DE GRUYTER Open Phys. 2018; 16:441–447

Research Article Open Access

Zaffar Iqbal, Abdul Rauf Nizami, Mobeen Munir, Amlish Rabia, and Shin Min Kang*

On right-angled spherical Artin monoid of type D_n

https://doi.org/10.1515/phys-2018-0061
Received March 9, 2018; accepted June 11, 2018

Abstract: Recently Berceanu and Iqbal proved that the growth rate of all the spherical Artin monoids is bounded above by 4. In this paper we compute the Hilbert series of the right-angled spherical Artin monoid $M(D_n^\infty)$ and graphically prove that growth rate is bounded by 4. We also discuss its recurrence relations and other main properties.

Keywords: Canonical words, Hilbert series, growth rate

PACS: 02.10.Ox, 02.40.Re, 02.20.Bb, 02.30.Lt.

1 Introduction

Coxeter groups are named after a British born Canadian geometer Harold Scott MacDonald Coxeter (1907 – 2003). Coxeter groups were introduced by Coxeter in 1934 as abstract form of reflection groups and defined as the groups with the generators $a_i, i \in I$ and relations $a_i^2 = 1$ and $a_ia_ja_i = a_ja_ia_j, i, j \in I$. Coxeter classified these groups into two categories, finite and infinite, in 1935. If the relation $a_i^2 = 1$ from the presentation of Coxeter group is removed then we get the presentation of Artin groups. Thus, we can say that Coxeter groups are quotient groups of the Artin groups. A Coxeter group is called finite Coxeter group if it has the presentation as a discrete, properly acting group of reflections of the sphere [1]. Therefore these groups are also called spherical Coxeter groups. In the list of spherical Coxeter groups, A_n is the first. The Artin group associated to A_n is the braid group. A Coxeter group is affine if it is infinite. It is generated by reflections in affine spaces and has the presentation as discrete, properly acting, affine reflection group [1]. Cardinality is invariant of graded algebraic structures. Hilbert series deals with the cardinality of elements in the graded algebraic structures.

In [2] Saito computed the growth series of the Artin monoids. In [3] Parry computed the growth series of the Coxeter groups. In [4] Mairesse and Mathéus gave the growth series of Artin groups of dihedral type. In [5] Iqbal gave a linear system for the reducible and irreducible words of the braid monoid MB_n, which leads to compute the Hilbert series of MB_n and in [6] computed the Hilbert series of the braid monoid MB_4 in band generators. In [7] Berceanu and Iqbal proved that the growth rate of all the spherical Artin monoids is less than 4. In the present paper we study the affine-type Coxeter group D_n^∞, and find the Hilbert series (or spherical growth series) of the associated right-angled affine Artin monoid $M(D_n^\infty)$. We also discuss its recurrence relations and the growth rate. In [8–13] authors presented some new ways to compute different solutions methods.

Let S be a set. A *Coxeter matrix* over S is a square matrix $M = (m_{st})_{s,t\in S}$ such that
- $m_{ss} = 1$ for all $s \in S$;
- $m_{st} = m_{ts} \in \{2, 3, 4, \dots, \infty\}$ for all $s, t \in S,\ s \neq t$.

A *Coxeter graph* Γ is a labeled graph defined by the following data:
- S is a set of vertices of Γ.
- Two vertices $s, t \in S, s \neq t$ are joined by an edge if $m_{st} \geq 3$. This edge is labeled by m_{st} if $m_{st} \geq 4$.

Remark 1.1. A Coxeter matrix $M=(m_{st})_{s,t\in S}$ is usually represented by its Coxeter graph $\Gamma(M)$.

Definition 1.2. Let $M = (m_{st})_{s,t\in S}$ be the Coxeter matrix and $\Gamma(M)$ its Coxeter graph. Then the group

$$W = \left\langle s \in S \mid s^2 = 1\,, (st)^{m_{st}} = 1, s, t \in S,\ s \neq t \right\rangle$$

is called the Coxeter group (of type $\Gamma(M)$).

In a simple way, we can write
$W = \left\langle s \in S \mid s^2 = 1,\ \underbrace{sts\cdots}_{m_{st}\text{ factors}} = \underbrace{tst\cdots}_{m_{st}\text{factors}}, s, t \in S,\ s \neq t \right\rangle$. We call Γ to be of *spherical type* if W is finite.

