Research Article

Sultan Z. Alamri, Sobhy E. Abd El-Bar, and Aly R. Seadawy*

The collinear equilibrium points in the restricted three body problem with triaxial primaries

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Abstract: The perturbed restricted three body problem has been reviewed. The mass of the primaries are assumed as triaxial. The locations of the collinear points have been computed. Series forms of these locations are obtained as new analytical results. In order to introduce a semi-analytical view, a Mathematica program has been constructed to graph the locations of collinear points versus the whole range of the mass ratio μ taking into account the triaxiality. The resultant figures have been analyzed

Keywords: Restricted three body problem, Collinear equilibrium points, perturbation, Triaxial, oblateness

PACS: 95.10.Ce, 45.50.Pk, 98.10.+z

1 Introduction

The restricted three-body problem (RTBP) is defined as a system where an infinitesimal mass, m_3, is attracted gravitationally by two finite arbitrary masses called primaries, m_1 and m_2, but their motion is not influenced. The term restricted comes from the fact that all masses are assumed to move in the same plane defined by the two revolving primaries which revolve around their center of mass in circular orbits. This definition is widely used in almost all classical books of celestial mechanics, *e.g.* Szebehely [1] and Murray and Drmott [2]. For a more generalized version of the problem many authors have amended the potential function with some relevant perturbing forces; *e.g.* considering oblate primaries instead of spherical masses. Or even more generalized as triaxial bodies, inclusion of the relativistic effects, assuming the primaries are emitters, and/or move in a resisting medium. Even if the RTBP is not integrable, a number of special solutions can be found in the rotating frame where the third body has zero velocity and zero acceleration. These solutions correspond to equilibrium positions in the rotating frame at which the gravitational forces and the centrifugal force associated with the rotation of the synodic frame all cancel, with the result that a particle located at one of these points appears stationary in the synodic frame. There are five equilibrium points in the circular RTBP, three of them are collinear points, namely L_1, L_2, L_3 and the another two are triangular points,namely L_4, L_5. The position of the infinitesimal body is displaced a little from the equilibrium point due to the some perturbations. If the resultant motion of the infinitesimal mass is a rapid departure from the vicinity of the point, we can call such a position of equilibrium point an “unstable” one, if however the body merely oscillates about the equilibrium point, it is said to be a “stable position” (in the sense of Lyapunov), Abd El-Salam [3]. In general, the dynamics of a circular and/or elliptical three-body problem is widely applicable toastrophysics, for example stellar/solar system dynamics and Earth-Moon system.This problem consequently received more attention from astronomers and dynamical system scientists. In spite of it, the solutions of this problem has been developed over the past centuries.

The literature concerning RTBPis extensive and it is worth highlighting here some relevant and recent studies dealingwith RTBP, with and without considering different perturbations: Sharma [4], Tsirogiannis *et al.* [5], Kushvah and Ishwar [6], Vishnu Namboori *et al.* [7], Mital *et al.* [8], Kumar and Ishwar [9], Rahoma and Abd El-Salam [10] and references therein. Ammar [11] analyzed solar radiation pressure effect on the positions and stability of the libration points in elliptic RTBP. Singh [12] formulated the triangular librationpoints nonlinear stability under the Coriolis effect and centrifugal forces as small perturbations in addition to the effect of primaries oblateness and radiation pressures. Singh and Umar [13] investigated the effect of luminous and oblate spheroids primaries on the loca-

***Corresponding Author: Aly R. Seadawy:** Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah, Saudi Arabia; Mathematics Department, Faculty of Science, Beni-Suef University, Egypt; Email: Aly742001@yahoo.com
Sultan Z. Alamri: Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah, Saudi Arabia
Sobhy E. Abd El-Bar: Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah, KSA; Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt

tions and stability of the collinear libration points. In another work, Singh and Umar [14] studied the effect of the big primary's triaxial and spherical shape of the companion on the locations and stability of the collinear libration points. They found that the position of collinear libration points and their stability are affected with their considered perturbations in addition to the eccentricity and the semi-major axis of the primaries orbit.

Katour *et al.* [15], Singh and Bello [16, 17], Abd El-Salam and Abd El-Bar [18], Abd El-Bar *et al.* [19] and Bello and Singh [20] were concerned with the relativistic RTBP in addition to some different perturbations—the primaries oblateness andradiation from one of the primaries—upon the equilibrium points locations and stability. They noticed that the stability regions of the concerned equilibrium points are varied (expanding or shrinking) related to the critical mass value and depending upon the value of their considered perturbations.

The fundamental structure in nature and science such as RTBP, plasma containment in tokamaks and stellarators for energy generation, population ecology,chaoticbehavior in biological systems, neural networks and solitonicfibre optical communication devices can be expound by nonlinear partial differential equations [21–37].

The aim of this study is the determination of the locations of the collinear equilibrium points,taking into consideration the fact that both primaries are triaxial. This paper will be organized as follows: Section 1 is an historical introduction,the equations of motion in an RTBP with triaxial primaries are formulated in Section 2, the computations related to the location of equilibrium points with the considered perturbations are introduced in section 3. Section 4 highlightssomenumerical simulations with a discussion and forthcoming works. The paper finishes with a conclusion in Section 5.

2 Equations of motion

We shall adopt the notation and terminology of Szebehely [1]. As a consequence, the distance between the primaries m_1 and m_2 does not change and is taken equal to one; the sum of the masses of the primaries is also taken as one. The unit of time is also chosen as to make the gravitational constant unity.

The equations of motion of the infinitesimal mass m_3 in the RTBP in a synodic co-ordinate system(x, y)in dimensionless variables in which the primary coordinates on the x-axis $(-\mu, 0)$, $(1-\mu, 0)$ are given by Brumberg (1972).

$$\ddot{x} - 2n\dot{y} = \frac{\partial U}{\partial x} - \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right), \quad (1)$$
$$\ddot{y} + 2n\dot{x} = \frac{\partial U}{\partial y} - \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{y}}\right)$$

where U is the potential–like function of the RTBP, which can be written as composed of two components (compound), namely the potential of the classical RTBP potential U are given by:

$$U = \frac{n^2}{2}\left[(1-\mu)r_1^2 + \mu r_2^2\right] + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^3} - \frac{3(1-\mu)(\sigma_1-\sigma_2)}{2r_1^5}y^2 + \frac{\mu(2\gamma_1-\gamma_2)}{2r_2^3} - \frac{3\mu(\gamma_1-\gamma_2)}{2r_2^5}y^2$$

with

$$r = \sqrt{(x^2+y^2)}, \quad r_1 = \sqrt{(x+\mu)^2+y^2}$$
$$r_2 = \sqrt{(x+\mu-1)^2+y^2}, \quad \mu = \frac{m_2}{m_1+m_2} \leq \frac{1}{2},$$

$m_1, m_2 (m_1 \geq m_2)$ being the masses of the primaries,

$$\sigma_1 = \frac{{a_1}^2 - {c_1}^2}{5R^2}, \quad \sigma_2 = \frac{{b_1}^2 - {c_1}^2}{5R^2}, \quad \sigma_1, \sigma_2 \ll 1,$$
$$\gamma_1 = \frac{{a_2}^2 - {c_2}^2}{5R^2}, \quad \gamma_2 = \frac{{b_2}^2 - {c_2}^2}{5R^2}, \quad \gamma_1, \gamma_2 \ll 1,$$

a_i, b_i, c_i, $i = 1, 2$ are the semi-axes of the triaxial of two primaries respectively and R the distance between the primaries.

The mean motion n of the primaries is given by

$$n^2 = 1 + \frac{3}{2}(2\sigma_1 - \sigma_2) + \frac{3}{2}(2\gamma_1 - \gamma_2)$$

Remark: The oblateness case of the primaries can be deduced directly from applying the condition $a_i = b_i$, $i = 1, 2$ or alternatively $\sigma_1 = \sigma_2$.

The libration points are obtained from equations of motion after setting $\ddot{x} = \ddot{y} = \dot{x} = \dot{y} = 0$. These points represent particular solutions of equations of motion

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = 0, \quad (2)$$

The explicit formulas are

$$U_x = n^2 x + (x+\mu)\left[-\frac{(1-\mu)}{r_1^3} - \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^5} + \frac{15(1-\mu)(\sigma_1-\sigma_2)}{2r_1^7}y^2\right] + (x+\mu-1)\left[-\frac{\mu}{r_2^3} - \frac{3\mu(2\gamma_1-\gamma_2)}{2r_2^5} + \frac{15\mu(\gamma_1-\gamma_2)}{2r_2^7}y^2\right] \quad (3)$$

and

$$U_y = n^2 y - \left[\frac{(1-\mu)}{r_1^3} + \frac{3(1-\mu)(2\sigma_1 - \sigma_2)}{2r_1^5} + \frac{3(1-\mu)(\sigma_1-\sigma_2)}{2r_1^5} \right] y \tag{4}$$

3 Location of collinear libration points

Any point of the collinear points must, by definition, have $z = y = 0$, and the solution of the classical RTBP satisfies the conditions of Abd El-Bar and Abd El-Salam [38]

$$\mathcal{B}_1 r_1 + \mathcal{B}_2 r_2 = 1, \quad r_1 = \mathcal{B}_1 (x+\mu), \quad r_2 = -\mathcal{B}_2 (\mu + x - 1) \tag{5}$$

where, we have (see Figures 1, 2, 3):

$$L_1: \ \mathcal{B}_1 = 1, \ \mathcal{B}_2 = 1, \quad L_2: \ \mathcal{B}_1 = 1, \ \mathcal{B}_2 = -1, \quad L_3: \ \mathcal{B}_1 = -1, \ \mathcal{B}_2 = 1$$

4 Location of L_1

The location geometry of L_1 can be visualized as given by Figure 1.

