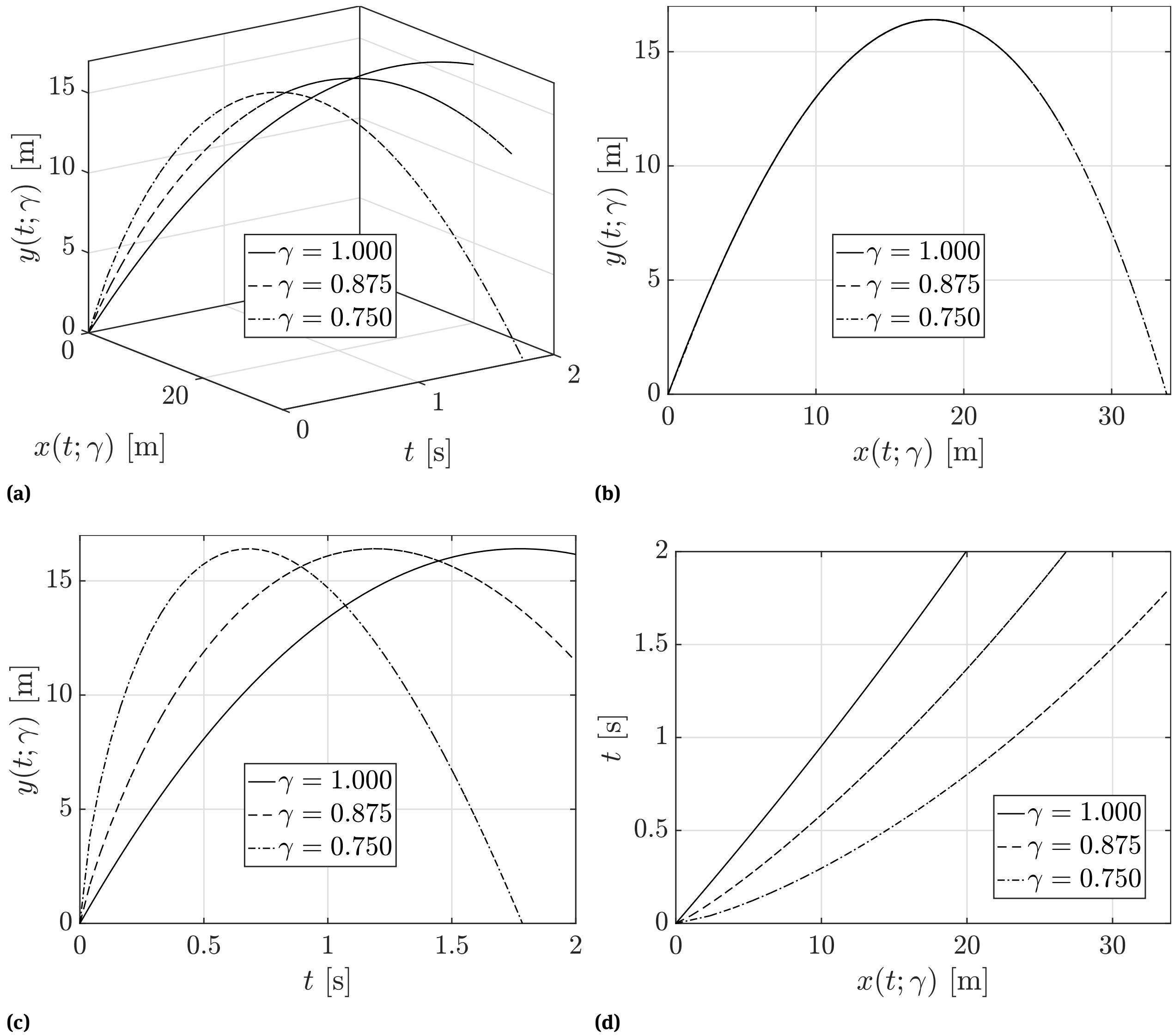


Figure 2: Projectile motion in terms of $x(t;\gamma)$, $y(t;\gamma)$ and t, within a resistive medium ($k = 0.1$ s^{-1}) and for different values of γ (*i.e.*, γ = 1.000, 0.875, and 0.750), with v_0 = 22 m/s, g = 9.81 m/s^2, and θ = 60°

Now, we fractionalize the equations (33) and (34). In the case of motion in a resistive medium, we can use k instead of σ in (16) [33], namely

$$\frac{d}{dt} \to k^{1-\gamma} t^{1-\gamma} \frac{d}{dt}, \qquad 0 < \gamma \leq 1. \tag{41}$$

Substituting the operator in (41) into equations (33) and (34), we have the conformable differential equations

$$\frac{dv_x}{dt} + k^{\gamma} t^{\gamma-1} v_x = 0, \tag{42}$$

$$\frac{dv_y}{dt} + k^{\gamma} t^{\gamma-1} v_y = -g k^{\gamma-1} t^{\gamma-1}. \tag{43}$$