Zaffar Iqbal: Department of Mathematics, University of Gujrat, Gujrat-Pakistan, E-mail: zaffar.iqbal@uog.edu.pk
Abdul Rauf Nizami: Faculty of Information Technology, University of Central Punjab, Lahore-Pakistan, E-mail: arnizami@ucp.edu.pk
Mobeen Munir: Division of Science and Technology, University of Education, Lahore-Pakistan, E-mail: mmunir@ue.edu.pk
Amlish Rabia: Department of Mathematics, University of Gujrat, Gujrat-Pakistan, E-mail: amlishrabia.ar@gmail.com
***Corresponding Author: Shin Min Kang:** Department of Mathematics and RINS, Gyeongsang National University, Jinju-Korea and Center for General Education, China Medical University, Taichung-Taiwan, E-mail: smkang@gnu.ac.kr

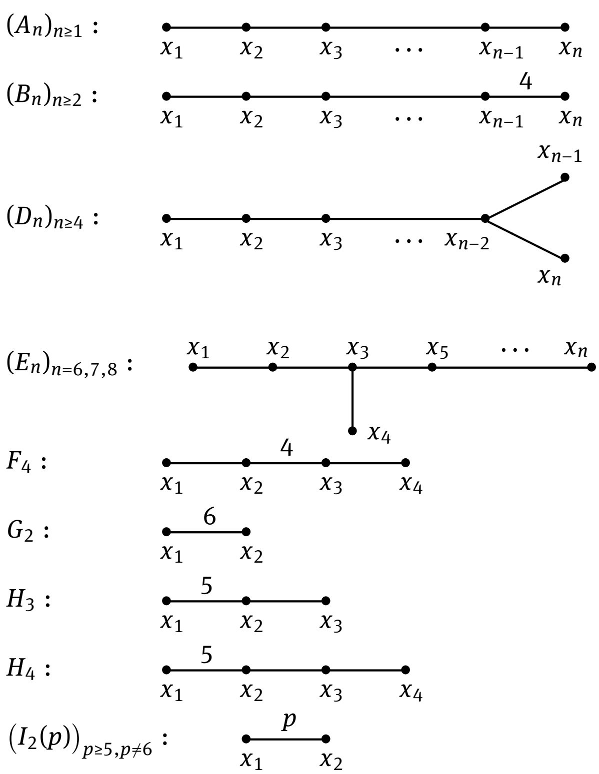


Figure 1: Spherical Coxeter graphs

An *Artin spherical monoid (or group)* is given by a finite union of connected Coxeter graphs from the well known classical list of Coxeter diagrams (see [1, 14]).

By convention m_{ij} is the label of the edge between x_i and x_j $(i \neq j)$; if there is no label then $m_{ij} = 3$. If there is no edge between x_i and x_j, then $m_{ij} = 2$.

To a given Coxeter graph X_n we associate the monoid $M(X_n)$ with generators (corresponds to vertices) $x_1, x_2, \dots, x_n$ and relations (corresponds to labels m_{ij} of the graphs) $\underbrace{x_i x_j x_i x_j \cdots}_{m_{ij} \text{ factors}} = \underbrace{x_j x_i x_j x_i \cdots}_{m_{ij} \text{ factors}}$, where $1 \leq j < i \leq n$.

The corresponding group $G(X_n)$ associated to X_n is defined by the same presentation. From now onward we will use X_n for $G(X_n)$ for simplicity.

Definition 1.3. If W is a Coxeter group with Coxeter matrix $M = (m_{st})_{s,t \in S}$, then the Artin group associated to W is defined by

$$\mathcal{A} = \left\langle s \in S \,\middle|\, \underbrace{sts \cdots}_{m_{st} \text{ factors}} = \underbrace{tst \cdots}_{m_{st} \text{ factors}} \right\rangle.$$

If W is finite then $\mathcal{A}$ is called a spherical Artin group.

Definition 1.4. In the spherical type Coxeter graphs, if all the labels $m_{st} \geq 3$ are replaced by ∞ then the associated groups (monoids) are called right-angled Artin groups (monoids).

Definition 1.5. [15] Let G be a finitely generated group and S be a finite set of generators of G. The *word lenth* $l_S(g)$ of an element $g \in G$ is the smallest integer n for which there exists $s_1, \dots, s_n \in S \cup S^{-1}$ such that $g = s_1 \cdots s_n$.

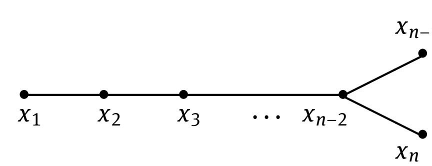
Definition 1.6. [15] Let G be a finitely generated group and S be a finite set of generators of G. The *growth function* of the pair (G, S) associates to an integer $k \geq 0$ the number a_k of elements $g \in G$ such that $l_S(g) = k$. The corresponding *spherical growth series (also known as the generating function)* or the *Hilbert series* is given by $\mathcal{H}_G(t) = \sum_{k=0}^{\infty} a_k t^k$.