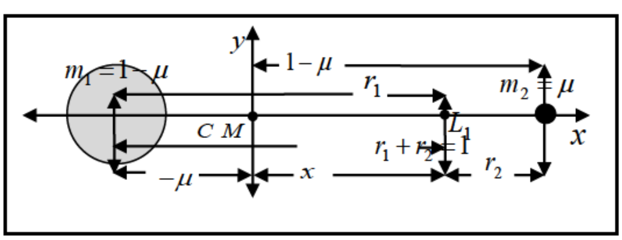

Figure 1: The location of L_1 and its corresponding parameters

Substituting from (5) with the corresponding values of $\mathcal{B}_1 = \mathcal{B}_2 = 1$, we get $r_1 + r_2 = 1$, $r_1 = x + \mu$, $r_2 = 1 - \mu - x$ and noting that $\frac{\partial r_1}{\partial x} = -\frac{\partial r_2}{\partial x} = 1$, equation (3) becomes:

$$U_x = n^2(1-\mu-r_2) - \left[\frac{(1-\mu)}{r_1^3} + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^5} - \frac{15(1-\mu)(\sigma_1-\sigma_2)}{2r_1^7} y^2 \right] + \left[\frac{\mu}{r_2^3} + \frac{3\mu(2\gamma_1-\gamma_2)}{2r_2^5} - \frac{15\mu(\gamma_1-\gamma_2)}{2r_2^7} y^2 \right] \tag{6}$$

Then it may be reasonable in our case to assume that positions of the equilibrium points L_1 are the same as given by classical RTBP but perturbed due to the triaxial primaries

$$r_1 = a_1 + \varepsilon_1, \quad r_2 = b_1 - \varepsilon_1, \quad a_1 = 1 - b_1 \tag{7}$$

from which we have

$$r_1 = (1-b_1)\left(1 + \frac{\varepsilon_1}{1-b_1}\right) \quad r_2 = b_1 \left(1 - \frac{\varepsilon_1}{b_1}\right) \tag{8}$$

where a_1 and b_1 are unperturbed positions of r_1 and r_2 respectively, and b_1 is given after some successive approximation by

$$b_1 = \alpha \left(1 - \frac{\alpha}{3} - \frac{\alpha^2}{9} + \frac{2}{27}\alpha^3 + \frac{2}{81}\alpha^4\right), \quad \alpha = \left(\frac{\mu}{3(1-\mu)}\right)^3 \tag{9}$$

Substituting from equations (7) into equation (6)

$$\begin{aligned} U_x = {} & n^2(1-\mu-b_1+\varepsilon_1) - \frac{1-\mu}{(1-b_1)^2}\left(1 - \frac{2\varepsilon_1}{1-b_1}\right) \\ & - \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2(1-b_1)^4}\left(1 - \frac{4\varepsilon_1}{1-b_1}\right) \\ & + \frac{15(1-\mu)(\sigma_1-\sigma_2)}{2(1-b_1)^6}\left(1 - \frac{6\varepsilon_1}{1-b_1}\right) y^2 \\ & + \frac{\mu}{b_1{}^2}\left(1 + \frac{2\varepsilon_1}{b_1}\right) + \frac{3\mu(2\gamma_1-\gamma_2)}{2b_1{}^4}\left(1 + \frac{4\varepsilon_1}{b_1}\right) \\ & - \frac{15\mu(\gamma_1-\gamma_2)}{2b_1{}^6}\left(1 + \frac{6\varepsilon_1}{b_1}\right) y^2 \end{aligned}$$

Since $y = 0$, then the above equation may be written as

$$\begin{aligned} U_x = {} & n^2(1-\mu-b_1+\varepsilon_1) - \frac{1-\mu}{(1-b_1)^2}\left(1 - \frac{2\varepsilon_1}{1-b_1}\right) \\ & - \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2(1-b_1)^4}\left(1 - \frac{4\varepsilon_1}{1-b_1}\right) \\ & + \frac{15(1-\mu)(\sigma_1-\sigma_2)}{2(1-b_1)^6}\left(1 - \frac{6\varepsilon_1}{1-b_1}\right) y^2 \\ & + \frac{\mu}{b_1{}^2}\left(1 + \frac{2\varepsilon_1}{b_1}\right) + \frac{3\mu(2\gamma_1-\gamma_2)}{2b_1{}^4}\left(1 + \frac{4\varepsilon_1}{b_1}\right) = 0 \end{aligned}$$

which can be solved for ε_1 yielding

$$\begin{aligned} \varepsilon_1 = {} & \left(n^2 + \frac{2(1-\mu)}{(1-b_1)^3} - \frac{2\mu}{b_1^3} - \frac{6\mu(2\gamma_1-\gamma_2)}{b_1^5} + \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{(1-b_1)^5}\right)^{-1} \left(-n^2 + n^2\mu + \frac{1-\mu}{(1-b_1)^2} - \frac{\mu}{b_1^2} \right. \\ & \left. + n^2 b_1 - \frac{3\mu(2\gamma_1-\gamma_2)}{2b_1^4} + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2(1-b_1)^4}\right) \end{aligned} \tag{10}$$

Setting

$$d_1 = \left(n^2 + \frac{2(1-\mu)}{(1-b_1)^3} - \frac{2\mu}{b_1^3} - \frac{6\mu(2\gamma_1-\gamma_2)}{b_1^5}\right.$$

$$+\frac{6\,(1-\mu)\,(2\sigma_1-\sigma_2)}{(1-b_1)^5}\Bigg)^{-1};$$

$$e_1=\frac{1}{b_1^2},\quad f_1=\frac{1}{b_1^4},\quad g_1=\frac{1}{(1-b_1)^2},\quad h_1=\frac{1}{(1-b_1)^4}$$

Using the above relations, equation (10) can be written in the form

$$\varepsilon_1=d_1\Bigg(n^2(-1+\mu+b_1)+(1-\mu)\,g_1-\mu e_1 - \frac{3\mu\,(2\gamma_1-\gamma_2)}{2}f_1+\frac{3\,(1-\mu)\,(2\sigma_1-\sigma_2)}{2}h_1\Bigg) \qquad (11)$$

where b_1, d_1, e_1, f_1, g_1 and h_1 are function of μ and they are given by

$$b_1=\left(\frac{\mu}{3}\right)^{1/3}-\frac{1}{3}\left(\frac{\mu}{3}\right)^{2/3}-\frac{1}{9}\left(\frac{\mu}{3}\right)+\frac{29}{27}\left(\frac{\mu}{3}\right)^{4/3} -\frac{52}{81}\left(\frac{\mu}{3}\right)^{5/3}-\frac{1}{3}\left(\frac{\mu}{3}\right)^{2}+\frac{62}{27}\left(\frac{\mu}{3}\right)^{7/3}-\frac{125}{81}\left(\frac{\mu}{3}\right)^{8/3} -\left(\frac{\mu}{3}\right)^{3}+\mathcal{O}\left(\frac{\mu}{3}\right)^{10/3}$$

$$d_1=\sum_{i=0}^{9}\mathcal{N}_{i1}\left(\frac{\mu}{3}\right)^{i/3}$$

$$e_1=\frac{5}{9}+\frac{2}{3}\left(\frac{3}{\mu}\right)^{\frac{1}{3}}+\left(\frac{3}{\mu}\right)^{\frac{2}{3}}-\frac{16}{9}\left(\frac{\mu}{3}\right)^{\frac{1}{3}}-\frac{50}{81}\left(\frac{\mu}{3}\right)^{\frac{2}{3}} -\frac{10}{243}\left(\frac{\mu}{3}\right)-\frac{608}{729}\left(\frac{\mu}{3}\right)^{\frac{4}{3}}-\frac{148}{243}\left(\frac{\mu}{3}\right)^{\frac{5}{3}}-\frac{947}{6561}\left(\frac{\mu}{3}\right)^{2} -\frac{22048}{19683}\left(\frac{\mu}{3}\right)^{\frac{7}{3}}-\frac{21044}{19683}\left(\frac{\mu}{3}\right)^{\frac{8}{3}}-\frac{87034}{177147}\left(\frac{\mu}{3}\right)^{3}+\ldots$$

$$f_1=-\frac{73}{27}+\frac{33^{1/3}}{\mu^{4/3}}+\frac{4}{\mu}+\frac{14}{33^{1/3}\mu^{2/3}}-\frac{68}{93^{2/3}\mu^{1/3}} -\frac{500\mu^{1/3}}{2433^{1/3}}+\frac{4\mu^{2/3}}{33^{2/3}}+\frac{1324\mu}{6561}-\frac{434\mu^{4/3}}{196833^{1/3}} +\frac{35180\mu^{5/3}}{590493^{2/3}}+\frac{13294\mu^2}{59049}+\frac{339800\mu^{7/3}}{15943233^{1/3}} +\frac{2601328\mu^{8/3}}{47829693^{2/3}}+\frac{3598016\mu^3}{14348907}$$

$$g_1=1+2\left(\frac{\mu}{3}\right)^{\frac{1}{3}}+\frac{7}{3}\left(\frac{\mu}{3}\right)^{\frac{2}{3}}+\frac{16}{9}\left(\frac{\mu}{3}\right)+\frac{76}{27}\left(\frac{\mu}{3}\right)^{\frac{4}{3}} +\frac{382}{81}\left(\frac{\mu}{3}\right)^{\frac{5}{3}}+\frac{46}{9}\left(\frac{\mu}{3}\right)^{2}+\frac{580}{81}\left(\frac{\mu}{3}\right)^{\frac{7}{3}}+\frac{976}{81}\left(\frac{\mu}{3}\right)^{\frac{8}{3}} +\frac{10882}{729}\left(\frac{\mu}{3}\right)^{3}+\ldots$$

$$h_1=1+\frac{4\mu^{1/3}}{3^{1/3}}+\frac{26\mu^{2/3}}{33^{2/3}}+\frac{116\mu}{27}+\frac{161\mu^{4/3}}{273^{1/3}}+\frac{740\mu^{5/3}}{813^{2/3}} +\frac{3316\mu^2}{729}+\frac{1540\mu^{7/3}}{2433^{1/3}}+\frac{2188\mu^{8/3}}{2433^{2/3}}+\frac{84460\mu^3}{19683}$$

where the coefficients $\mathcal{N}_{i1}$ are given appendix (D-L$_1$)

Substituting back into equation (11), r_2 is being a function of b_1, d_1, e_1, f_1, g_1 and h_1. It can be written in the form

$$r_2=b_1-d_1\Bigg(n^2\,(-1+\mu+b_1)+(1-\mu)\,g_1-\mu e_1 -\frac{3\mu\,(2\gamma_1-\gamma_2)}{2}f_1+\frac{3\,(1-\mu)\,(2\sigma_1-\sigma_2)}{2}h_1\Bigg) \qquad (12)$$

since the location of L_1 is given by

$$\xi_{0,L_1}=1-\mu+r^2 \qquad (13)$$

from equation (12) and (13) we get

$$\xi_{0,L_1}=1-\sum_{i=-3}^{9}\mathcal{N}^{\star}_{i1}\left(\frac{\mu}{3}\right)^{i/3} \qquad (14)$$

where the coefficients $\mathcal{N}^{\star}_{i1}$ are given appendix (B-L$_1$)

4.1 Location of L_2

The geometry of L_2 can be visualized as given by Figure 2.