These expressions are ordinary linear, homogeneous and non-homogeneous differential equations, hence, they can be solved easily using the ordinary methods. The analytical solutions that satisfy initial conditions (35) and (36) are:

$$v_x(t;\gamma) = v_0 \cos\theta \; e^{\left(-\frac{k^{\gamma}}{\gamma} t^{\gamma}\right)}, \tag{44}$$

$$v_y(t;\gamma) = -\frac{g}{k} + \left(v_0 \sin\theta + \frac{g}{k}\right) \; e^{\left(-\frac{k^{\gamma}}{\gamma} t^{\gamma}\right)}. \tag{45}$$

In the case $\gamma = 1$, these equations become (37) and (38). Once again, replacing (41) in the equations (44) and (45), and integrating them, we get the parametric formulae as:

$$x(t;\gamma) = \frac{v_0 \cos\theta}{k}\left(1 - e^{-\frac{k^{\gamma}}{\gamma} t^{\gamma}}\right), \tag{46}$$

$$y(t;\gamma) = -\frac{gk^{\gamma-2}}{\gamma}t^{\gamma} + \frac{1}{k}\left(v_0 \sin\theta + \frac{g}{k}\right)\left(1 - e^{-\frac{k\gamma}{\gamma}t^{\gamma}}\right). \quad (47)$$

Moreover, if $\gamma = 1$, then equations (46) and (47) become the ordinary equations (39) and (40). For the particular case of angle $\theta = 90°$, we have

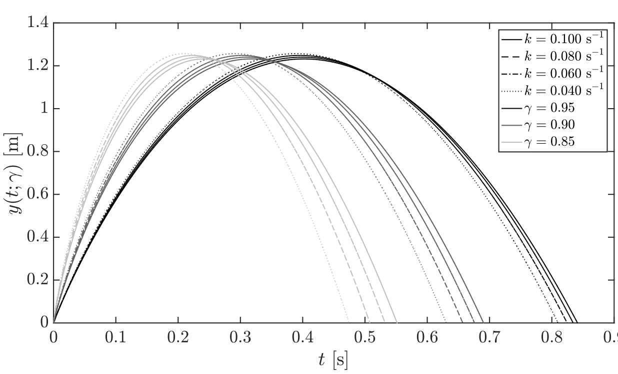
$$x(t;\gamma) = 0, \quad (48)$$

$$y(t;\gamma) = -\frac{gk^{\gamma-2}}{\gamma}t^{\gamma} + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)\left(1 - e^{-\frac{k\gamma}{\gamma}t^{\gamma}}\right). \quad (49)$$

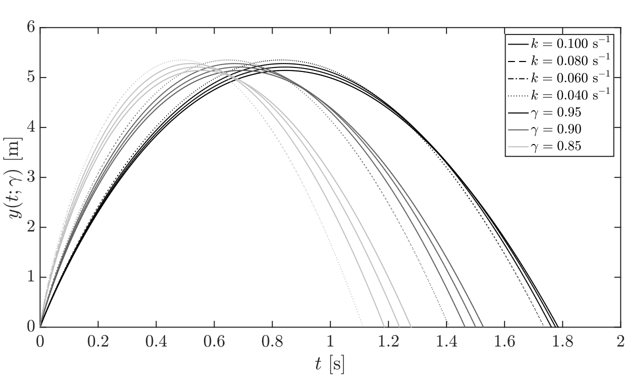
Therefore, we only have a vertical conformable trajectory [34].

As an illustrative example, Figure 2 presents the projectile motion for a defined medium with $k = 0.1\ \mathrm{s}^{-1}$, in terms of $x(t;\gamma)$, $y(t;\gamma)$ and t. The main tri-dimensional view of this behavior is displayed with its planes.

In addition, the vertical displacement for different values of γ, k, velocities and angles are shown in the Figure 3. It is observed that the trajectory varies with γ (cf. Figure 1), but there are negligible changes for different values of k.



(a) $v_0 = 10$ m/s, and $\theta = 30°$



(b) $v_0 = 12$ m/s, and $\theta = 60°$

Figure 3: The vertical displacement of a body for different values of γ, k, and initial conditions

5 Conclusion

In this work, we have considered free and resistive two-dimensional projectile motion in the conformable derivative sense. New formulae for the trajectory (23), height (25), range (26), flight time (27), optimal angle (29), and maximum range (32) were obtained. Also, calculations were performed in the case of a resistive medium (46) and (47). The obtained formulae are qualitatively different from the solutions previously obtained in [26], [27], [32] and [33]. Moreover, we found that the new formulae depend on the conformable derivative order $0 < \gamma \leq 1$, and they became ordinary in the particular case $\gamma = 1$. We noticed that the obtained trajectories never exceeded the ordinary ones where $\gamma = 1$, unlike the results obtained in other studies [27].

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