The affine (or infinite) Coxeter groups form another important series of Coxeter groups. These well-known affine Coxeter groups are $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{G}_2$ and $\widetilde{I}_1$ (for details, see [14]). In [7] we proved that the universal upper bound for all the spherical Artin monoids is less than 4.

In this work we discuss right-angled affine Artin monoids, specifically, we study the affine monoid $M(D_n^\infty)$ and compute its Hilbert series. We show that the growth of $M(D_n^\infty)$ series is bounded above by 4. Along with Hilbert series we also compute the recurrence relations related to $M(D_n^\infty)$. Based on some computations with mathematical softwares Derive 6 and Mathematica, we give a conjecture of the growth rate of $M(D_n^\infty)$.

The monoid $M(D_n^\infty)$ is represented by its Coxeter graph as follows:

$(D_n)_{n\geq 4}$:



Here $x_1, x_2, \dots, x_n$ are vertices of the graph and all the labels are ∞. If all the labels in a Coxeter diagram are replaced by ∞, then there is no relation between the adjacent edges. Hence we have the associated right-angled Artin groups and the associated right-angled Artin monoids denoted by $G(D_n^\infty)$ and $M(D_n^\infty)$, respectively.

Definition 1.7. Let X be a nonempty set and $X^\star$ be the free monoid on X. Let w_1 and $w_2 \in X^\star$, where $w_1 = x_1 x_2 \cdots x_r$, $w_2 = y_1 y_2 \cdots y_r$ with $x_i, y_i \in X$. Then $w_1 < w_2$ *length-lexicographically* (or *quasi-lexicographically*) if there is a $k \leq r$ such that $x_k < y_k$ and $x_i = y_i$ for all $i < k$.

Let $\alpha = \beta$ be a relation in a monoid M and $\alpha_1 = uw$ and $\alpha_2 = wv$ be the words in M.

Then the word of the form $\alpha_1 \times_w \alpha_2 = uwv$ is said to be an *ambiguity*. If $\alpha_1 v = u\alpha_2$ (in the length-lexicographic order) then we say that the ambiguity uwv is solvable. A presentation of M is *complete* if and only if all the ambiguities are solvable. Corresponding to the relation $\alpha = \beta$, the changes $\gamma\alpha\delta \longrightarrow \gamma\beta\delta$ give a rewriting system. A complete presentation is equivalent to a confluent rewriting system. In a complete presentation of a monoid a word containing α will be called *reducible word* and a word that does not contain α will be called an *irreducible word* or *canonical word*. For example $x_2x_1x_2 = x_1x_2x_1$ is a basic relation in the braid monoid MB_3. A word $v = x_2^2x_1x_2$ contains $\alpha = x_2x_1x_2$. Hence v is a reducible word. Then $x_2^2x_1x_2 = x_2x_1x_2x_1 = x_1x_2x_1^2$ is the canonical form of v [16].

In a presentation of a monoid we fix a total order $x_1 < x_2 < \cdots < x_n$ on the generators. Hence clearly we have

Lemma 1.8. *The presentation of the monoid $M(D_n^\infty)$ has generators $x_1, x_2, \ldots, x_n$ and has relations $x_ix_j = x_jx_i$, $3 \le j+2 \le i \le n-1$, and $x_nx_k = x_kx_n$, $k \ne n-2$.*

2 Results and discussion

Here we present our main results. First we compute recurrence relations for monoid $M(D_n^\infty)$. Then we formulate our results about Hilbert series of monoid $M(D_n^\infty)$. At the end we formulate a conjecture about the growth rate of these monoids, graphically proving that it is bounded by 4.

2.1 Recurrence relations of the monoid $M(D_n^\infty)$

In this section we discuss few interesting results relating the recurrence relations of $M(D_n^\infty)$. First we talk about the solution of the system of linear recurrences.

Consider a system [17] of linear recurrences

$$\begin{aligned} u_1(t+1) &= a_{11}(t)u_1(t) + \cdots + a_{1n}(t)u_n(t) + f_1(t) \\ u_2(t+1) &= a_{11}(t)u_1(t) + \cdots + a_{1n}(t)u_n(t) + f_2(t) \\ &\vdots \\ u_n(t+1) &= a_{11}(t)u_1(t) + \cdots + a_{1n}(t)u_n(t) + f_n(t). \end{aligned}$$

This system can be written as $u(t+1) = A(t)u(t) + f(t)$, where

$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}, \; A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix} \text{ and } f(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

The solution (which we need in our work) of the homogenous equation $u(t+1) = A(t)u(t)$ is given by $u(t) = c_1\lambda_1^t u^1 + \cdots + c_k\lambda_k^t u^k$, where $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of $A(t)$ and u^i is an eigenvector corresponding to λ_i. The largest eigenvalue is the growth rate of the sequence $(u_i(t))$ (by the definition of growth rate).

