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Neighborhood condition for all fractional (g, f, n', m)-critical deleted graphs

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Abstract: In data transmission networks, the availability of data transmission is equivalent to the existence of the fractional factor of the corresponding graph which is generated by the network. Research on the existence of fractional factors under specific network structures can help scientists design and construct networks with high data transmission rates. A graph G is named as an all fractional (g, f, n', m)-critical deleted graph if the remaining subgraph keeps being an all fractional (g, f, m)-critical graph, despite experiencing the removal of arbitrary n' vertices of G. In this paper, we study the relationship between neighborhood conditions and a graph to be all fractional (g, f, n', m)-critical deleted. Two sufficient neighborhood conditions are determined, and furthermore we show that the conditions stated in the main results are sharp.

Keywords: graph, data transmission network, all fractional factor, all fractional (g, f, n', m)-critical deleted graph

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1 Introduction

The classic fractional factor problem is usually regarded as an extension of illustrious cardinality matching which is one of the hot topics in graph theory and operations research. Its extensive applications can be found in many domains like combinatorial polyhedron, scheduling and designing of network. For instance, we send some large data packets to several destinations through channels of the data transmission network, and the efficiency will be improved if the large data packets can be categorized into

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more smaller ones. The problem of feasibility allocations of data packets can be considered as the existence of fractional flow in the network, and it can be transformed to the fractional factor problem in the network graph.

In particular, the whole network can be modelled as a graph in which each vertex represents a site and each edge denotes a channel. In normal networks, the route of data transmission is picked out based on the shortest path between vertices. A few advances of data transmission in networks have been manifested in recent years. Rolim et al. [1] improved the data transmission by studying an urban sensing problem in view of opportunistic networks. Vahidi et al. [2] considered unmanned aerial vehicles and proposed the high-mobility airborne hyperspectral data transmission algorithm. Miridakis et al. [3] adopted a cost-effective solution to study the dual-hop cognitive secondary relaying system. Streaming data transmission on a discrete memoryless channel was discussed by Lee et al. [4]. But, in the context of software definition network (SDN), the path between vertices in data transmission relies on the current network flow computation. With minimum transmission congestion in the current moment, the transmission route is chosen. In this view, the framework of data transmission problem in SDN equals to the existence of the fractional factor avoiding certain subgraphs.

Throughout this paper, we only consider the simple graph which corresponds to a data transmission network. Let G=(V(G), thenE(G)) be a graph with vertex set V(G) and edge set E(G). For any $x\in V(G)$, we denote $d_G(x)$ and $N_G(x)$ by the degree and the open neighborhood of x in G, separately. Set $N_G[x]=N_G(x)\cup\{x\}$ as the closed neighborhood of x in G. We denote by G[S] the subgraph of G induced by G[S], and $G-S=G[V(G)\setminus S]$ for any G[S] for any G[S]. For two vertex-disjoint subsets G[S], we set G[S], we set G[S], we set G[S], we denote the minimum degree of G[S] by G[S] in G[S] we denote the minimum degree of G[S] by G[S] in G[S] as a simple expression, we take G[S] to express G[S] for G[S]. More terminologies and notations used but undefined in our article can be found in book Bondy and Mutry G[S].

Let g and f be integer-valued functions on V(G) satisfying $0 \le g(x) \le f(x)$ for any $x \in V(G)$. A fractional (g,f)-factor can be considered as a function h that gives

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to every edge of a graph G a real number in [0,1] with $g(x) \le \sum_{e \in E(x)} h(e) \le f(x)$ for each vertex x. If g(x) = f(x)

for any $x \in V(G)$, a fractional (g, f)-factor is a fractional *f*-factor. If g(x) = a and f(x) = b for any $x \in V(G)$, then a fractional (g, f)-factor is a fractional [a, b]-factor. Furthermore, a fractional (g, f)-factor becomes a fractional kfactor if g(x) = f(x) = k (where $k \ge 1$ is an integer) for any $x \in V(G)$. Then, we always assume that *n* is the order of *G*, namely, n = |V(G)|.

If *G* has a fractional *p*-factor for every $p:V(G)\to \mathbb{N}$ such that $g(x) \le p(x) \le f(x)$ for any $x \in V(G)$, G possesses all fractional (g, f)-factors. If g(x) = a, f(x) = b for each $x \in V(G)$ and G possesses all fractional (g, f)-factors, then G has all fractional [a, b]-factors.

Lu [6] determined the sufficient and necessary condition for a graph that has all fractional (g, f)-factors.

Theorem 1 (Lu [6]) Assume *G* to be a graph, $g, f : V(G) \rightarrow$ \mathbb{Z}^+ be integer functions so that $g(x) \leq f(x)$ for all $x \in V(G)$, so G has all fractional (g, f)-factors. If and only if for any subset $S \subseteq V(G)$, we have

$$g(S)-f(T)+\sum_{x\in T}d_{G-S}(x)\geq 0,$$

where $T = \{x \in V(G) - S | d_{G-S}(x) < f(x) \}$.

Obviously, this sufficient and necessary condition equals to the following version.

Theorem 2 Assume *G* be a graph, $g, f : V(G) \to \mathbb{Z}^+$ be integer functions so that $g(x) \le f(x)$ for all $x \in V(G)$. As a result, G has all fractional (g, f)-factors. If and only if

$$g(S) - f(T) + \sum_{x \in T} d_{G-S}(x) \ge 0$$

for arbitrary non-disjoint subsets S, $T \subset V(G)$.

Set g(x) = a, f(x) = b for each $x \in V(G)$, we immediately get the below corollary from Theorem 1.

Corollary 1 Let G be a graph and $a \le b$ be two positive integers. Then G has all fractional [a, b]-factors if and only if for any subset $S \subseteq V(G)$, we have

$$a|S| - b|T| + \sum_{x \in T} d_{G-S}(x) \ge 0,$$

where $T = \{x \in V(G) - S | d_{G-S}(x) < b\}$.

Also, this sufficient and necessary condition in Corollary 1 for all fractional [a, b]-factors equals to the following version.