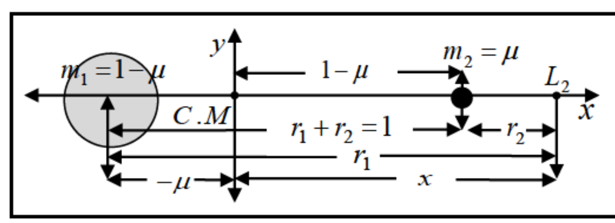

Figure 2: The location of L_2 and its corresponding parameters

Follow the same procedure as with L_1, with the corresponding values of$\mathcal{B}_1 = 1$, $\mathcal{B}_2 = -1$. Substituting into (5) we get

$$r_1-r_2=1,\quad r_1=x+\mu,\quad r_2=x+\mu-1, \qquad (15)$$
$$\frac{\partial r_1}{\partial x}=\frac{\partial r_2}{\partial x}=1$$

Substituting from (15) into (3) we get

$$U_x=n^2(1-\mu+r_2)-\left[\frac{(1-\mu)}{r_1^2}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^4}-\frac{15(1-\mu)(\sigma_1-\sigma_2)}{2r_1^6}y^2\right] -\left[\frac{\mu}{r_2^2}+\frac{3\mu\,(2\gamma_1-\gamma_2)}{2r_2^4}-\frac{15\mu\,(\gamma_1-\gamma_2)}{2r_2^6}y^2\right] \qquad (16)$$

Then it may be reasonable in our case to assume that position of the equilibrium point L_2 is the same as given by classical (RTBP) but perturbed due to the triaxial primaries

$$r_1=a_2+\varepsilon_2,\quad r_2=b_2+\varepsilon_2,\quad a_2=1+b_2. \qquad (17)$$

where a_2 and b_2 are the unperturbed positions of r_1 and r_2 respectively, and b_2 is given after some successive approximation by the relation

$$b_2 = \alpha\left(1 + \frac{\alpha}{3} - \frac{\alpha^2}{9} - \frac{2}{27}\alpha^3 + \frac{2}{81}\alpha^4\right), \tag{18}$$

$$\alpha = \left(\frac{\mu}{3\,(1-\mu)}\right)^3$$

Substituting from equations (17) into equation (16) we get

$$\begin{aligned}
U_x &= n^2(1-\mu+b_2+\varepsilon_2) - \frac{1-\mu}{(1+b_2)^2}\left(1-\frac{2\varepsilon_2}{1+b_2}\right) \\
&- \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2(1+b_2)^4}\left(1-\frac{4\varepsilon_2}{1+b_2}\right) \\
&+ \frac{15(1-\mu)(\sigma_1-\sigma_2)}{2(1+b_2)^6}\left(1-\frac{6\varepsilon_2}{1+b_2}\right)y^2 \\
&- \frac{\mu}{{b_2}^2}\left(1-\frac{2\varepsilon_2}{b_2}\right) - \frac{3\mu(2\gamma_1-\gamma_2)}{2{b_2}^4}\left(1-\frac{4\varepsilon_2}{b_2}\right) \\
&+ \frac{15\mu(\gamma_1-\gamma_2)}{2{b_2}^6}\left(1-\frac{6\varepsilon_2}{b_2}\right)y^2
\end{aligned}$$

Since $y = 0$, then the above equation may be written as

$$\begin{aligned}
U_x &= n^2(1-\mu+b_2+\varepsilon_2) - \frac{1-\mu}{(1+b_2)^2}\left(1-\frac{2\varepsilon_2}{1+b_2}\right) \\
&- \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2(1+b_2)^4}\left(1-\frac{4\varepsilon_2}{1+b_2}\right) \\
&- \frac{\mu}{{b_2}^2}\left(1-\frac{2\varepsilon_2}{b_2}\right) - \frac{3\mu(2\gamma_1-\gamma_2)}{2{b_2}^4}\left(1-\frac{4\varepsilon_2}{b_2}\right) = 0
\end{aligned} \tag{19}$$

Equation (19) can be solved for ε_2 yielding

$$\begin{aligned}
\varepsilon_2 &= \left(n^2 + \frac{2\mu}{b_2^3} + \frac{2\,(1-\mu)}{(1+b_2)^3} + \frac{6\mu\,(2\gamma_1-\gamma_2)}{b_2^5}\right. \\
&\left.+\frac{6\,(1-\mu)\,(2\sigma_1-\sigma_2)}{(1+b_2)^5}\right)^{-1} \times \left(n^2(\mu-b_2-1) + \frac{\mu}{b_2^2}\right. \\
&\left.+\frac{1-\mu}{(1+b_2)^2} + \frac{3\mu(2\gamma_1-\gamma_2)}{2b_2^4} + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2(1+b_2)^4}\right)
\end{aligned} \tag{20}$$

Setting

$$\begin{aligned}
d_2 &= \left(n^2 + \frac{2\mu}{b_2^3} + \frac{2\,(1-\mu)}{(1+b_2)^3} + \frac{6\mu\,(2\gamma_1-\gamma_2)}{b_2^5}\right. \\
&\left.+\frac{6\,(1-\mu)\,(2\sigma_1-\sigma_2)}{(1+b_2)^5}\right)^{-1},
\end{aligned}$$

$$e_2 = \frac{1}{b_2^2}, \quad f_2 = \frac{1}{b_2^4}, \quad g_2 = \frac{1}{(1+b_2)^2}, \quad h_2 = \frac{1}{(1+b_2)^4}.$$

Using the above relations, equation (20) can be written in the form

$$\begin{aligned}
\varepsilon &= d_2\left[n^2(\mu-b_2-1) + (1-\mu)g_2 + \mu e_2\right. \\
&+ \frac{9(1-\mu)A_1}{2}h_2 + \frac{9\mu A_2}{2}f_2 + \frac{3\mu(2\gamma_1-\gamma_2)}{2}f_2 \\
&\left.+ \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2}h_2\right]
\end{aligned} \tag{21}$$

where the coefficients $\mathcal{N}$ are given appendix (D-L$_2$) and where, d_2, e_2, f_2, g_2 and h_2 are functions of μ andare given by

$$\begin{aligned}
b_2 &= \left(\frac{\mu}{3}\right)^{\frac{1}{3}} + \frac{1}{3}\left(\frac{\mu}{3}\right)^{\frac{2}{3}} - \frac{1}{9}\left(\frac{\mu}{3}\right) + \frac{25}{27}\left(\frac{\mu}{3}\right)^{\frac{4}{3}} + \frac{56}{81}\left(\frac{\mu}{3}\right)^{\frac{5}{3}} \\
&- \frac{1}{3}\left(\frac{\mu}{3}\right)^2 + \frac{46}{27}\left(\frac{\mu}{3}\right)^{\frac{7}{3}} + \frac{145}{81}\left(\frac{\mu}{3}\right)^{\frac{8}{3}} - \left(\frac{\mu}{3}\right)^3 + \ldots
\end{aligned}$$

$$d_2 = \sum_{i=0}^{9} \mathcal{N}_{i2}\left(\frac{\mu}{3}\right)^{i/3}$$

$$\begin{aligned}
e_2 &= \frac{5}{9} - \frac{2}{3}\left(\frac{3}{\mu}\right)^{1/3} + \left(\frac{3}{\mu}\right)^{2/3} - \frac{20}{9}\left(\frac{\mu}{3}\right)^{1/3} + \frac{58}{81}\left(\frac{\mu}{3}\right)^{2/3} \\
&+ \frac{10}{243}\left(\frac{\mu}{3}\right) - \frac{932}{729}\left(\frac{\mu}{3}\right)^{4/3} + \frac{196}{243}\left(\frac{\mu}{3}\right)^{5/3} \\
&+ \frac{673}{6561}\left(\frac{\mu}{3}\right)^2 - \frac{39296}{19683}\left(\frac{\mu}{3}\right)^{7/3} + \frac{30796}{19683}\left(\frac{\mu}{3}\right)^{8/3} \\
&+ \frac{42916}{177147}\left(\frac{\mu}{3}\right)^3 + \ldots
\end{aligned}$$

$$\begin{aligned}
f_2 &= \frac{143}{27} + \left(\frac{3}{\mu}\right)^{4/3} - \frac{4}{\mu} + \frac{14}{9}\left(\frac{3}{\mu}\right)^{2/3} - \frac{148}{27}\left(\frac{3}{\mu}\right)^{1/3} \\
&- \frac{1012}{243}\left(\frac{\mu}{3}\right)^{1/3} - \frac{10538\mu^2}{6561}\left(\frac{\mu}{3}\right)^2 - \frac{456872}{53144}\left(\frac{\mu}{3}\right)^{7/3} \\
&+ \frac{3600328}{531441}\left(\frac{\mu}{3}\right)^{7/3} - \frac{2258600}{531441}\left(\frac{\mu}{3}\right)^2
\end{aligned}$$

$$\begin{aligned}
g_2 &= 1 - 2\left(\frac{\mu}{3}\right)^{1/3} + \frac{7}{3}\left(\frac{\mu}{3}\right)^{2/3} - \frac{16}{9}\left(\frac{\mu}{3}\right) - \frac{32}{27}\left(\frac{\mu}{3}\right)^{4/3} \\
&+ \frac{347}{81}\left(\frac{\mu}{3}\right)^{5/3} - \frac{50}{9}\left(\frac{\mu}{3}\right)^2 - \frac{52}{81}\left(\frac{\mu}{3}\right)^{7/3} + \frac{104}{9}\left(\frac{\mu}{3}\right)^{8/3} \\
&- \frac{12826}{729}\left(\frac{\mu}{3}\right)^3 + \ldots
\end{aligned}$$

$$\begin{aligned}
h_2 &= 1 - 4\left(\frac{\mu}{3}\right)^{1/3} + \frac{26}{3}\left(\frac{\mu}{3}\right)^{2/3} - \frac{1169}{9}\left(\frac{\mu}{3}\right) + \frac{89}{9}\left(\frac{\mu}{3}\right)^{4/3} \\
&+ \frac{196}{27}\left(\frac{\mu}{3}\right)^{5/3} - \frac{2948}{81}\left(\frac{\mu}{3}\right)^2 + \frac{1460}{27}\left(\frac{\mu}{3}\right)^{7/3} \\
&- \frac{532}{27}\left(\frac{\mu}{3}\right)^{8/3} - \frac{64588}{729}\left(\frac{\mu}{3}\right)^3
\end{aligned}$$

Substituting back into equation (21), r_2 is a function of b_2, d_2, e_2, f_2, g_2 and h_2 and can be written in the form

$$\begin{aligned}
r_2 &= b_2 + d_2\left[n^2(\mu-b_2-1) + (1-\mu)g_2 + \mu e_2\right. \\
&\left.+ \frac{3\mu(2\gamma_1-\gamma_2)}{2}f_2 + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2}h_2\right]
\end{aligned} \tag{22}$$

since the location of L_2 is given by

$$\xi_{0,L_2} = 1 - \mu + r_2 \tag{23}$$

from equation (22) and (23) we get

$$\xi_{0,L_2} = 1 - \sum_{i=-3}^{9} \mathcal{N}^{\star}_{i2} \left(\frac{\mu}{3}\right)^{i/3}$$

where the coefficients $\mathcal{N}^{\star}_{i2}$ are given appendix (A-L$_2$)

4.2 Location of L_3

The geometry of L_3 can be visualized as given by Figure 2.