Let c_k = #{canonical words of length k} and $c_{k;i}$ = #{canonical words starting with x_i of length k}. Then by Lemma 1.8 we have the following

Lemma 2.1. *The monoid $M(D_n^\infty)$ satisfies the following recurrence relations*

a) $c_0 = 1$, $c_{1;i} = 1$, $c_k = \sum_{i=1}^{n} c_{k;i}$ $(k \ge 1)$.

b) $c_{k;i}$ $(k \ge 2)$ *is given by following recurrence relations*

$$c_{k;i} = \begin{cases} \sum_{i=1}^{n} c_{k-1;i}, & i = 1, \\ \sum_{i=j-1}^{n} c_{k-1;i} \; (j = 2, 3, \ldots, n-1), \\ c_{k-1;i-2} + c_{k-1;i}, & i = n. \end{cases}$$

From the above equations it is clear that $c_{k;1} = c_{k;2}$.

Let M_n be the matrix of order $n \times n$ of the system of linear recurrences given in Lemma 2.1. Then

$$M_n = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 1 \end{bmatrix}$$

and its characteristic polynomial $D_n(\lambda)$ is

$$\begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & \lambda-1 & \cdots & -1 & -1 & -1 & -1 \\ 0 & -1 & \cdots & -1 & -1 & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda-1 & -1 & -1 \\ 0 & 0 & \cdots & 0 & -1 & \lambda-1 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 0 & \lambda-1 \end{vmatrix}.$$

Lemma 2.2. *[7] The polynomials $\mathcal{A}_n(\lambda)$ for the monoid $M(A_n^{\infty})$ satisfy the recurrence*

$$\mathcal{A}_n(\lambda) = \lambda\mathcal{A}_{n-1}(\lambda) - \lambda\mathcal{A}_{n-2}(\lambda) \quad (n \geq 2) \tag{2.1}$$

with $\mathcal{A}_0(\lambda) = 1$ and $\mathcal{A}_1(\lambda) = \lambda - 1$ the initial values.

Theorem 2.3. *The polynomials $\left(D_n(\lambda)\right)_{n\geq 4}$ satisfy the recurrence*

$$D_n(\lambda) = (\lambda - 1)\mathcal{A}_{n-1}(\lambda) - \lambda^2\mathcal{A}_{n-4}(\lambda)\,,\ n \geq 4 \tag{2.2}$$

with $D_0(\lambda) = 1$, $D_1(\lambda) = \lambda - 1$, $D_2(\lambda) = \lambda^2 - 2\lambda$ and $D_3(\lambda) = \lambda(\lambda^2 - 3\lambda + 1)$ as the initial values.

Proof. The characteristic polynomial of M_n is given (as above) by

$$D_n(\lambda) = \begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & \lambda-1 & \cdots & -1 & -1 & -1 & -1 \\ 0 & -1 & \cdots & -1 & -1 & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda-1 & -1 & -1 \\ 0 & 0 & \cdots & 0 & -1 & \lambda-1 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 0 & \lambda-1 \end{vmatrix}.$$

Now by decomposing $D_n(\lambda)$ by last row as sum of two determinants, we have

$$D_n(\lambda) = \begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & \lambda-1 & \cdots & -1 & -1 & -1 & -1 \\ 0 & -1 & \cdots & -1 & -1 & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda-1 & -1 & -1 \\ 0 & 0 & \cdots & 0 & -1 & \lambda-1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \lambda-1 \end{vmatrix}$$

$$+ \begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & \lambda-1 & \cdots & -1 & -1 & -1 & -1 \\ 0 & -1 & \cdots & -1 & -1 & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda-1 & -1 & -1 \\ 0 & 0 & \cdots & 0 & -1 & \lambda-1 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 0 & 0 \end{vmatrix}.$$

Expanding both the determinants by last rows, respectively, we get

$$D_n(\lambda) = (\lambda - 1)\mathcal{A}_{n-1}(\lambda) - V(\lambda),$$

where

$$V(\lambda) = \begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & \lambda-1 & \cdots & -1 & -1 & -1 & -1 \\ 0 & -1 & \cdots & -1 & -1 & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda-1 & -1 & -1 \\ 0 & 0 & \cdots & 0 & -1 & -1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & \lambda-1 & -1 \end{vmatrix}.$$