Corollary 2 Let *G* be a graph and $a \le b$ be two positive integers. Then G has all fractional [a, b]-factors if and only if

$$a|S|-b|T|+\sum_{x\in T}d_{G-S}(x)\geq 0$$

for arbitrary non-disjoint subsets S, $T \subset V(G)$.

Zhou and Zhang [7] presented the sufficient and necessary situation for a graph with all fractional (g, f)-factors excluding a subgraph H.

Theorem 3 (Zhou and Zhang [7]) Let G be a graph and Hbe a subgraph of G. Let $g, f : V(G) \to \mathbb{Z}^+$ be two integervalued functions with $g(x) \le f(x)$ for each $x \in V(G)$. Then G admits all fractional (g, f)-factors excluding H if and only if

$$g(S) - f(T) + \sum_{x \in T} d_{G-S}(x) \ge \sum_{x \in T} d_H(x) - e_H(S, T)$$

for arbitrary subset *S* of V(G), where $T = \{x \in V(G) - (G)\}$ $S|d_{G-S}(x) - d_H(x) + e_H(x, S) < f(x)\}.$

Clearly, this sufficient and necessary condition has the following equal version.

Theorem 4 Let G be a graph and $g, f : V(G) \to \mathbb{Z}^+$ be two integer-valued functions with $g(x) \le f(x)$ for each $x \in V(G)$. Let H be a subgraph of G. Then G admits all fractional (g, f)-factors excluding H if and only if

$$g(S) - f(T) + \sum_{x \in T} d_{G-S}(x) \ge \sum_{x \in T} d_H(x) - e_H(S, T)$$

for arbitrary non-disjoint subsets S, $T \subset V(G)$.

Set g(x) = a, f(x) = b for each $x \in V(G)$, we immediate get the Corollary below.

Corollary 3 Let *G* be a graph, and *a* and *b* be two integers where $1 \le a \le b$. Let *H* be a subgraph of *G*. Then *G* admits all fractional [a, b]-factors excluding H if and only if

$$a|S| + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - b) \ge 0,$$

where $T = \{x \in V(G) - S | d_{G-S}(x) - d_H(x) + e_H(x, S) < b\}.$

Again, this sufficient and necessary condition can be stated as follows.

Corollary 4 Let a and b be two integers with $1 \le a \le b$, and let *G* be a graph. Let *H* be a subgraph of *G*. So *G* admits all fractional [a, b]-factors excluding H if and only if

$$a|S| + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - b) \ge 0$$

for arbitrary non-disjoint subsets S, $T \subset V(G)$.

Zhou and Sun [8] introduced the concept of all fractional (a, b, n')-critical graphs. A graph G is named as an all fractional (a, b, n')-critical graph if the remaining graph of G has all fractional [a, b]-factors, even though any n' vertices of G are deleted. Also, they presented the necessary and valuable situation for a graph to be all fractional (a, b, n')-critical.

Theorem 5 (Zhou and Sun [8]) Let a, b and n' be nonnegative integers with $1 \le a \le b$, and let G be a graph of order n with $n \ge a + n' + 1$. Then *G* is all fractional (a, b, n')-critical if and only if for any $S \subset V(G)$ with $|S| \ge n'$,

$$a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| \ge an',$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) < b\}.$

Equally, the above necessary and sufficient condition can be re-written as follows.

Theorem 6 Assume a, b and n' be nonnegative integers with $1 \le a \le b$, hence suppose G is a graph of order n with $n \ge a + n' + 1$. Then, G is all fractional (a, b, n')-critical if and only if

$$a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| \ge an'$$

for arbitrary non-disjoint subsets $S, T \subset V(G)$ with $|S| \ge n'$.

More results from this topic, regarding fractional factor, fractional deleted graphs, fractional critical and other network applications can be found in Zhou et al. [9], Jin [10], Gao and Wang [11] and [12], Gao et al. [13], [14] and [15], and Guirao and Luo [16].

In this paper, we first introduce some new concepts. A graph G is named as an all fractional (g,f,m)-deleted graph if the remaining graph of G has all fractional (g,f)-factors, when any m edges of G are deleted. If for any $x \in V(G)$, we have g(x) = a and f(x) = b, then an all fractional (g,f,m)-deleted graph becomes an all fractional (a,b,m)-deleted graph, i.e., a graph G is named as an all fractional (a,b,m)-deleted graph if the remaining graph of G has all fractional [a,b]-factors, when any m edges of G are deleted. Following this definition, we clearly see that Theorem 3 and Theorem 4 show the necessary and sufficient condition of all fractional (g,f,m)-deleted graph with |E(H)| = m respectively, and Corollary 3 and Corollary 4 present the necessary and sufficient condition of all fractional (a,b,m)-deleted graph respectively.

Next, we combine two concepts, all fractional (g, f, m)-deleted graph and all fractional (g, f, m)-critical graph together. A graph G is named as an all fractional (g, f, n', m)-critical deleted graph if the remaining graph of G is still an all fractional (g, f, m)-deleted graph, when any n' vertices of G are deleted. If g(x) = a, f(x) = b for each $x \in V(G)$, every fractional (g, f, n', m)-critical deleted graph becomes all fractional (a, b, n', m)-critical deleted graph, it means, after deleting any n' vertices of G the remaining graph of G is still an all fractional (a, b, m)-deleted graph.

The concept of all fractional (g, f, n', m)-critical deleted graph reflects the feasibility of data transmission in data transmission networks, in the case of some sites or channels being damaged. At the same time, it also pro-

vides theoretical support for SDN: in the peak data transmission, some sites and communications are often in the situation of information congestion, so we regard the site and channel in a blocked state as the vertices and edges that need to be deleted. Then, we consider the existence of fractional factors in the resulting subgraph. Thus, it inspires us to study the sufficient condition of all fractional (g, f, n', m)-critical deleted graph from the graph structure prospect. It will imply which structures of network can ensure the success of data transmission, and the theoretical results obtained in our article can help us carry out the network design.

In this paper, we explore relations between the neighborhood union condition and all fractional (g, f, n', m)-critical deleted graphs. The first key result can be formulated as follows.