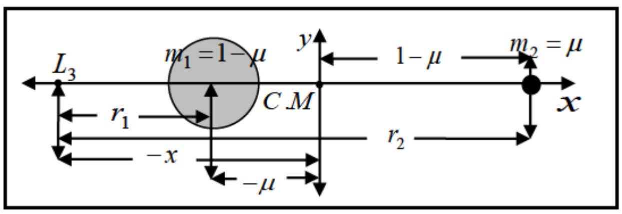

Figure 3: The location of L_3 and its corresponding parameters

Follow the same procedure as in L_1, with the corresponding values of $\mathcal{B}_1 = -1$, $\mathcal{B}_2 = 1$, substituting into (5) we get

$$r_2 - r_1 = 1, \quad r_1 = -(x+\mu), \qquad (24)$$
$$r_2 = 1 - \mu - x, \quad \frac{\partial r_1}{\partial x} = \frac{\partial r_2}{\partial x} = -1$$

Hence substituting from equation (24) into (5), we get

$$U_x = n^2(1-\mu-r_2) + \left[\frac{(1-\mu)}{r_1^2} + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^4} - \frac{15(1-\mu)(\sigma_1-\sigma_2)}{2r_1^6}y^2\right] + \left[\frac{\mu}{r_2^2} + \frac{3\mu(2\gamma_1-\gamma_2)}{2r_2^4} - \frac{15\mu(\gamma_1-\gamma_2)}{2r_2^6}\right] \qquad (25)$$

Then it may be reasonable in our case to assume that position of the equilibrium point L_3 is the same as given by classical restricted three-body problem but perturbed due tothe triaxial primaries

$$r_1 = a_3 + \varepsilon_3, \quad r_2 = b_3 + \varepsilon_3, \quad a_3 = b_3 - 1. \qquad (26)$$

where a_3 and b_3 are the unperturbed values of r_1 and r_2 respectively, and b_3 is given after some successive approximation by the relation

$$b_3 = 2 - \frac{7\mu}{12} - \frac{161\mu^3}{1728} + \ldots \qquad (27)$$

Substituting from equations (26) into equation (25) and retaining terms up to the first order in the small quantities ε_3 we get

$$\begin{aligned} U_x = {} & n^2(1-\mu-b_3-\varepsilon_3) + \frac{1-\mu}{(b_3-1)^2}\left(1-\frac{2\varepsilon_3}{b_3-1}\right) \\ & + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2(b_3-1)^4}\left(1-\frac{4\varepsilon_3}{b_3-1}\right) \\ & - \frac{15(1-\mu)(\sigma_1-\sigma_2)}{2(b_3-1)^6}\left(1-\frac{6\varepsilon_3}{b_3-1}\right)y^2 \\ & + \frac{\mu}{{b_3}^2}\left(1-\frac{2\varepsilon_3}{b_3}\right) + \frac{3\mu(2\gamma_1-\gamma_2)}{2{b_3}^4}\left(1-\frac{4\varepsilon_3}{b_3}\right) \\ & - \frac{15\mu(\gamma_1-\gamma_2)}{2{b_3}^6}\left(1-\frac{6\varepsilon_3}{b_3}\right)y^2 \end{aligned}$$

Since $y = 0$, then the above equation may be written as

$$\begin{aligned} U_x = {} & n^2(1-\mu-b_3-\varepsilon_3) + \frac{1-\mu}{(b_3-1)^2}\left(1-\frac{2\varepsilon_2}{b_3-1}\right) \\ & + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2(b_3-1)^4}\left(1-\frac{4\varepsilon_3}{b_3-1}\right) + \frac{\mu}{{b_3}^2}\left(1-\frac{2\varepsilon_3}{b_3}\right) \\ & + \frac{3\mu(2\gamma_1-\gamma_2)}{2{b_3}^4}(1-\frac{4\varepsilon_3}{b_3}) = 0 \end{aligned} \qquad (28)$$

Equation (28) can be solved for ε_3 to yield

$$\begin{aligned} \varepsilon_3 = {} & \left(-n^2 - \frac{2(1-\mu)}{(b_3-1)^3} - \frac{2\mu}{b_3^3} - \frac{6\mu(2\gamma_1-\gamma_2)}{b_3^5} - \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{(b_3-1)^5}\right)^{-1} \times \left(-n^2 + n^2\mu - \frac{1-\mu}{(-1+b_3)^2}\right. \\ & \left. - \frac{\mu}{b_3^2} + n^2 b_3 - \frac{3\mu(2\gamma_1-\gamma_2)}{2b_3^4} - \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2(-1+b_3)^4}\right) \end{aligned} \qquad (29)$$

now, letting

$$d_3 = \left(-n^2 - \frac{2(1-\mu)}{(b_3-1)^3} - \frac{2\mu}{b_3^3} - \frac{6\mu(2\gamma_1-\gamma_2)}{b_3^5} - \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{(b_3-1)^5}\right)$$

$$e_3 = \frac{1}{b_3^2}, \quad f_3 = \frac{1}{b_3^4}, \quad g_3 = \frac{1}{(b_3-1)^2}, \quad h_3 = \frac{1}{(b_3-1)^4}$$

Using the above relations, equation (29) can be written in the form

$$\varepsilon_3 = d_3\left(-n^2(1-\mu) + n^2 b_3 - (1-\mu)g_3 - \mu e_3 - \frac{3\mu(2\gamma_1-\gamma_2)}{2}f_3 - \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2}h_3\right) \qquad (30)$$

where d_3, e_3, f_3, g_3 and h_3 are functions of μ. They are given by

$$d_3 = \sum_{i=-2}^{9} \mathcal{N}_{i3}\left(\frac{\mu}{3}\right)^{i/3}$$

$$e_3 = \frac{1}{4} + \frac{7\mu}{48} + \frac{49\mu^2}{768} + \frac{665\mu^3}{13824} + \ldots$$
$$f_3 = \frac{1}{16} + \frac{7\mu}{96} + \frac{245\mu^2}{4608} + \frac{2359\mu^3}{55296} + \ldots$$
$$g_3 = \frac{1}{9} + \frac{7\mu}{162} + \frac{49\mu^2}{3888} + \frac{2135\mu^3}{209952} + \ldots$$
$$h_3 = \frac{1}{81} + \frac{7\mu}{729} + \frac{245\mu^2}{52488} + \frac{791\mu^3}{236196} + \ldots$$

where the coefficients $\mathcal{N}_{i1}$ are given appendix (D-L_3).

Substituting back into equation (29) to yield r_2 as a function of b_3, d_3, e_3, f_3, g_3 and h_3. It can be rewritten in the form

$$r_2 = b_3 + d_3\Bigg(-n^2(1-\mu) + b_3 - (1\mu)g_3 - \mu e_3 - \frac{3\mu(2\gamma_1-\gamma_2)}{2}f_3 - \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2}h_3\Bigg) \quad (31)$$

since the location of L_3 is given by

$$\xi_{0,L_3} = 1 - \mu - r_2 \quad (32)$$

from equation (31) and (32) we get

$$\xi_{0,L_3} = 1 - \sum_{i=-3}^{9} \mathcal{N}^{\star}_{i3}\ \mu^{i/3} \quad (33)$$

where the coefficients $\mathcal{N}^{\star}_{i3}$ are given Appendix (A-L_3)

5 Numerical representations

In the following section, we will draw the locations of collinear points L_i, $i = 1,2,3$ versus the mass ratio $\mu \in (0, 0.1)$ taking into account the effect oftrixial primaries.In all casesthe black curve represents the classical unperturbed RTBP in which the primaries are considered spheres. The blue and the red curves represent the increasing magnitude in the perturbing parameters.

5.1 Analysis of L_1 location

Considering the massive primary as oblate, it can be seen in Figure 4a that the location of is shifted towards the massive primary, *i.e.* towards the barycenter. Since any equilibrium point emerges from the balance between the gravitational field and the rotational field of the primaries, we can conclude that the resultant of these forces is to perturb the location of L_1towards the massive primary. This is very logical dynamical effect, since the additional mass bulge due to oblateness of the massive primarycauses gravitational attraction towards the center. The effect is noticeable for a relatively higher mass ratio. In view of Figure 4b, and considering the less massive primary as oblate, the location of L_1 is largely shifted towards the less massive primary, *i.e.* away from the barycenter. The effect of perturbation is much bigger than that in Figure 4a due to the close proximity of the point to the less massive primary.

In view of the Figure 4c and Figure 4d, the dynamical effects are towards and away from the barycenter, respectively. The size of perturbation is small comparedto the effects of oblateness due to the magnitude of the perturbing parameter.