By subtracting last column from second last column we get

$$V(\lambda) = \begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 & -1 & 0 \\ -1 & \lambda-1 & \cdots & -1 & -1 & -1 & 0 \\ 0 & -1 & \cdots & -1 & -1 & -1 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & -1 & \lambda-1 & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & -1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \lambda-1 & -\lambda \end{vmatrix}$$

and expanding by last row we get

$$= -\lambda \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & \lambda-1 & -1 & \cdots & -1 & -1 & -1 \\ 0 & -1 & \lambda-1 & \cdots & -1 & -1 & -1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & \lambda-1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & -1 \end{vmatrix}.$$

Subtracting last column from second last column we have

$$V(\lambda) = -\lambda \begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & 0 & -1 \\ -1 & \lambda-1 & \cdots & -1 & 0 & -1 \\ 0 & -1 & \cdots & -1 & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & -1 & \lambda & -1 \\ 0 & 0 & \cdots & 0 & 0 & -1 \end{vmatrix}.$$

Simplifying last determinant, we finally have

$$V(\lambda) = \lambda^2 \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda-1 & -1 & \cdots & -1 & -1 \\ 0 & -1 & \lambda-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & \lambda-1 \end{vmatrix}.$$

Therefore we have the result

$$D_n(\lambda) = (\lambda - 1)\mathcal{A}_{n-1}(\lambda) - \lambda^2\mathcal{A}_{n-4}(\lambda).$$

□

2.2 Hilbert series of $M(D_n^\infty)$

Now we compute the Hilbert series of $M(D_n^\infty)$. For this we need to fix some notations first. Let $H_M^{(n)}(t) = \sum_{k\geq 0} c_k t^k$ denote the Hilbert series of $M(D_n^\infty)$, where $c_k = \#\{$canonical words of length $k\}$. Let $H_{M;i}^{(n)}(t) = \sum_{k\geq 0} c_{k;i}^{(n)} t^k$ denote the Hilbert series of $M(D_n^\infty)$ of words starting with x_i, where $c_{k;i} = \#\{$canonical words starting with x_i of length $k\}$.

Theorem 2.4. *The Hilbert series of $M(D_n^\infty)$ is represented by the following system of equations:*

(1) $H_M^{(n)}(t) = 1 + \sum_{i=1}^{n} H_{M;i}^{(n)}(t)$

(2) $H_{M;1}^{(n)}(t) = H_{M;2}^{(n)}(t)$

(3) $H_{M;j}^{(n)}(t) = t + t \sum_{i=j-1}^{n} H_{M;i}^{(n)}(t) \quad (j = 2, 3, \ldots, n-1)$

(4) $H_{M;n}^{(n)}(t) = t + tH_{M;n-2}^{(n)}(t) + tH_{M;n}^{(n)}(t)$

Proof. (1) From Lemma 2.1 we have $c_k = \sum_{i=1}^{n} c_{k;i}, (k \geq 1)$. Therefore $H_M^{(n)}(t) = \sum_{k\geq 0} c_k t^k = c_0 + \sum_{k\geq 1} c_k t^k = 1 + \sum_{k\geq 1}\sum_{i=1}^{n} c_{k;i} t^k = 1 + \sum_{i=1}^{n}\sum_{k\geq 1} c_{k;i} t^k = 1 + \sum_{i=1}^{n} H_{M;i}^{(n)}(t)$.
(2) is obvious as $c_{k;1} = c_{k;2}$.
(3) From Lemma 2.1 we have, $c_{k;j} = \sum_{i=j-1}^{n} c_{k-1;i}$ $(j = 2, 3, \ldots, n-1)$. Hence $H_{M;j}^{(n)}(t) = \sum_{k\geq 1} c_{k;j}(t) t^k = c_{1;j}(t)t + \sum_{k\geq 2} c_{k;j}(t) t^k = t + \sum_{k\geq 2}\sum_{i=j-1}^{n} c_{k-1;i}(t) t^k = t + t\sum_{i=j-1}^{n}\sum_{k\geq 2} c_{k-1;i}(t) t^{k-1} = t + t\sum_{i=j-1}^{n} H_{M;i}^{(n)}(t)$.