Theorem 7 Assume a, b, n' and m are four non-negative integers satisfying $2 \le a \le b$. Assume G to be a graph with $\delta(G) \ge \frac{(a+b-1)^2+4b}{4a} + m + n'$ and $n \ge \frac{2(a+b)(a+b+m-1)}{a} + n'$. Let g and f be two integer functions which are defined on V(G) such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If $|N_G(u) \cup N_G(v)| \ge \frac{bn+an'}{a+b}$ for arbitrary two nonadjacent vertices u and v in G, then G is all fractional (g, f, n', m)-critical deleted.

Set n' = 0 in Theorem 7, then we get the following corollary.

Corollary 5 Assume a, b and m be three non-negative integers satisfying $2 \le a \le b$. Assume G is a graph with $\delta(G) \ge \frac{(a+b-1)^2+4b}{4a} + m$ and $n \ge \frac{2(a+b)(a+b+m-1)}{a}$. Let g and f be two integer functions whose definition can be found on V(G) such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If $|N_G(u) \cup N_G(v)| \ge \frac{bn}{a+b}$ for arbitrary two nonadjacent vertices u and v in G, then G is all fractional (g, f, m)-deleted.

By setting m = 0 in Theorem 7, the following corollary is obtained.

Corollary 6 Assume a, b and n' to be three non-negative integers satisfying $2 \le a \le b$. Let G be a graph with $\delta(G) \ge \frac{((a+b-1)^2+4b)}{4a}+n'$ and $n \ge \frac{2(a+b)(a+b-1)}{a}+n'$. Let g and f be two integer functions whose definition can be found on V(G) such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If $|N_G(u) \cup N_G(v)| \ge \frac{bn+an'}{a+b}$ for arbitrary two nonadjacent vertices u and v in G, then G is all fractional (g, f, n')-critical.

Set g(x) = a and f(x) = b for any $x \in V(G)$, then we get the following condition for all fractional (a, b, n', m)-critical deleted graph.

Corollary 7 Let a, b, n' and m be four non-negative integers satisfying $2 \le a \le b$. Let G be a graph with $\delta(G) \ge \frac{(a+b-1)^2+4b}{4a} + m + n'$ and $n \ge \frac{2(a+b)(a+b+m-1)}{a} + n'$. If $|N_G(u) \cup N_G(v)| \ge \frac{bn+an'}{a+b}$ for arbitrary two nonadjacent

vertices u and v in G, then G is all fractional (a, b, n', m)critical deleted.

Set n' = 0 in Corollary 7, then the corollary below can be obtained.

Corollary 8 Let *a*, *b* and *m* be three non-negative integers satisfy $2 \le a \le b$. Let G be a graph with $\delta(G) \ge \frac{(a+b-1)^2+4b}{4a}+m$ and $n \ge \frac{2(a+b)(a+b+m-1)}{a}$. If $|N_G(u) \cup N_G(v)| \ge \frac{bn}{a+b}$ for arbitrary two nonadjacent vertices u and v in G, then G is all fractional (a, b, m)-deleted.

Set m = 0 in Corollary 7, then the following corollary is obtained.

Corollary 9 Let a, b and n' be three non-negative integers satisfy $2 \le a \le b$. Let *G* be a graph with $\delta(G) \ge \frac{(a+b-1)^2+4b}{4a} +$ n' and $n \ge \frac{2(a+b)(a+b-1)}{a} + n'$. If $|N_G(u) \cup N_G(v)| \ge \frac{bn+an'}{a+b}$ for arbitrary two nonadiacent vertices *u* and *v* in *G*, then *G* is all fractional (a, b, n')-critical.

Our second main result is stated as follows.

Theorem 8 Let a, b, n' and m be four non-negative integers satisfying $2 \le a \le b$. Let G be a graph with order $n \ge \frac{(a+2b-3)(a+b-2)}{a} + \frac{an'}{a-1} + m$. Let g and f be two integer functions whose definition can be found on V(G) such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. Suppose that

$$N_G(X)=V(G) \quad \text{if} \quad |X|\geq \lfloor \frac{(an-a-an'-2m)n}{(a+b-1)(n-1)} \rfloor;$$

or

$$N_G(X) \ge \frac{(a+b-1)(n-1)}{an-a-an'-2m}|X|$$

if
$$|X| < \lfloor \frac{(an-a-an'-2m)n}{(a+b-1)(n-1)} \rfloor$$

for any subset $X \subseteq V(G)$. Then G is all fractional (g, f, n', m)-critical deleted.

Set n' = 0 in Theorem 8, then we yield the following

Corollary 10 Let a, b, and m be three non-negative integers satisfy $2 \le a \le b$. Let G be a graph with order $n \ge \frac{(a+2b-3)(a+b-2)}{a} + m$. Let g and f be two integer functions defined on V(G) such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. Suppose that

$$N_G(X) = V(G)$$
 if $|X| \ge \lfloor \frac{(an-a-2m)n}{(a+b-1)(n-1)} \rfloor$;

or

$$N_G(X) \ge \frac{(a+b-1)(n-1)}{an-a-2m}|X|$$

if
$$|X| < \lfloor \frac{(an-a-2m)n}{(a+b-1)(n-1)} \rfloor$$

for any subset $X \subseteq V(G)$. Then G is all fractional (g, f, m)-

Set m = 0 in Theorem 8, then the Corollary below can be inferred.

Corollary 11 Let a, b and n' be three non-negative integers satisfy $2 \le a \le b$. Let G be a graph with order $n \ge$ $\frac{(a+2b-3)(a+b-2)}{a} + \frac{an'}{a-1}$. Let g and f be two integer functions defined on V(G) such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. Suppose that

$$N_G(X) = V(G)$$
 if $|X| \ge \lfloor \frac{(an - a - an')n}{(a + b - 1)(n - 1)} \rfloor$;

or

$$N_G(X) \geq \frac{(a+b-1)(n-1)}{an-a-an'}|X|$$

if
$$|X| < \lfloor \frac{(an-a-an')n}{(a+b-1)(n-1)} \rfloor$$

for any subset $X \subseteq V(G)$. Then G is all fractional (g, f, n')critical.