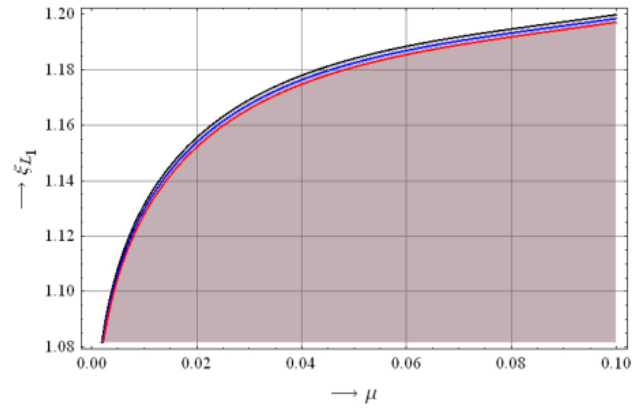

Figure 4a: The location of L_1 against under the effect of an oblate massive primary and spherical,less massive, primary

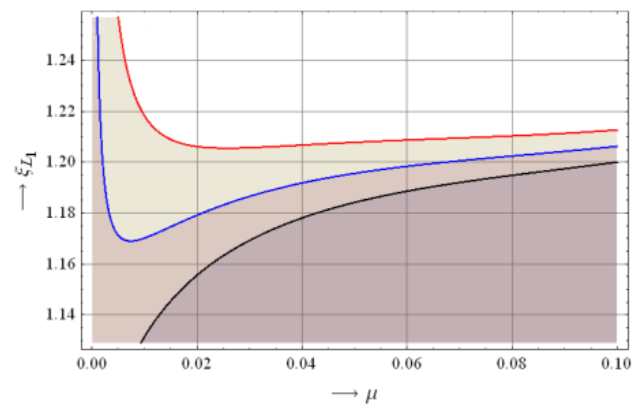

Figure 4b: The location of against μ under the effect of an oblate less massive primary and sphere massive primary

5.2 Analysis of L_2 location

From Figure 4e, considering the massive primary as oblate, the location of L_2 is shifted towards the barycenter for mass ratios larger that the classical critical mass ratio, namely the Routian value $\mu_c = 0.03841$. In Figure 4f, con-

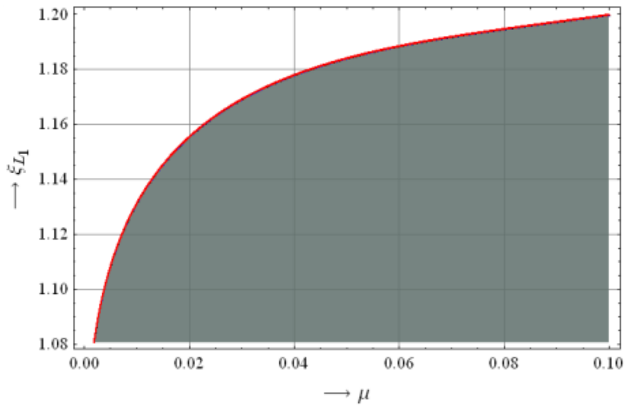

Figure 4c: The location of L_1 against μ under the effect of triaxial massive primary and sphere less massive primary

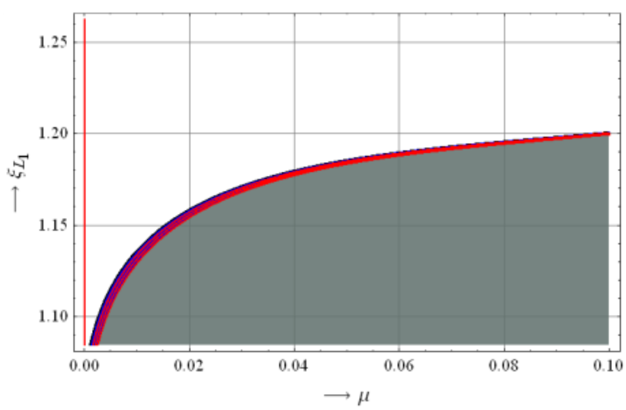

Figure 4d: The location of L_1 against μ under the effect of an oblate massive primary and sphere less massive primary

sidering the less massive primary as oblate, the location of L_2 is also largely shifted towards to thebarycenter. The effect of perturbation is much bigger than that shown in Figure 4e due to the close proximity of the point to the less massive primary.

Referring to Figure 4g and Figure 4h, the dynamical effects are towards and away from the barycenter respectively. These effects could be easily interpreted by considering balancing between the gravitational and rotational fields. The size of perturbation is small comparedto the effects of oblateness due to the magnitude of the perturbing parameter.

5.3 Analysis of L_3 location

The Figures 4i-4l show the locations of L_3considering the massive primary as oblate in 4i and the less massive primary as oblate in Figure 4j. Figure 4k and 4l show the perturbed location of L_3 due to the triaxiality of the primaries.

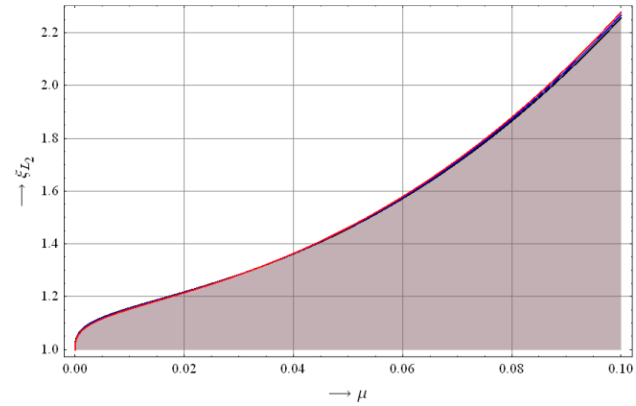

Figure 4e: The location of L_2 against μ under the effect of an oblate massive primary and sphere less massive primary

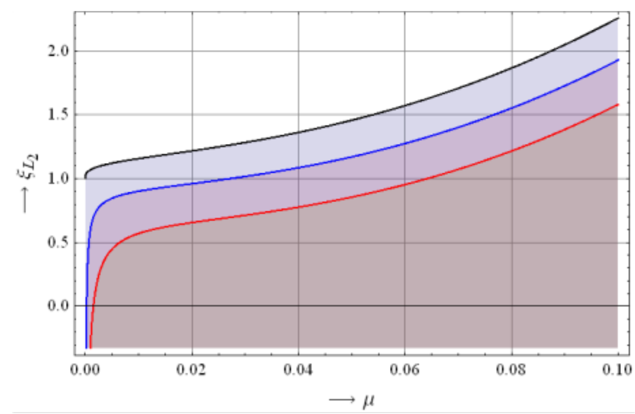

Figure 4f: The location of L_2 against μ under the effect of an oblate less massive primary and sphere massive primary

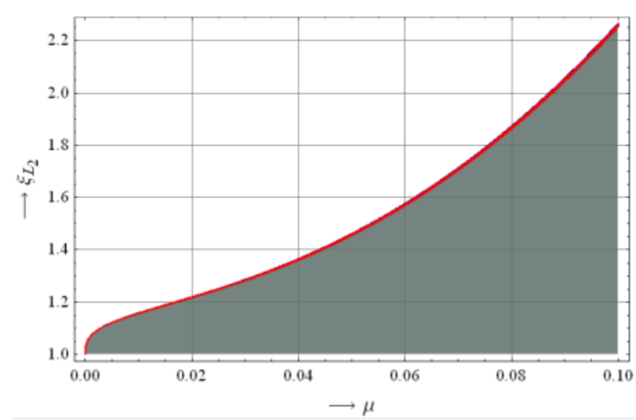

Figure 4g: The location of L_2 against μ under the effect of triaxial massive primary and sphere less massive primary

It is very clear that the dynamics are dominated by the massive primary.

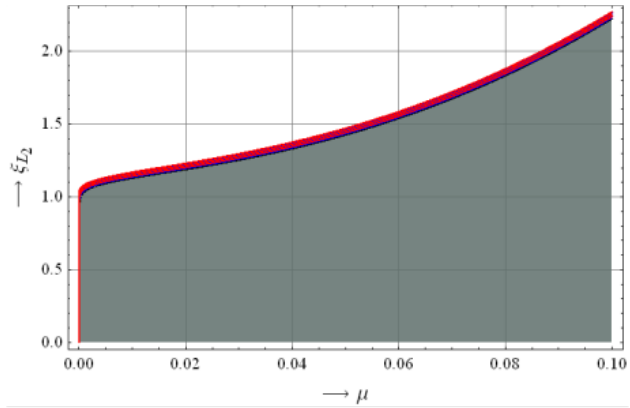

Figure 4h: The location of L_2 against μ under the effect of an oblate massive primary and sphere less massive primary

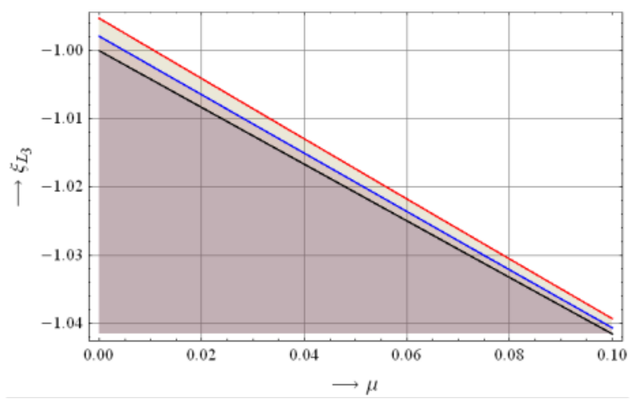

Figure 4i: The location of L_3 against μ under the effect of an oblate massive primary and sphere less massive primary

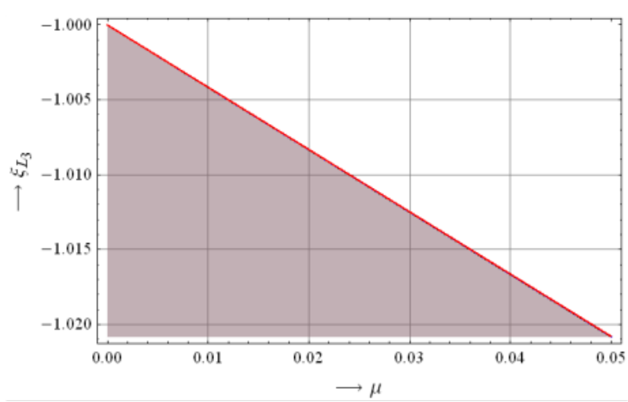

Figure 4j: The location of L_3 against μ under the effect of an oblate less massive primary and sphere massive primary

6 Conclusion

We have treated the perturbed RTBP with the triaxiality of both primaries up to the terms of order $\mathcal{O}\left(1/r_{1,2}^{7}\right)$ perturbations.As expected,these perturbations bring deviations