Similarly we can easily prove (4). □

The system of equations in Theorem 2.4 can be written in matrix form as $W_n X = B$, where

$$W_n = \begin{bmatrix} 1-t & -t & \cdots & -t & -t & -t & -t \\ -t & 1-t & \cdots & -t & -t & -t & -t \\ 0 & -t & \cdots & -t & -t & -t & -t \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -t & 1-t & -t & -t \\ 0 & 0 & \cdots & 0 & -t & 1-t & -t \\ 0 & 0 & \cdots & 0 & -t & 0 & 1-t \end{bmatrix}, X = \begin{bmatrix} H_{M;1}^{(n)}(t) \\ H_{M;2}^{(n)}(t) \\ H_{M;3}^{(n)}(t) \\ \vdots \\ H_{M;n}^{(n)}(t) \end{bmatrix}, \text{ and } B = \begin{bmatrix} t \\ t \\ t \\ \vdots \\ t \end{bmatrix}.$$

Lemma 2.5. *In the monoid $M(D_n^\infty)$*

$$\det(W_n) = t^n D_n\left(\frac{1}{t}\right).$$

Proof. The result follows immediately by factoring out t from each row of $\det(W_n)$. □

Lemma 2.6. *In $M(D_n^\infty)$*

$$H_{H;m}^{(n)}(t) = \begin{cases} \dfrac{t^{m-1}\mathcal{A}_{m-2}\left(\frac{1}{t}\right)}{t^n D_n\left(\frac{1}{t}\right)}, & 2 \leq m \leq n-1 \\ \dfrac{t^{m-2}(1-t)\mathcal{A}_{m-3}\left(\frac{1}{t}\right)}{t^n D_n\left(\frac{1}{t}\right)}, & m = n \end{cases}$$

Proof. The system explained in Theorem 2.4 of n equations in n variables $H_{M;i}^{(n)}(t)$, $1 \leq i \leq n$ is already written in the form $W_n X = B$, where $X = [H_{M;1}^{(n)}(t), \ldots, H_{M;n}^{(n)}(t)]^t$, $\det(W_n) = t^n D_n(\frac{1}{t})$ and $B = [t, \ldots, t]^t$. Here we have two cases:

Case I: $2 \leq m \leq n-1$. By using Cramer's rule we have

$$H_{M;m}^{(n)}(t) = \frac{T_m}{\det(W_n)},$$

where T_m is a determinant obtained by replacing mth column of W_n by column of B. That is, $[H_{M;m}^{(n)}(t)$ is now becomes

$$\frac{1}{t^n D_n(\frac{1}{t})}\begin{vmatrix} 1-t & -t & \cdots & -t & t & -t & -t \\ -t & 1-t & \cdots & -t & t & -t & -t \\ 0 & -t & \cdots & -t & t & -t & -t \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -t & t & -t & -t \\ 0 & 0 & \cdots & 0 & t & 1-t & -t \\ 0 & 0 & \cdots & 0 & t & 0 & 1-t \end{vmatrix}.$$

Let C_i denote the ith column of T_m. Adding C_m in $C_{m+1}, C_{m+2}, \ldots, C_n$ of T_m and simplifying we have a determinant of order m. Hence

$$H_{M;m}^{(n)}(t) = \frac{1}{t^n D_n(\frac{1}{t})}\begin{vmatrix} 1-t & -t & \cdots & -t & t & 0 & 0 \\ -t & 1-t & \cdots & -t & t & 0 & 0 \\ 0 & -t & \cdots & -t & t & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -t & t & 0 & 0 \\ 0 & 0 & \cdots & 0 & t & 1 & 0 \\ 0 & 0 & \cdots & 0 & t & 0 & 1 \end{vmatrix}.$$

Now simplifying it we finally have

$$H_{M;m}^{(n)}(t) = \frac{t^{m-1}\mathcal{A}_{m-2}(\frac{1}{t})}{t^n D_n(\frac{1}{t})}.$$

Case II: $m = n$.
Again using the Cramer's rule, $H^{(n)}_{M;n}(t)$ becomes

$$\frac{1}{t^nD_n(\frac{1}{t})}\begin{vmatrix} 1-t & -t & \cdots & -t & -t & -t & t \\ -t & 1-t & \cdots & -t & -t & -t & t \\ 0 & -t & \cdots & -t & -t & -t & t \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -t & 1-t & -t & t \\ 0 & 0 & \cdots & 0 & -t & 1-t & t \\ 0 & 0 & \cdots & 0 & -t & 0 & t \end{vmatrix}.$$

Let C_i denote the ith column of D_n. Adding C_n in C_{n-1} and C_{n-2}, we get

$$H^{(n)}_{M;n}(t) = \frac{1}{t^nD_n(\frac{1}{t})}\begin{vmatrix} 1-t & -t & \cdots & -t & 0 & 0 & t \\ -t & 1-t & \cdots & -t & 0 & 0 & t \\ 0 & -t & \cdots & -t & 0 & 0 & t \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -t & 1 & 0 & t \\ 0 & 0 & \cdots & 0 & 0 & 1 & t \\ 0 & 0 & \cdots & 0 & 0 & 0 & t \end{vmatrix}.$$

Therefore

$$H^{(n)}_{M;n}(t) = \frac{t^{m-2}(1-t)\mathcal{A}_{m-3}\left(\frac{1}{t}\right)}{t^nD_n\left(\frac{1}{t}\right)}\quad, m = n$$

□

Now we have our main result.