Set g(x) = a and f(x) = b for any $x \in V(G)$, then we get the following condition for an all fractional (a, b, n', m)critical deleted graph.

Corollary 12 Let a, b, n' and m be four non-negative integers satisfying $2 \le a \le b$. Let G be a graph with order $n \ge \frac{(a+2b-3)(a+b-2)}{a} + \frac{an'}{a-1} + m$. Suppose that

$$N_G(X) = V(G)$$
 if $|X| \ge \lfloor \frac{(an-a-an'-2m)n}{(a+b-1)(n-1)} \rfloor$;

or

$$N_G(X) \geq \frac{(a+b-1)(n-1)}{an-a-an'-2m}|X|$$

if
$$|X| < \lfloor \frac{(an-a-an'-2m)n}{(a+b-1)(n-1)} \rfloor$$

for any subset $X \subset V(G)$. Then G is all fractional (a, b, n', m)-critical deleted.

Set n' = 0 in Corollary 12, then we have the following corollary.

Corollary 13 Let a, b and m be three non-negative integers satisfy $2 \le a \le b$. Let G be a graph with order $n \ge$ $\frac{(a+2b-3)(a+b-2)}{a} + m$. Suppose that

$$N_G(X) = V(G)$$
 if $|X| \ge \lfloor \frac{(an-a-2m)n}{(a+b-1)(n-1)} \rfloor$;

or

$$N_G(X) \geq \frac{(a+b-1)(n-1)}{an-a-2m}|X|$$

if
$$|X| < \lfloor \frac{(an-a-2m)n}{(a+b-1)(n-1)} \rfloor$$

for any subset $X \subseteq V(G)$. Then G is all fractional (a, b, m)deleted.

Set m = 0 in Corollary 12, then we deduce the following corollary.

Corollary 14 Let a, b and n' be three non-negative integers satisfying $2 \le a \le b$. Let G be a graph with order $n \ge \frac{(a+2b-3)(a+b-2)}{a} + \frac{an'}{a-1}$. Suppose that

$$N_G(X) = V(G)$$
 if $|X| \ge \lfloor \frac{(an-a-an')n}{(a+b-1)(n-1)} \rfloor$;

or

$$N_G(X) \ge \frac{(a+b-1)(n-1)}{an-a-an'}|X|$$

if $|X| < \lfloor \frac{(an-a-an')n}{(a+b-1)(n-1)} \rfloor$

for any subset $X \subseteq V(G)$. Then G is all fractional (a, b, n')-critical deleted.

2 Proof of main results

2.1 The necessary and sufficient condition of all fractional (g, f, n', m)-critical deleted graphs

To prove the main results in our paper, we first determine the necessary and sufficient condition of all fractional (g, f, n', m)-critical deleted graphs which is manifested as follows.

Theorem 9 a, b, m and n' are assumed to be nonnegative integers satisfying $1 \le a \le b$, and G is let to be a graph of order n with $n \ge b + n' + m + 1$. Let $g, f : V(G) \to \mathbb{Z}^+$ be two valued functions with $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$, and H will be a subgraph of G with m edges. After that, G is all fractional (g, f, n', m)-critical deleted if and only if for any $S \subseteq V(G)$ with $|S| \ge n'$,

$$g(S) - f(T) + \sum_{x \in T} d_{G-S}(x)$$

$$\geq \max_{U \subseteq S, |U| = n', H \subseteq E(G-U), |H| = m} \{g(U) + \sum_{x \in T} d_H(x) - e_H(S, T)\},$$

where

$$T = \{x : x \in V(G) \setminus S, d_{G-S}(x) - d_H(x) + e_H(x, S) < f(x)\}.$$
 (1)

The equal version of Theorem 9 is stated as follows.

Theorem 10 a, b, m and n' are assumed to be nonnegative integers satisfying $1 \le a \le b$, and let G be a graph of order n with $n \ge b + n' + m + 1$. Let g, $f : V(G) \to \mathbb{Z}^+$ be two valued functions with $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$, and H be a subgraph of G with m edges. Then, G is all fractional (g, f, n', m)-critical deleted if and only if

$$g(S) - f(T) + \sum_{x \in T} d_{G-S}(x)$$

$$\geq \max_{U \subseteq S, |U| = n', H \subseteq E(G-U), |H| = m} \{g(U) + \sum_{x \in T} d_H(x) - e_H(S, T)\},$$

for any non-disjoint subsets S, $T \subset V(G)$ with $|S| \ge n'$.

Set g(x) = a and f(x) = b for every $x \in V(G)$, and the following corollary on the necessary and sufficient condition of all fractional (a, b, n', m)-critical deleted graph is obtained.

Corollary 15 Assume a, b, m and n' be nonnegative integers satisfying $1 \le a \le b$, and G be a graph of order n with $n \ge a + n' + m + 1$. H is assumed to be a subgraph of G with m edges. G is all fractional (a, b, n', m)-critical deleted if and only if for any $S \subset V(G)$ with $|S| \ge n'$,

$$a|S| + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - b) \ge an',$$

where $T = \{x \in V(G) - S | d_{G-S}(x) - d_H(x) + e_H(x, S) < b\}.$

Clearly, the above necessary and sufficient condition has the equal version which is stated as follows.

Corollary 16 Assume a, b, m and n' to be nonnegative integers satisfying $1 \le a \le b$, and G a graph of order n with $n \ge a + n' + m + 1$. H is assumed to be a subgraph of G with m edges. G is all fractional (a, b, n', m)-critical deleted if and only if

$$|a|S| + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - b) \ge an^{\frac{1}{2}}$$

for any non-disjoint subsets S, $T \subseteq V(G)$ with $|S| \ge n'$.

Now, we present the detailed proof of Theorem 9.