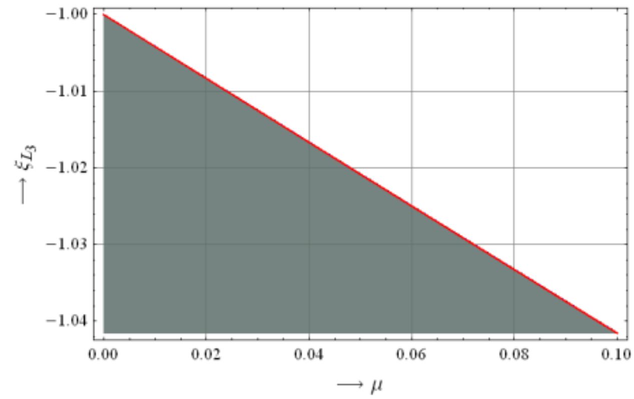

Figure 4k: The location of L_3 against μ under the effect of triaxial massive primary and sphere less massive primary

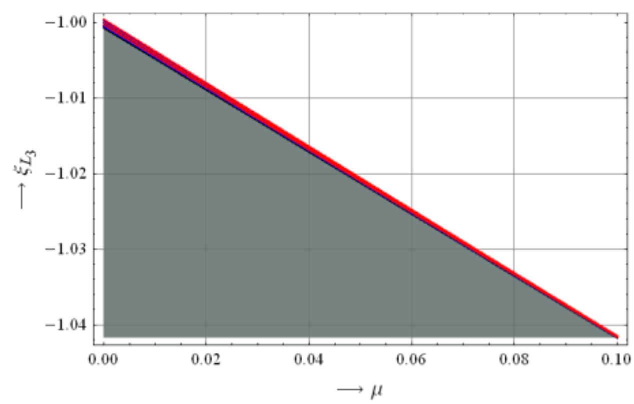

Figure 4l: The location of L_3 ratio μ under the effect of an oblate massive primary and sphere less massive primary

of the locations of the equilibrium points from classical RTBP. In this work, we computed and illustrated these deviations in collinear points explicitly as functions in the mass ratio. We analyzed the oblate RTBP as special cases of a triaxialproblem. All the dynamical effects could be properly interpreted in view of balancing between the gravitational and rotational fields. It is observed that the dynamics of L_3 are clearly dominated by the massive primary, while the dynamics of L_1 and L_2 are dominatedby the less massive primary due to their close proximity to it. When investigating the triaxiality effects, it is noticed that the size of perturbation is small compared to the effects of oblateness due to the magnitude of the perturbing parameter.

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Appendix (D-L$_1$)

$$\mathcal{N}_{01} = 2 - \frac{n^2}{4} + 305\gamma_1 - \frac{305\gamma_2}{2} - 3\sigma_1 + \frac{3\sigma_2}{2} + \left(\frac{3}{\mu}\right)^{1/3}\left(-75\gamma_1 + \frac{75}{2}\gamma_2\right) + \left(\frac{3}{\mu}\right)^{2/3}\left(9\gamma_1 - \frac{9}{2}\gamma_2\right)$$

$$\mathcal{N}_{11} = -15 + \frac{5n^2}{2} - \frac{2515\gamma_1}{3} + \frac{2515\gamma_2}{6} + 15\sigma_1 - \frac{15}{2}\sigma_2$$

$$\mathcal{N}_{21} = 59 - \frac{145n^2}{12} + \frac{15865\gamma_1}{9} - \frac{15865\gamma_2}{18} - 35\sigma_1 + \frac{35\sigma_2}{2}$$

$$\mathcal{N}_{31} = -\frac{2873}{18} + \frac{335n^2}{9} - \frac{78860\gamma_1}{27} + \frac{39430\gamma_2}{27} + \frac{172\sigma_1}{3} - \frac{86\sigma_2}{3}$$

$$\mathcal{N}_{41} = \frac{2884}{9} - \frac{315}{4}n^2 + \frac{301145\gamma_1}{81} - \frac{301145\gamma_2}{163} - \frac{215\sigma_1}{3} + \frac{215\sigma_2}{6}$$

$$\mathcal{N}_{51} = -\frac{8575}{18} + 108n^2 - \frac{790940\gamma_1}{243} + \frac{395470\gamma_2}{243} + \frac{502\sigma_1}{9} - \frac{251\sigma_2}{9}$$

$$\mathcal{N}_{81} = -\frac{462913}{729} + \frac{26615n^2}{54} - \frac{31442824\gamma_1}{6561} + \frac{15721412\gamma_2}{6561} + \frac{704\sigma_1}{9} - \frac{352\sigma_2}{9}$$

$$\mathcal{N}_{91} = \frac{2806807}{2187} - \frac{145255n^2}{243} + \frac{56532350\gamma_1}{59049} - \frac{28266175\gamma_2}{59049} - \frac{2320\sigma_1}{243} + \frac{1160\sigma_2}{243}$$

Appendix (D-L$_2$)

$$\mathcal{N}_{02} = -1 - \frac{n^2}{4} - 305\gamma_1 + \frac{305\gamma_2}{2} - 3\sigma_1 + \frac{3\sigma_2}{2} + \left(\frac{3}{\mu}\right)^{1/3}\left(-75\gamma_1 + \frac{75}{2}\gamma_2\right) + \left(\frac{3}{\mu}\right)^{2/3}\left(-9\gamma_1 + \frac{9}{2}\gamma_2\right)$$

$$\mathcal{N}_{12} = -12 - \frac{5n^2}{2} - \frac{2245\gamma_1}{3} + \frac{2245\gamma_2}{6} - 15\sigma_1 + \frac{15}{2}\sigma_2$$

$$\mathcal{N}_{22} = -59 - \frac{145n^2}{12} - \frac{10465\gamma_1}{9} + \frac{10465\gamma_2}{18} - 35\sigma_1 + \frac{35\sigma_2}{2}$$

$$\mathcal{N}_{32} = -\frac{2999}{18} - \frac{335n^2}{9} - \frac{29450\gamma_1}{27} + \frac{14725\gamma_2}{27} - \frac{118\sigma_1}{3} + \frac{59\sigma_2}{3}$$

$$\mathcal{N}_{42} = -\frac{2757}{3} - \frac{335n^2}{4} - \frac{44105\gamma_1}{81} + \frac{44105\gamma_2}{163} - \frac{35\sigma_1}{3} + \frac{35\sigma_2}{6}$$

$$\mathcal{N}_{52} = -\frac{22175}{54} - \frac{469n^2}{3} - \frac{80030\gamma_1}{243} + \frac{40015\gamma_2}{243} + \frac{128\sigma_1}{9} - \frac{64\sigma_2}{9}$$

$$\mathcal{N}_{62} = -\frac{123127}{243} - \frac{3305n^2}{12} - \frac{690895\gamma_1}{729} + \frac{690895\gamma_2}{1458} - \frac{85\sigma_1}{27} + \frac{85\sigma_2}{54}$$

$$\mathcal{N}_{72} = -\frac{59750}{81} - \frac{4360n^2}{9} - \frac{3651725\gamma_1}{2187} + \frac{3651725\gamma_2}{4374} - \frac{215\sigma_1}{9} + \frac{215\sigma_2}{18}$$

$$\mathcal{N}_{82} = -\frac{831071}{729} - \frac{44755n^2}{54} - \frac{12965156\gamma_1}{6561} + \frac{6482578\gamma_2}{6561} - \frac{44\sigma_1}{9} + \frac{22\sigma_2}{9}$$

$$\mathcal{N}_{92} = -\frac{3651845}{2187} - \frac{332240n^2}{243} - \frac{55392640\gamma_1}{59049} + \frac{27696320\gamma_2}{59049} - \frac{2270\sigma_1}{243} + \frac{1135\sigma_2}{243}$$

Appendix (D-L$_3$)

$$\mathcal{N}_{03} = \frac{1}{4}\left(2 - 3n^2 - 12\sigma_1 + 6\sigma_2\right)$$

$$\mathcal{N}_{33} = \frac{1}{64}\left(-84 + 560n^2 - 18\gamma_1 + 9\gamma_2 + 2256\sigma_1 - 1128\sigma_2\right)$$

$$\mathcal{N}_{63} = \frac{7}{1536}\left(4032 + 10000n^2 - 810\gamma_1 + 405\gamma_2 + 34680\sigma_1 - 17340\sigma_2\right)$$

$$\mathcal{N}_{93} = \frac{7}{24576}\left(384932 + 491120n^2 - 65610\gamma_1 + 32805\gamma_2 + 1090560\sigma_1 - 545280\sigma_2\right)$$

Appendix (A-L$_1$)

$$\mathcal{N}^{\star}_{01} = 2 - \frac{9n^2}{4} + \frac{n^4}{4} + 494\gamma_1 - \frac{805n^2\gamma_1}{2} + 5139\gamma_1^2 - 247\gamma_2 + \frac{805n^2\gamma_2}{4} + \frac{75\gamma_2\sigma_2}{2} + \frac{5139\gamma_2^2}{4} + 3\sigma_1 + \frac{9n^2\sigma_1}{4} + 150\gamma_1\sigma_1 - 75\gamma_2\sigma_1 - 9\sigma_1^2 - \frac{9\sigma_2^2}{4} - \frac{3\sigma_2}{2} - \frac{9n^2\sigma_2}{8} - 75\gamma_1\sigma_2 + 9\sigma_1\sigma_2 - 5139\gamma_1\gamma_2$$

$$\mathcal{N}^{\star}_{-11} = -102\gamma_1 + \frac{345}{4}n^2\gamma_1 - 1971\gamma_1^2 + 51\gamma_2 - \frac{345}{8}n^2\gamma_2 + 1971\gamma_1\gamma_2 - \frac{1971}{4}\gamma_2^2 - 90\gamma_1\sigma_1 + 45\gamma_2\sigma_1 + 45\gamma_1\sigma_2 - \frac{45}{2}\gamma_2\sigma_2$$

$$\mathcal{N}^{\star}_{-21} = (9\gamma_1 - 9n^2\gamma_1 + 567\gamma_1^2 - \frac{9}{2}\gamma_2 + \frac{9}{2}n^2\gamma_2 - 567\gamma_1\gamma_2 + \frac{567}{4}\gamma_2^2 + 27\gamma_1\sigma_1 - \frac{27}{2}\gamma_2\sigma_1 - \frac{27}{2}\gamma_1\sigma_2 + \frac{27}{4}\gamma_2\sigma_2$$

$$\mathcal{N}^{\star}_{-31} = -81\gamma_1^2 + 81\gamma_1\gamma_2 - \frac{81\gamma_2^2}{4}$$