Theorem 2.7. *The Hilbert series of the monoid $M(D_n^\infty)$ is*

$$H^{(n)}_M(t) = \frac{1}{t^nD_n(\frac{1}{t})}\ .$$

Proof. From Theorem 2.4 we have

$$\begin{aligned}
H^{(n)}_M(t) &= 1 + \sum_{i=1}^{n} H^{(n)}_{M;i}(t) \\
&= 1 + H^{(n)}_{M;1}(t) + H^{(n)}_{M;2}(t) + \cdots + H^{[n]}_{M;n-1}(t) + H^{(n)}_{M;n}(t) \\
&= \frac{1}{t^nD_n(\frac{1}{t})}\Bigg(t^nD_n\left(\frac{1}{t}\right) + 2t + t^2\mathcal{A}_1\left(\frac{1}{t}\right) + \cdots \\
&\quad + t^{n-3}\mathcal{A}_{n-4}\left(\frac{1}{t}\right) + 2t^{n-2}\mathcal{A}_{n-3}\left(\frac{1}{t}\right) + t^{n-1}\mathcal{A}_{n-3}\left(\frac{1}{t}\right)\Bigg) \\
&= \frac{1}{t^nD_n(\frac{1}{t})}\Bigg(t^{n-1}\mathcal{A}_{n-1}\left(\frac{1}{t}\right) - t^n\mathcal{A}_{n-1}\left(\frac{1}{t}\right) \\
&\quad - t^{n-2}\mathcal{A}_{n-4}\left(\frac{1}{t}\right) + 2t + t^2\mathcal{A}_1\left(\frac{1}{t}\right) + \cdots \\
&\quad + t^{n-3}\mathcal{A}_{n-4}\left(\frac{1}{t}\right) + 2t^{n-2}\mathcal{A}_{n-3}\left(\frac{1}{t}\right) + t^{n-1}\mathcal{A}_{n-3}\left(\frac{1}{t}\right)\Bigg) \\
&= \frac{1}{t^nD_n(\frac{1}{t})}\Bigg(t^{n-2}\mathcal{A}_{n-2}\left(\frac{1}{t}\right) - t^{n-2}\mathcal{A}_{n-3}\left(\frac{1}{t}\right) \\
&\quad - t^{n-1}\mathcal{A}_{n-2}\left(\frac{1}{t}\right) + t^{n-1}\mathcal{A}_{n-3}\left(\frac{1}{t}\right) - t^{n-2}\mathcal{A}_{n-4}\left(\frac{1}{t}\right) \\
&\quad + 2t + t^2\mathcal{A}_1\left(\frac{1}{t}\right) + t^3\mathcal{A}_2\left(\frac{1}{t}\right) + \cdots + t^{n-3}\mathcal{A}_{n-4}\left(\frac{1}{t}\right) \\
&\quad + 2t^{n-2}\mathcal{A}_{n-3}\left(\frac{1}{t}\right) - t^{n-1}\mathcal{A}_{n-3}\left(\frac{1}{t}\right)\Bigg) \\
&= \frac{1}{t^nD_n(\frac{1}{t})}\Bigg(2t + t^2\mathcal{A}_1\left(\frac{1}{t}\right) + t^3\mathcal{A}_2\left(\frac{1}{t}\right) + \cdots \\
&\quad + t^{n-4}\mathcal{A}_{n-5}\left(\frac{1}{t}\right) + t^{n-3}\mathcal{A}_{n-3}\left(\frac{1}{t}\right)\Bigg) \\
&= \frac{1}{t^n\mathcal{L}_n(\frac{1}{t})}\left(2t + t^2\mathcal{A}_2\left(\frac{1}{t}\right)\right) \\
&= \frac{1}{t^nD_n(\frac{1}{t})}\left(2t + t\mathcal{A}_1\left(\frac{1}{t}\right) - t\mathcal{A}_0\left(\frac{1}{t}\right)\right) \\
&= \frac{1}{t^nD_n(\frac{1}{t})}
\end{aligned}$$

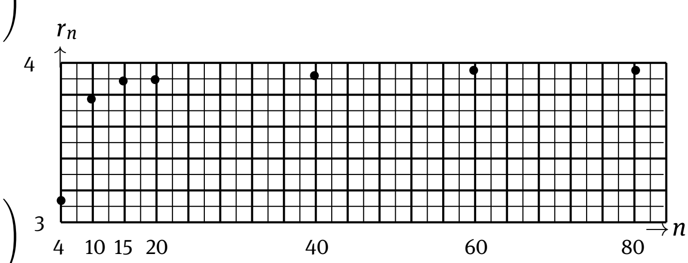
□

2.3 Conjecture on the upper bound of growth rate of $M(D_n^\infty)$

Now we compute the growth rates of $M(D_n^\infty)$ and construct the graph using these growth rates. We see that the growth rate of $M(D_n^\infty)$ is bounded above by 4. Let r_n be the growth rate (or the maximal root of the polynomial $D_n(\lambda)$).