Proof of Theorem 9. Assume $U \subseteq S \subseteq V(G)$ with |U| = n'. Let $S' = S \setminus U$ and G' = G - U. Then we have G' - S' = G - S. Moreover, let

$$T' = \{x : x \in V(G') \setminus S', d_{G'-S'}(x) - d_H(x) + e_H(x, S') < f(x)\},$$
 (2)

then by the definition of (1), we get T' = T. Furthermore, we infer

$$g(S') + \sum_{x \in T'} (d_{G'-S'}(x) - d_H(x))$$

$$+e_H(x, S') - f(x))$$

$$= g(S) + \sum_{x \in T} (d_{G-S}(x) - d_H(x))$$

$$+e_H(x, S) - f(x)) - g(U),$$

so that

$$g(S') + \sum_{x \in T'} (d_{G'-S'}(x) - d_H(x)$$

$$+e_H(x, S') - f(x)) \ge 0$$

$$\Leftrightarrow g(S) - f(T) + \sum_{x \in T} d_{G-S}(x)$$

$$\ge \max_{U \subseteq S, |U| = n', H \subseteq E(G-U), |H| = m} \{g(U) \}$$

$$+\sum_{x\in T}d_{H}(x)-e_{H}(S,T)$$
.

First, assume G be all fractional (g, f, n', m)-critical deleted, and let S, S', U, T' and G' be described as the above. So G' = G - U is all fractional (g, f, m)-deleted, thus in terms of Theorem 3, we have

$$g(S') + \sum_{x \in T'} (d_{G'-S'}(x) - d_H(x) + e_H(x, S') - f(x)) \ge 0.$$

Hence, using (3), we infer that

$$g(S) - f(T) + \sum_{x \in T} d_{G-S}(x)$$

$$\geq \max_{U \subseteq S, |U| = n', H \subseteq E(G-U), |H| = m} \{g(U) + \sum_{x \in T} d_H(x) - e_H(S, T)\}$$

holds for any $S \subset V(G)$ with $|S| \ge n'$.

Conversely, suppose that

$$g(S) - f(T) + \sum_{x \in T} d_{G-S}(x)$$

$$\geq \max_{U \subseteq S, |U| = n', H \subseteq E(G-U), |H| = m} \{g(U) + \sum_{x \in T} d_H(x) - e_H(S, T)\}$$

holds for any $S \subset V(G)$ with $|S| \ge n'$. Let $U \subset V(G)$ satisfy |U| = n' and G' = G - U. For any $S' \subseteq V(G')$, define T'according to (2) and let $S = S' \cup U$. Thus, $S' = S \setminus U$, and in light of (3) we have

$$g(S') + \sum_{x \in T'} (d_{G'-S'}(x) - d_H(x) + e_H(x, S') - f(x)) \ge 0$$

holds for any $S' \subseteq V(G')$. It implies that G' = G - U is an all fractional (g, f, m)-deleted graph by Theorem 3. And, this establishes for any $U \subseteq V(G)$ with |U| = n', and therefore G is all fractional (g, f, n', m)-critical deleted.

In all, this completes the proof of Theorem 9.

2.2 Proof of Theorem 7

In this part, we mainly present the detailed proof of Theo-

Suppose *G* satisfies the hypothesis of Theorem 7, but is not an all fractional (g, f, n', m)-critical deleted graph. Then via Theorem 10 and the fact that $\sum_{x \in T} d_H(x)$ – $e_H(T, S) \le 2m$ for any $H \subseteq E(G)$ with m edges, there is a non-disjoint subset S, $T \subseteq V(G)$ satisfying

$$a|S| - b|T| + \sum_{x \in T} d_{G-S}(x) - an' - 2m$$

$$\leq a(|S| - n') + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - b)$$

$$\leq g(S - U) + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - f(x)) \leq -1. \tag{4}$$

S and *T* are chosen such that |T| is minimum. The subsets *S* and $T \setminus \{x\}$ satisfy (4), when there is a $x \in T$ which can satisfy $d_{G-S}(x) \ge g(x)$, This is conflicted with the selection rule of *S* and *T*. It infers that $d_{G-S}(x) \le g(x) - 1 \le b - 1$ for anv $x \in T$.

Note that $|T| \neq \emptyset$, otherwise we have $\sum_{x \in T} d_H(x)$ – $e_H(T, S) = 0$ and a(|S| - n') < 0, contradicting (4). We write $N_T(x) = N_G(x) \cap T$ and $N_T[x] = (N_G(x) \cap T) \cup \{x\}$ for any vertex $x \in T$.

Let

$$d_1 = \min\{d_{G-S}(x)|x \in T\}$$

and x_1 be a vertex in T with $d_{G-S}(x_1) = d_1$. If $T - N_T[x_1] \neq \emptyset$, we further set

$$d_2 = \min\{d_{G-S}(x)|x \in T - N_T[x_1]\},$$

and x_2 is a vertex in $T - N_T[x_1]$ with $d_{G-S}(x_2) = d_2$. Hence, we have $d_1 \le d_2 \le b-1$ and $|N_G(x_1) \cup N_G(x_2)| \le |S| + d_1 + d_2$.

The following discussion is divided into two cases.

Case 1.
$$T = N_T[x_1]$$
.

Case 1. $T = N_T[x_1]$. Since $\delta(G) \ge \frac{(a+b-1)^2+4b}{4a} + m + n'$, $|T| \le d_1 + 1 \le b$ and $d_1 + |S| \ge d_G(x_1) \ge \delta(G)$, by (4) and the definition of d_1 , we

$$an' + 2m - 1$$

$$\geq a|S| - b|T| + \sum_{x \in T} d_{G-S}(x)$$

$$\geq a|S| - b|T| + d_1|T|$$

$$\geq a(\delta(G) - d_1) - (b - d_1)|T|$$

$$\geq a(\delta(G) - d_1) - (b - d_1)(d_1 + 1)$$

$$= d_1^2 - (a + b - 1)d_1 + a\delta(G) - b$$

$$\geq (d_1 - \frac{a + b - 1}{2})^2 + an' + 2m \geq an' + 2m,$$

a contradiction.

Case 2. $T - N_T[x_1] \neq \emptyset$.