$$\mathcal{N}^{\star}_{11} = -18 + \frac{79n^2}{4} - \frac{11n^4}{4} - \frac{4720\gamma_1}{3} + \frac{15319n^2\gamma_1}{12} - 11556\gamma_1^2 + \frac{2360\gamma_2}{3} - \frac{15319n^2\gamma_2}{24} + 11556\gamma_1\gamma_2 - 2079\gamma_2^2 - 3\sigma_1 - \frac{27}{2}n^2\sigma_1 - 361\gamma_1\sigma_1$$

$$\mathcal{N}^{\star}_{21} = 75 - \frac{268n^2}{3} + \frac{44n^4}{3} + \frac{70511\gamma_1}{18} - \frac{28270n^2\gamma_1}{9} + 22125\gamma_1^2 - \frac{70511\gamma_2}{36} + \frac{14135n^2\gamma_2}{9} - 22125\gamma_1\gamma_2 + \frac{22125}{4}\gamma_2^2 - 2\sigma_1 + \frac{153}{4}n^2\sigma_1 + \frac{2155\gamma_1\sigma_1}{3}$$

$$\mathcal{N}^{\star}_{31} = -\frac{4073}{18} + \frac{1121n^2}{3} - \frac{1831n^4}{36} - \frac{217631\gamma_1}{27} + \frac{337225n^2\gamma_1}{54} - 36168\gamma_1^2 + \frac{217631\gamma_2}{54} - \frac{337225n^2\gamma_2}{108} + 36168\gamma_1\gamma_2 - 9042\gamma_2^2 + \frac{9\sigma_1}{2} - \frac{327n^2\sigma_1}{4} - \frac{8372\gamma_1\sigma_1}{9} + \frac{4186\gamma_2\sigma_1}{9} + 53\sigma_1^2 - \frac{9\sigma_2}{4} + \frac{327n^2\sigma_2}{8} + \frac{4186\gamma_1\sigma_2}{9} - \frac{2093\gamma_2\sigma_2}{9} - 53\sigma_1\sigma_2 + \frac{53\sigma_2^2}{4}$$

$$\mathcal{N}^{\star}_{41} = \frac{3271}{9} - \frac{2006n^2}{3} + \frac{4571n^4}{36} + \frac{2253133\gamma_1}{162} - \frac{1630015n^2\gamma_1}{162} + \frac{154879\gamma_1^2}{3} - \frac{15}{4}\sigma_2^2 - \frac{2253133\gamma_2}{324} + \frac{1630015n^2\gamma_2}{324} - \frac{154879\gamma_1\gamma_2}{3} + \frac{154879\gamma_2^2}{12} - \frac{55\sigma_1}{3} + 49n^2\sigma_1 + 15\sigma_1\sigma_2$$

$$\mathcal{N}^{\star}_{51} = -\frac{28247}{27} + \frac{33547n^2}{27} - \frac{24991n^4}{108} - \frac{4893158\gamma_1}{243} + \frac{3138185n^2\gamma_1}{243} - \frac{572404\gamma_1^2}{9} + \frac{2446579\gamma_2}{243} - \frac{3138185n^2\gamma_2}{486} + \frac{572404\gamma_1\gamma_2}{9} - \frac{143101\gamma_2^2}{9} + \frac{641\sigma_1}{18} - \frac{797n^2\sigma_1}{4} - \frac{130655\gamma_1\sigma_1}{108} + \frac{130655\gamma_2\sigma_1}{162} + 12\sigma_1^2 - \frac{641\sigma_2}{36} + \frac{797n^2\sigma_2}{8} + \frac{130655\gamma_1\sigma_2}{162} - \frac{130655\gamma_2\sigma_2}{324} - 12\sigma_1\sigma_2 + 3\sigma_2^2$$

$$\mathcal{N}^{\star}_{61} = \frac{787937}{486} - \frac{854033n^2}{486} + \frac{22696n^4}{81} + \frac{35047541\gamma_1}{1458} - \frac{9005530n^2\gamma_1}{729} + \frac{15905525\gamma_1^2}{243} - \frac{35047541\gamma_2}{2916} + \frac{4502765n^2\gamma_2}{729} - \frac{15905525\gamma_1\gamma_2}{234} + \frac{15905525\gamma_2^2}{972} - \frac{8557\sigma_1}{162} + \frac{589n^2\sigma_1}{3} + \frac{589n^2\sigma_1}{3} + \frac{350242\gamma_1\sigma_1}{243} - \frac{175121\gamma_2\sigma_1}{243} - 78\sigma_1^2 + \frac{8557\sigma_2}{324} - \frac{589n^2\sigma_2}{18} - \frac{175121\gamma_1\sigma_2}{243} + \frac{175121\gamma_2\sigma_2}{486} + 78\sigma_1\sigma_2 - 14\sigma_2^2$$

$$\mathcal{N}^{\star}_{71} = -\frac{954265}{486} + \frac{271853n^2}{162} - \frac{1273n^4}{12} - \frac{49027529\gamma_1}{2187} + \frac{59331695n^2\gamma_1}{8748} - \frac{12948929\gamma_1^2}{243} + \frac{49027529\gamma_2}{4374} - \frac{59331695n^2\gamma_2}{17496} + \frac{12948929\gamma_1\gamma_2}{243} - \frac{12948929\gamma_2^2}{972} + \frac{2923\sigma_1}{54} - \frac{257n^2\sigma_1}{2} - \frac{799070\gamma_1\sigma_1}{729} + \frac{399535\gamma_2\sigma_1}{729} + 15\sigma_1^2 - \frac{2923\sigma_2}{108} + \frac{257n^2\sigma_2}{4} + \frac{399535\gamma_1\sigma_2}{729} - \frac{399535\gamma_2\sigma_2}{18122} - 15\sigma_1\sigma_2 + \frac{15\sigma_2^2}{4}$$

$$\mathcal{N}^{\star}_{81} = \frac{2402459}{1458} - \frac{1313483n^2}{2916} - \frac{11426n^4}{27} + \frac{17992143\gamma_1}{13122} + \frac{12927712n^2\gamma_1}{65561} + \frac{20836592\gamma_1^2}{729} - \frac{177992143\gamma_2}{26244} - \frac{6463856n^2\gamma_2}{65561} - \frac{20836592\gamma_1\gamma_2}{729} + \frac{5209148\gamma_2^2}{729} - \frac{14753\sigma_1}{486} + \frac{461n^2\sigma_1}{36} - \frac{8007568\gamma_1\sigma_1}{2187} + \frac{4003784\gamma_2\sigma_1}{2187} - 6\sigma_1^2 + \frac{14753\sigma_2}{972} - \frac{461n^2\sigma_2}{72} + \frac{4003784\gamma_1\sigma_2}{2187} - \frac{2001892\gamma_2\sigma_2}{2187} + 6\sigma_1\sigma_2 - \frac{3\sigma_2^2}{2}$$

$$\mathcal{N}^{\star}_{91} = -\frac{11914}{27} - \frac{3940255n^2}{2187} + \frac{1084129n^4}{972} - \frac{164982886\gamma_1}{19683} - \frac{626757695n^2\gamma_1}{78732} + \frac{223007333\gamma_1^2}{6561} + \frac{82491443\gamma_2}{19683} + \frac{626757695n^2\gamma_2}{157464} - \frac{223007333\gamma_1\gamma_2}{6561} + \frac{223007333\gamma_2^2}{26244} - \frac{913\sigma_1}{1458} + \frac{5333n^2\sigma_1}{81} + \frac{188329745\gamma_1\sigma_1}{6561} - \frac{188329745\gamma_2\sigma_1}{13122} + \frac{913\sigma_2}{2916} - \frac{5333n^2\sigma_2}{162} - \frac{188329745\gamma_1\sigma_2}{13122} + \frac{188329745\gamma_2\sigma_2}{26244}$$

Appendix (A-L$_2$)

$$\begin{aligned}
\mathcal{N}^{\star}_{02} = {} & 1 - \frac{3n^2}{4} - \frac{n^4}{4} + 479\gamma_1 - \frac{727n^2\gamma_1}{2} + 3681\gamma_1^2 \\
& - \frac{479\gamma_2}{2} + \frac{727n^2\gamma_2}{4} - 3681\gamma_1\gamma_2 + \frac{3681\gamma_2^2}{4} + 6\sigma_1 \\
& - \frac{9n^2\sigma_1}{4} + 348\gamma_1\sigma_1 - 174\gamma_2\sigma_1 - 3\sigma_2 + \frac{9n^2\sigma_2}{8} \\
& + \frac{9\sigma_2^2}{4} - 174\gamma_1\sigma_2 + 87\gamma_2\sigma_2 - 9\sigma_1\sigma_2 + 9\sigma_1^2
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}^{\star}_{-12} = {} & 93\gamma_1 - \frac{327}{4}n^2\gamma_1 + 1971\gamma_1^2 - \frac{93}{2}\gamma_2 + \frac{327}{8}n^2\gamma_2 \\
& - 1971\gamma_1\gamma_2 + \frac{1971}{4}\gamma_2^2 + 144\gamma_1\sigma_1 - 72\gamma_2\sigma_1 \\
& - 72\gamma_1\sigma_2 + 36\gamma_2\sigma_2
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}^{\star}_{-22} = {} & 9\gamma_1 - 9n^2\gamma_1 + 567\gamma_1^2 - \frac{9}{2}\gamma_2 + \frac{9}{2}n^2\gamma_2 - 567\gamma_1\gamma_2 \\
& + \frac{567}{4}\gamma_2^2 + 27\gamma_1\sigma_1 - \frac{27}{2}\gamma_2\sigma_1 - \frac{27}{2}\gamma_1\sigma_2 + \frac{27}{4}\gamma_2\sigma_2
\end{aligned}$$

$$\mathcal{N}^{\star}_{-32} = 81\gamma_1^2 - 81\gamma_1\gamma_2 + \frac{81\gamma_2^2}{4}$$