We compute few initial growth rates (using Mathematica and Derive 6) for $M(D_n^\infty)$. We have the following few initial values of r_n: $r_3 = 2.6180$, $r_4 = 3.1478$, $r_5 = 3.4142$, $r_6 = 3.5618$, $r_7 = 3.6657$, $r_8 = 3.7320$, $r_9 = 3.7793$, $r_{10} = 3.8144$, $r_{11} = 3.8413$, $r_{12} = 3.8624$, $r_{13} = 3.8793$, $r_{14} = 3.8932$, $r_{15} = 3.9046$, $r_{16} = 3.9142$, $r_{17} = 3.9224$, $r_{18} = 3.9294$, $r_{19} = 3.9354$, $r_{20} = 3.9407$.

We also compute $r_{40} = 3.9817$, $r_{60} = 3.9911$, $r_{80} = 3.9947$, $r_{100} = 3.9965$, and $r_{120} = 3.9975$ using Mathematica. We have the following graph representing the growth rate of $M(D_n^\infty)$:



We observe that the growth rate for $M(D_n^\infty)$ approaching 4 as n approaches ∞. Hence at the end we have

Conjecture: The growth rate of $M(D_n^\infty)$ is bounded above by 4.

3 Conclusions

We computed the Hilbert series of the right-angled affine Artin monoid $M(D_n^\infty)$ for the first time. We also formulated new recurrence relations for this monoid. In the end we graphically proved that growth rate is bounded by 4 as it is the case in [7] with all spherical Artin monoids. We also posed it as an open problem for its rigorous proof.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All the authors jointly worked on deriving the results and approved the final manuscript.

Acknowledgement: Authors are extremely thankful for reviewer's comments.

References

[1] Bourbaki N., Chapitres 4-6, In: Bourbaki N., Groupes et algèbres de Lie, Elem. Math., Hermann, 1968.
[2] Saito K., Growth functions for Artin monoids, Proc. Japan Acad., Ser. A, 2009, 85, 84-88.
[3] Parry W., Growth series of Coxeter groups and Salem numbers, J. Algebra, 1993, 154, 406-415.
[4] Mairesse J., Mathéus F., Growth series for Artin groups of dihedral type, Int. J. Algebr. Comput., 2006, 16, 1087-1107.
[5] Iqbal Z., Hilbert series of positive braids, Algebr. Colloq., 2011, 18, 1017-1028.
[6] Iqbal Z., Yousaf S., Hilbert series of braid monoid MB_4 in band generators, Turk. J. Math, 2014, 38, 977-984.
[7] Berceanu B., Iqbal Z., Universal upper bound for the growth of Artin monids, Commun. Algebra, 2015, 43, 1967-1982.
[8] Akgül A., Inc M., Hashemi S., Group preserving scheme and reproducing kernel method for the Poisson-Boltzmann equation for semiconductor devices, Nonlinear Dynam., 2017, 88, 2817-2829.
[9] Hashemi S., Inc M., Karatas E., Akgül A., A Numerical Investigation on Burgers Equation by MOL-GPS Method, Journal of Advanced Physics, 2017, 6, 413-417.
[10] Inc M., Akgül A. The reproducing kernel Hilbert space method for solving Troesch's problem, Journal of the Association of Arab Universities for Basic and Applied Sciences, 2013, 14, 19-27.
[11] Sakar M.S., Akgül A., Baleanu D., On solutions of fractional Riccati differential equations, Adv. Differ. Equ-ny., 2017, 39.
[12] Inc M., Akgül A., Approximate solutions for MHD squeezing fluid flow by a novel method, Bound. Value Probl., 2014, 18
[13] Akgül A., New Reproducing Kernel Functions, Math. Probl. Eng., 2015.
[14] Coxeter H.S.M., Regular complex polytopes, Cambridge Univ. Press, 2nd edition, 1991.
[15] Harpe P.D., Topics in geometric group theory. America: The University of Chicago Press, 2000.
[16] Cohn P.M., Further algebra and applications, Springer-Verlag, London, 2003.
[17] Kelley W.G., Peterson A.C., Difference equations: An Introduction With Applications, Second Edition, Academic Press, New York, 2001.