Let $p = |N_T[x_1]|$, then we get $d_1 \ge p - 1$. In view of $T-N_T[x_1] \neq \emptyset$, we obtain $|T| \geq p+1$. By virtue of $n-|S|-|T| \geq$ $0, d_1 - d_2 \le 0, b - d_2 \ge 1$, and (4), we yield

$$(n - |S| - |T|)(b - d_2)$$

$$\geq a|S| - b|T| + \sum_{x \in T} d_{G-S}(x) - 2m - an' + 1$$

$$\geq a|S| - b|T| + d_1p + d_2(|T| - p) - 2m - an' + 1$$

$$= a|S| + (d_1 - d_2)p + (d_2 - b)|T| - 2m - an' + 1$$

$$\geq a|S| + (d_1 - d_2)(d_1 + 1) + (d_2 - b)|T|$$
$$-2m - an' + 1.$$

It reveals

$$(n-|S|)(b-d_2) \ge a|S| + (d_1-d_2)(d_1+1) - 2m - an' + 1.$$

By arranging the above inequality, we infer

$$n(b-d_2)-(a+b-d_2)|S|+(d_2-d_1)(d_1+1)+2m+an'-1 \ge 0$$
(5)

Since vertices x_1 and x_2 are non-adjacent, using the neighborhood condition of the Theorem 7, we deduce

$$\frac{bn + an'}{a + b'} \le |N_G(x_1) \cup N_G(x_2)| \le |S| + d_1 + d_2,$$

which implies

$$|S| \ge \frac{bn + an'}{a + b} - (d_1 + d_2).$$
 (6)

In terms of (5), (6), $d_1 \le d_2 \le b - 1$, and $n \ge \frac{2(a+b)(a+b+m-1)}{a} + n'$, we get

$$0 \leq n(b-d_2) - (a+b-d_2)|S| + (d_2-d_1)(d_1+1) + 2m + an' - 1$$

$$\leq n(b-d_2) - (a+b-d_2)(\frac{bn+an'}{a+b} - (d_1+d_2)) + (d_2-d_1)(d_1+1) + 2m + an' - 1$$

$$= \frac{-and_2}{a+b} + (a+b-1)d_2 - d_2^2 + (a+b-1)d_1$$

$$-d_1^2 - (an' - \frac{d_2an'}{a+b}) + 2m + an' - 1$$

$$\leq \frac{-and_1}{a+b} + (a+b-1)d_1 - d_1^2 + (a+b-1)d_1$$

$$-d_1^2 + 2m + \frac{d_1an'}{a+b} - 1$$

$$= \frac{-and_1}{a+b} + 2(a+b)d_1 - 2d_1^2 + 2m + \frac{d_1an'}{a+b} - 1$$

$$\leq 2d_1 - 2d_1^2 - 1 \leq -1.$$

a contradiction.

Therefore, the desired result is proved.

2.3 Proof of Theorem 8

In the following part, we prove the second main result in this paper.

Assume that G satisfies the hypothesis of Theorem 8, but is not a fractional all (g,f,n',m)-critical deleted graph. Then via Theorem 10 and the fact that $\sum_{x\in T} d_H(x) - e_H(T,S) \le 2m$ for any $H\subseteq E(G)$ with m edges, there is a non-disjoint subset $S,T\subseteq V(G)$ satisfying

$$a|S| - b|T| + \sum_{x \in T} d_{G-S}(x) - an' - 2m$$

$$\leq a(|S| - n') + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - b)$$

$$\leq g(S - U) + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - f(x)) \leq -1.$$
(7)

As depicted in the former subsection, S and T are selected such that |T| is the minimum. We get $T \neq \emptyset$ and $d_{G-S}(x) \leq g(x) - 1 \leq b - 1$ for any $x \in T$.

The Claim below is firstly shown here.

Claim 1 *G* is assumed to be a graph of order *n* which satisfies the hypothesis of Theorem 8. After that, we obtain

$$\delta(G) \geq \frac{(b-1)n + a + an' + 2m}{a+b-1}.$$

Proof of Claim 1. Assume x to be a vertex of a graph G satisfying $d_G(x) = \delta(G)$, and $X = V(G) \setminus N_G(x)$. Clearly, $x \notin N_G(X)$, and hence $N_G(X) \neq V(G)$. Combining this with the hypothesis of Theorem 8, we infer

$$N_G(X) \ge \frac{(a+b-1)(n-1)}{an-a-an'-2m}|X|.$$

By virtue of $|X| = n - \delta(G)$ and $|N_G(X)| \le n - 1$, we deduce

$$|N_G(X)| \ge \frac{(a+b-1)(n-1)}{an-a-an'-2m}|X|$$

$$= \frac{(a+b-1)(n-1)}{an-a-an'-2m}(n-\delta(G)),$$

which implies

$$\delta(G) \geq \frac{(b-1)n + a + an' + 2m}{a+b-1}.$$

Claim 1 is proved.

By setting $d=\min\{d_{G-S}(x):x\in T\}$, we have $0\leq d\leq b-1$. Select $x_1\in T$ with $d_{G-S}(x_1)=d$. In light of $\delta(G)\leq d_G(x_1)\leq d_{G-S}(x_1)+|S|=d+|S|$ and Claim 1, we yield

$$|S| \ge \delta(G) - d \ge \frac{(b-1)n + a + an' + 2m}{a+b-1} - d.$$
 (8)

In what follows, we discuss three cases for the value of d.

Case 1. $2 \le d \le b - 1$.