$$\begin{aligned}
\mathcal{N}^{\star}_{12} = {} & 12 - \frac{41n^2}{4} - \frac{11n^4}{4} + \frac{4354\gamma_1}{3} - \frac{11617n^2\gamma_1}{12} \\
& + 8283^{2/3}\gamma_1^2 - \frac{2177\gamma_2}{3} + \frac{11617n^2\gamma_2}{24} - 2484\gamma_1\gamma_2 \\
& + 621\gamma_2^2 + 42\sigma_1 - \frac{27}{2}n^2\sigma_1 + 283\gamma_1\sigma_1 - \frac{283\gamma_2\sigma_1}{2} \\
& + \frac{27}{4}n^2\sigma_2 - \frac{283\gamma_1\sigma_2}{2} + 9\sigma_1^2 - 21\sigma_2 + \frac{283\gamma_2\sigma_2}{4} \\
& - 9\sigma_1\sigma_2 + \frac{9}{4}\sigma_2^2
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}^{\star}_{22} = {} & 71 - \frac{170n^2}{3} - \frac{44n^4}{3} + \frac{48017\gamma_1}{18} - \frac{14182n^2\gamma_1}{9} \\
& - 18233^{1/3}\gamma_1^2 - \frac{48017\gamma_2}{36} + \frac{7091n^2\gamma_2}{9} + 5469\gamma_1\gamma_2 \\
& - \frac{5469}{4}\gamma_2^2 + 110\sigma_1 - \frac{153}{4}n^2\sigma_1 - \frac{1649\gamma_1\sigma_1}{3} \\
& + \frac{1649\gamma_2\sigma_1}{6} + 3\sigma_1^2 - 55\sigma_2 + \frac{153}{8}n^2\sigma_2 + \frac{1649\gamma_1\sigma_2}{6} \\
& - \frac{1649\gamma_2\sigma_2}{12} - 3\sigma_1\sigma_2 + \frac{3}{4}\sigma_2^2
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}^{\star}_{32} = {} & \frac{4133}{18} - \frac{6377n^2}{36} - \frac{1777n^4}{36} + \frac{65279\gamma_1}{27} \\
& - \frac{68593n^2\gamma_1}{54} - 16752\gamma_1^2 - \frac{65279\gamma_2}{54} + \frac{68593n^2\gamma_2}{108} \\
& + 16752\gamma_1\gamma_2 - 4188\gamma_2^2 + \frac{757\sigma_1}{9} - \frac{201n^2\sigma_1}{4} \\
& - \frac{16360\gamma_1\sigma_1}{9} + \frac{8180\gamma_2\sigma_1}{9} - 55\sigma_1^2\frac{757\sigma_2}{12} + \frac{201n^2\sigma_2}{8} \\
& + \frac{8180\gamma_1\sigma_2}{9} - \frac{4090\gamma_2\sigma_2}{9} + 55\sigma_1\sigma_2 - \frac{55\sigma_2^2}{4}
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}^{\star}_{42} = {} & \frac{2711}{6} - \frac{1019n^2}{3} - \frac{4229n^4}{36} - \frac{166043\gamma_1}{162} \\
& + \frac{63197n^2\gamma_1}{162} - \frac{55721\gamma_1^2}{3} + \frac{166043\gamma_2}{324} - \frac{63197n^2\gamma_2}{324} \\
& + \frac{55721\gamma_1\gamma_2}{3} - \frac{55721\gamma_2^2}{12} - 44\sigma_1 - 3n^2\sigma_1 \\
& - \frac{50278\gamma_1\sigma_1}{27} + \frac{25139\gamma_2\sigma_1}{27} - 15\sigma_1^2 + 22\sigma_2 + \frac{3n^2\sigma_2}{2} \\
& + \frac{25139\gamma_1\sigma_2}{27} - \frac{25139\gamma_2\sigma_2}{54} + 15\sigma_1\sigma_2 - \frac{15}{4}\sigma_2^2
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}^{\star}_{52} = {} & \frac{15115}{27} - \frac{10601n^2}{27} - \frac{23497n^4}{108} - \frac{1449466\gamma_1}{243} \\
& + \frac{453517n^2\gamma_1}{243} - \frac{27212\gamma_1^2}{9} + \frac{724733\gamma_2}{243} \\
& - \frac{453517n^2\gamma_2}{486} + \frac{27212\gamma_1\gamma_2}{9} - \frac{6803\gamma_2^2}{9} - \frac{1951\sigma_1}{6} \\
& + \frac{349n^2\sigma_1}{4} + \frac{15743\gamma_1\sigma_1}{81} - \frac{15743\gamma_2\sigma_1}{162} - 12\sigma_1^2 \\
& + \frac{1951\sigma_2}{12} - \frac{349n^2\sigma_2}{8} - \frac{15743\gamma_1\sigma_2}{162} + \frac{15743\gamma_2\sigma_2}{324} \\
& + 12\sigma_1\sigma_2 - 3\sigma_2^2
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}^{\star}_{62} = {} & \frac{177809}{486} - \frac{122333n^2}{486} - \frac{28960n^4}{81} - \frac{10534933\gamma_1}{1458} \\
& + \frac{538370n^2\gamma_1}{729} + \frac{3877025\gamma_1^2}{243} + \frac{10534933\gamma_2}{2916} \\
& - \frac{269185n^2\gamma_2}{729} - \frac{3877025\gamma_1\gamma_2}{243} + \frac{3877025\gamma_2^2}{972} \\
& - \frac{62437\sigma_1}{162} + \frac{281n^2\sigma_1}{3} + \frac{563392\gamma_1\sigma_1}{243} \\
& - \frac{281696\gamma_2\sigma_1}{243} + 84\sigma_1^2 + \frac{62437\sigma_2}{324} - \frac{281n^2\sigma_2}{6} \\
& - \frac{281696\gamma_1\sigma_2}{243} + \frac{140848\gamma_2\sigma_2}{234} - 84\sigma_1\sigma_2 + 21\sigma_2^2
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}^{\star}_{72} = {} & \frac{8141}{162} - \frac{80365n^2}{486} - \frac{21469n^4}{36} - \frac{6513922\gamma_1}{2187} \\
& - \frac{22227281n^2\gamma_1}{8748} + \frac{4233653\gamma_1^2}{243} + \frac{3256961\gamma_2}{2187} \\
& + \frac{22227281n^2\gamma_2}{17496} - \frac{4233653\gamma_1\gamma_2}{243} + \frac{4233653\gamma_2^2}{972} \\
& - \frac{19063\sigma_1}{162} - \frac{57n^2\sigma_1}{2} + \frac{1420652\gamma_1\sigma_1}{729} \\
& - \frac{710326\gamma_2\sigma_1}{729} + 45\sigma_1^2 + \frac{19063\sigma_2}{324} \\
& + \frac{57n^2\sigma_2}{4} - \frac{710326\gamma_1\sigma_2}{729} + \frac{355163\gamma_2\sigma_2}{729} - 45\sigma_1\sigma_2 \\
& + \frac{45\sigma_2^2}{4}
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}^{\star}_{82} = {} & \frac{100105}{1458} - \frac{1357369n^2}{2916} - \frac{27773n^4}{27} + \frac{13081537\gamma_1}{13122} \\
& - \frac{29852018n^2\gamma_1}{6561} + \frac{3497408\gamma_1^2}{729} - \frac{13081537\gamma_2}{36244} \\
& + \frac{14926009n^2\gamma_2}{6561} - \frac{3497408\gamma_1\gamma_2}{729} + \frac{874352\gamma_2^2}{729} \\
& + \frac{16265\sigma_1}{162} - \frac{4037n^2\sigma_1}{36} - \frac{11987038\gamma_1\sigma_1}{2187}
\end{aligned}$$

$$+\frac{5993519\gamma_2\sigma_1}{2187}+6\sigma_1^2-\frac{16265\sigma_2}{324}+\frac{4037n^2\sigma_2}{72}+\frac{5993519\gamma_1\sigma_2}{2187}-\frac{5993519\gamma_2\sigma_2}{4374}-6\sigma_1\sigma_2+\frac{3\sigma_2^2}{2}$$

$$\mathcal{N}_{92}^{*}=\frac{969374}{2187}-\frac{730099n^2}{729}-\frac{1680113n^4}{972}-\frac{429534110\gamma_1}{19683}-\frac{1932138199n^2\gamma_1}{73732}-\frac{269045573\gamma_1^2}{6561}+\frac{214767055\gamma_2}{19683}+\frac{1932138199n^2\gamma_2}{157464}+\frac{269045573\gamma_1\gamma_2}{6561}-\frac{269045573\gamma_2^2}{26244}-\frac{2665\sigma_1}{1458}-\frac{4963n^2\sigma_1}{81}-\frac{282835739\gamma_1\sigma_1}{6561}+\frac{282835739\gamma_2\sigma_1}{13122}+\frac{2665\sigma_2}{2916}+\frac{4963n^2\sigma_2}{162}+\frac{282835739\gamma_1\sigma_2}{13122}-\frac{282835739\gamma_2\sigma_2}{26244}$$

Appendix (A-L$_3$)

$$\mathcal{N}_{03}^{*}=2+\frac{1}{8}\left(-2+3n^2+12\sigma_1-6\sigma_2\right)\left(-2+2n^2-6\sigma_1+3\sigma_2\right)$$

$$\mathcal{N}_{33}^{*}=\frac{5}{4}-\frac{1}{64}\left(-84+560n^2-18\gamma_1+9\gamma_2+2256\sigma_1-1128\sigma_2\right)\left(-1+n^2-3\sigma_1+\frac{3\sigma_2}{2}\right)-\frac{1}{128}\left(2-3n^2-12\sigma_1+6\sigma_2\right)\left(-40+40n^2-18\gamma_1+9\gamma_2-384\sigma_1+192\sigma_2\right)$$

$$\mathcal{N}_{63}^{*}=\frac{7}{1536}\left(4032+10000n^2-810\gamma_1+405\gamma_2+34680\sigma_1-17340\sigma_2\right)\left(-1+n^2-3\sigma_1+\frac{3\sigma_2}{2}\right)+\frac{63}{768}(6\gamma_1-3\gamma_2+88\sigma_1 44\sigma_2)\left(2-3n^2-12\sigma_1+6\sigma_2\right)-\frac{1}{2048}\left(-84+560n^2-18\gamma_1+9\gamma_2+2256\sigma_1-1128\sigma_2\right)\times\left(-40+40n^2-18\gamma_1+9\gamma_2-384\sigma_1+192\sigma_2\right)$$

$$\mathcal{N}_{93}^{*}=-\frac{7}{49152}\left(-593128+20824n^2+595488n^4+174420\gamma_1-17100n^2\gamma_1-10692\gamma_1^2-87210\gamma_2+8550n^2\gamma_2+10692\gamma_1\gamma_2-2673\gamma_2^2-1543128\sigma_1+367008n^2\sigma_1+367308\gamma_1\sigma_1-183654\gamma_2\sigma_1+1230336\sigma_1^2+771564\sigma_2-183504n^2\sigma_2-183654\gamma_1\sigma_2+91827\gamma_2\sigma_2-1230336\sigma_1\sigma_2+307584\sigma_2^2\right)$$