By means of (8) and $|S| + |T| \le n$, we obtain

$$a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| - an' - 2m$$

$$\geq a|S| + d|T| - b|T| - an' - 2m$$

$$\geq a|S| - (b - d)(n - |S|) - an' - 2m$$

$$= (a+b-d)|S| - (b-d)n - an' - 2m$$

$$\geq (a+b-d)(\frac{(b-1)n+a+an'+2m}{a+b-1}-d)$$
$$-(b-d)n-an'-2m.$$

Let $h(d) = (a + b - d)(\frac{(b-1)n + a + an' + 2m}{a + b - 1} - d) - (b - d)n - a + b - d$ an' - 2m, then we have

$$a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| - an' - 2m \ge h(d).$$
 (9)

Using $2 \le d \le b - 1$ and $n \ge \frac{(a+2b-3)(a+b-2)}{a} + \frac{an'}{a-1} + m$, we get

$$h'(d) = -(\frac{(b-1)n + a + an' + 2m}{a+b-1} - d)$$

$$-(a+b-d) + n$$

$$= 2d + \frac{an - a - an' - 2m}{a+b-1} - a - b$$

$$\ge 4 + \frac{an - a - an' - 2m}{a+b-1} - a - b > 0.$$

Combining this with $2 \le d \le b - 1$, we infer

$$h(d) \ge f(2). \tag{10}$$

By virtue of (7), (9) and (10), we derive

$$an' + 2m - 1 \ge h(d) \ge h(2)$$

$$= (a + b - 2)(\frac{(b - 1)n + a + an' + 2m}{a + b - 1} - 2)$$

$$-(b - 2)n.$$

which implies

$$n \leq \frac{(a+2b-3)(a+b-2)-1}{a} + \frac{2m}{a} + n'$$

<
$$\frac{(a+2b-3)(a+b-2)}{a} + \frac{an'}{a-1} + m,$$

which conflicts with $n \ge \frac{(a+2b-3)(a+b-2)}{a} + \frac{an'}{a-1} + m$.

Case 2. d = 1.

In this case, we first prove the following two claims. Claim 2 $|T| \leq \lfloor \frac{(an-a-an-2m)n}{(a+b-1)(n-1)} \rfloor$.

Proof of Claim 2. Assume that $|T| \ge \lfloor \frac{(an-a-an'-2m)n}{(a+b-1)(n-1)} \rfloor + 1$. Followed from $d_{G-S}(x_1) = d = 1$, we have

$$|T \setminus N_G(x_1)| \ge |T| - 1 \ge \lfloor \frac{(an - a - an' - 2m)n}{(a + b - 1)(n - 1)} \rfloor. \tag{11}$$

In view of (11) and the hypothesis of Theorem 8, we obtain

$$N_G(T \setminus N_G(x_1)) = V(G). \tag{12}$$

On the other hand, it is clear that

$$x_1 \notin N_G(T \setminus N_G(x_1)),$$

which conflicts with (12).

Claim 3
$$|T| \leq \frac{an-a-an'-2m}{a+b-1}$$
.

Proof of Claim 3. Assume that $|T| > \frac{an-a-an'-2m}{a+b-1}$. In terms of (8) and d = 1, we yield

$$n \ge |S| + |T|$$
>
$$\frac{(b-1)n + a + an' + 2m}{a+b-1} - 1$$

$$+ \frac{an - a - an' - 2m}{a+b-1}$$
=
$$n-1,$$

which means

$$|S| + |T| = n. \tag{13}$$

According to (7), (13) and Claim 2, we get

$$an' + 2m - 1$$

$$\geq a|S| + \sum_{x \in T} d_{G-S}(x) - b|T|$$

$$\geq a|S| + |T| - b|T|$$

$$= a(n - |T|) - (b - 1)|T|$$

$$= an - (a + b - 1)|T|$$

$$\geq an - (a + b - 1) \frac{(an - a - an' - 2m)n}{(a + b - 1)(n - 1)}$$

$$= an - \frac{(an - a - an' - 2m)n}{n - 1}$$

$$\geq an' + 2m,$$

a contradiction. Hence the Claim 3 is hold.

Now, we set $t' = |\{x : x \in T, d_{G-S}(x) = 1\}|$. It is obvious that $t' \ge 1$ and $|T| \ge t'$. In light of (8), d = 1 and Claim 3, we obtain

$$\begin{aligned} a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| - an' - 2m \\ & \geq a|S| + 2|T| - t' - b|T| - an' - 2m \\ & = a|S| - (b-2)|T| - t' - an' - 2m \\ & \geq a(\frac{(b-1)n + a + an' + 2m}{a+b-1} - 1) - an' - 2m \\ & - (b-2)\frac{an - a - an' - 2m}{a+b-1} - t' \\ & = \frac{an - a - an' - 2m}{a+b-1} - t' \\ & \geq |T| - t' \geq 0, \end{aligned}$$

which conflicts with (7).

Case 3. d = 0.

Let $r = |\{x : x \in T, d_{G-S}(x) = 0\}| \text{ and } Y = V(G) \setminus S$. Obviously, we have $r \ge 1$ and $N_G(Y) \ne V(G)$. Combining these with the hypothesis of Theorem 8, we derive

$$|n-r| \ge |N_G(Y)| \ge \frac{(a+b-1)(n-1)}{an-a-an'-2m}|Y|$$

$$= \frac{(a+b-1)(n-1)}{an-a-an'-2m}(n-|S|),$$

which implies,

$$|S| \ge n - \frac{(n-r)(an-a-an'-2m)}{(a+b-1)(n-1)}.$$
 (14)

In view of $n \ge \frac{(a+2b-3)(a+b-2)}{a} + \frac{an'}{a-1} + m$, we ensure that

$$\frac{an - a - an' - 2m}{n - 1} > 1. {(15)}$$

By means of (7), (14), (15) and $|S| + |T| \le n$, we infer

$$an' + 2m - 1$$

$$\geq a|S| + \sum_{x \in T} d_{G-S}(x) - b|T|$$

$$\geq a|S| - (b-1)|T| - r$$

$$\geq a|S| - (b-1)(n-|S|) - r$$

$$= (a+b-1)|S| - (b-1)n - r$$

$$\geq (a+b-1)(n - \frac{(n-r)(an-a-an'-2m)}{(a+b-1)(n-1)})$$

$$-(b-1)n - r$$

$$= an - \frac{(n-r)(an-a-an'-2m)}{n-1} - r$$

$$\geq an - \frac{(n-1)(an-a-an'-2m)}{n-1} - 1$$

$$= an - (an-a-an'-2m) - 1$$

$$= an' + 2m + a - 1 > an' + 2m,$$

a contradiction.

In conclusion, we complete the proof of Theorem 8.

3 Sharpness

The most likely in the sense that we can't replace $\frac{bn+an'}{a+b}$ by $\frac{bn+an'}{a+b}-1$ is the lower bound on the condition $|N_G(x)|$ $|N_G(y)| \ge \frac{bn+an'}{a+b}$ in Theorem 7. We present this by constructing a graph $G = (bt+n')K_1 \lor (at+1)K_1$, where a and b are two nonnegative integers with $2 \le a \le b$ and t is a large sufficiently positive integer. Obviously,

$$|V(G)| = n = (a+b)t + n' + 1$$

and

$$\frac{bn + an'}{a + b} > |N_G(x) \cup N_G(y)| = bt + n' > \frac{bn + an'}{a + b} - 1$$

for any non-adjacent vertices $x, y \in V((at+1)K_1)$. Set $S = V((bt+n')K_1)$, $T = V((at+1)K_1)$. Let $U \subset S$ with |U| = n'. Let H be a subgraph of T with m edges. Obviously, |S| = bt+n', |T| = at+1, $\sum_{x \in T} d_{G-S}(x) = 0$ and $\sum_{x \in T} d_H(x) - e_H(S,T) = 0$. Furthermore, let g(x) = a for any $x \in S$, and f(x) = b for any $x \in T$. Hence, we obtain

$$g(S-U) + \sum_{x \in T} (d_{G-S}(x) - d_H(x))$$

$$+e_H(x, S) - f(x)$$

= $g(S - U) - f(T) = abt - b(at + 1) = -b < 0.$

By Theorem 10, G is not all fractional (g, f, n', m)-critical deleted.

The bound of the hypothesis in Theorem 8 is the best possible in some sense, i.e., it can't be replaced by $N_G(X)$ = V(G) or $|N_G(X)| \ge \frac{(a+b-1)(n-1)}{an-a-an'-2m}|X|$ for any $X \subseteq V(G)$ (i.e., the restriction condition on |X| is necessary). We explain this by constructing a graph G as follows. Let $V(G) = S \cup T$ with $S \cap T = \emptyset$, |S| = (b-1)t + n' and |T| = at + 1, and $T = \{x_1, x_2, \dots, x_{2t'}\}\$, where $n' \ge 0$, $b \ge a \ge 2$ are integers, a and t are odd integers. Obviously, 2t' = at + 1. For $y \in S$, we write $N_G(y) = V(G) \setminus \{y\}$. For $x \in T$, we write $N_G(x) = S \cup \{x'\}$, where $\{x, x'\} = \{x_{2i-1}, x_{2i}\}$ for some $1 \le i \le t'$. It is obvious that n = (b-1)t + n' + at + 1. Next, we show that $N_G(X) = V(G)$ or $|N_G(X)| \ge \frac{(a+b-1)(n-1)}{an-a-an'-2m}|X|$ holds for any $X \subseteq V(G)$. Apparently, $N_G(X) = V(G)$ if $|X \setminus S| \ge 2$, or $|X \setminus S| = 1$ and $|X \setminus T| \ge 1$ for any $X \subseteq V(G)$. If |X| = 1 and $X \subseteq S$, then $|N_G(X)| = |V(G)| - 1 =$ If |X| = 1 and $X \subseteq S$, then $|N_G(X)| = |V(G)| - 1 = n - 1 > \frac{(a+b-1)(n-1)}{at(a+b-1)} = \frac{(a+b-1)(n-1)}{an-a-an'-2m} = \frac{(a+b-1)(n-1)}{an-a-an'-2m} |X|$. Thus, we may suppose $X \subseteq T$. Note that $|N_G(X)| = |S| + |X| = (b-1)t+n'+|X|$. Thus, $|N_G(X)| \ge \frac{(a+b-1)(n-1)}{an-a-an'-2m} |X|$ holds if and only if $(b-1)t+n'+|X| \ge \frac{(a+b-1)(n-1)}{b(n-1)-bn'-2m} |X|$, which is equivalent to $|X| = a^{-1} C |x| = a^{-1} C |x|$. alent to $|X| \le at$. Clearly, $|N_G(X)| \ge \frac{(a+b-1)(n-1)}{b(n-1)-bn'-2m}|X|$ holds for any $X \subseteq T$ with $X \neq T$. If X = T, then $N_G(X) = V(G)$. As a result, $N_G(X) = V(G)$ or $|N_G(X)| \ge \frac{(a+b-1)(n-1)}{an-a-an'-2m}|X|$ holds for any $X \subset V(G)$. Then, we show that G is not all fractional (g, f, n', m)-critical deleted. For above S and T, it is obvious that |S| > n' and $d_{G-S}(x) = 1$ for each $x \in T$. Let g(x) = a for any $x \in S$, and f(x) = b for any $x \in T$. Let H be a subgraph of T with m edges. Therefore, in terms of $\sum_{x \in T} d_{G-S}(x) - \sum_{x \in T} d_H(x) + e_H(S, T) \ge 0 \text{ holds for any}$ H, we obtain

$$g(S - U) + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - f(x))$$

$$= a(|S| - n') + (\sum_{x \in T} d_{G-S}(x) - \sum_{x \in T} d_H(x) + e_H(S, T)) - b|T|$$

$$\geq a(b - 1)t - b(at + 1)$$

$$= -at - bt < 0.$$

It follows from Theorem 10 that G is not all fractional (g, f, n', m)-critical deleted.

4 Conclusions

In recent years, the problem of fractional factor in graphs has raised much attention in the field of graph theory and computer networks. In this paper, we consider the theoretical problems in data transmission networks when some sites and channels are not available in the certain time. The relationship between neighborhood conditions and a graph to be all fractional (g, f, n',m)- critical deleted is discussed. Two sufficient neighborhood conditions are obtained, and the sharpness of conditions is presented. The theoretical conclusions we yield in this paper have potential applications in network design and information transmission.

Conflict of Interests

The authors hereby declare that there is no conflict of interests regarding the publication of this paper.

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