Research Article

Piotr Żenczykowski*

A Note on Koide's Doubly Special Parametrization of Quark Masses

Abstract: Three charged lepton masses may be expressed in terms of a Z_3-symmetric parametrization relevant for the discussion of Koide's formula. After disregarding the overall scale parameter, the observed pattern of lepton masses can be described extremely well if the remaining two parameters acquire the unexpectedly simple values of 1 and 2/9. We argue that an analogue of this doubly special feature of the parametrization can also be seen in the quark sector provided that the mass of the strange quark is taken to be around 160 MeV, as might be expected in the low-energy regime.

1 Introduction

Despite the general acceptance of the Higgs mechanism, a truly successful resolution of the problem of mass (*i.e.* the actual prediction of the masses of fundamental fermions) still eludes our understanding. Hopefully, the search for such an understanding could be guided by the analysis of the observed pattern of particle masses. The identified regularities might then serve as the analogues of the Balmer formula, thus helping us decipher the new physics.

2 Koide's parametrization

Koide discovered one of the most interesting regularities. Namely, it appears that the three charged lepton (L) masses satisfy an empirical relation [1]:

$$\frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{1 + k_L^2}{3}, \qquad (1)$$

*Corresponding Author: Piotr Żenczykowski: Division of Theoretical Physics, the Henryk Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences, Radzikowskiego 152, 31-342 Kraków, Poland; Email: piotr.zenczykowski@ifj.edu.pl

with k_L being almost exactly 1.

Indeed, if one takes $k_L = 1$ and uses the central values of experimental electron and muon masses (the pole masses $m_e = 0.510998928(11)$ MeV and $m_\mu = 105.65836715(35)$ MeV [2]), the larger of the two solutions of Eq. (1) is

$$m_\tau = 1776.9689 \text{ MeV}, \qquad (2)$$

while the experimental τ mass m_τ is

$$m_\tau = 1776.82 \pm 0.16 \text{ MeV}. \qquad (3)$$

The parameter k_L entering into Eq. (1) may be defined through the following Z_3-symmetric parametrization [3] of three arbitrary masses (m_1, m_2, m_3):

$$\sqrt{m_j} = \sqrt{M_f} \left(1 + \sqrt{2}\, k_f \cos\left(\frac{2\pi j}{3} + \delta_f \right) \right), \qquad (4)$$
$$(j = 1, 2, 3),$$

(with $f = L$) in terms of three parameters: the overall mass scale M_f, the average spread k_f of the three masses (relative to the scale M_f) and the pattern/phase parameter δ_f. As δ_f is free, we assume that $m_1 \leq m_2 \leq m_3$. The above parametrization is particularly suited to Koide's formula, as the latter appears independent of parameter δ_f. Furthermore, from (4) one can derive a counterpart of Eq. (1) that depends on δ_L but is independent of k_L [4]:

$$\frac{\sqrt{3}\,(\sqrt{m_\mu} - \sqrt{m_e})}{2\sqrt{m_\tau} - \sqrt{m_\mu} - \sqrt{m_e}} = \tan \delta_L. \qquad (5)$$

It was observed by Brannen and Rosen [5, 6] that the three charged lepton masses satisfy Eq.(5) with δ_L being almost exactly 2/9. Indeed, if one assumes that δ_L is 2/9, one can use Eq.(5) and the experimental values of electron and muon masses to predict the mass of the tauon:

$$m_\tau = 1776.9664 \text{ MeV}. \qquad (6)$$

This prediction of the tauon mass is as good as that given in Eq.(2) by Koide's original formula. For this reason Koide's parametrization may be rightly called 'doubly special'. While the origin of values $k_L = 1$ and $\delta_L = 2/9$ is unknown, the appearance of such numbers suggests the existence of a fairly simple underlying algebraic framework.

Although the two predictions of Eqs (2) and (6) are incompatible with each other, the degree of their incompatibility is extremely small. Thus, once the overall scale M_L is properly adjusted, the choice of $k_L = 1$ and $\delta_L = 2/9$ gives an almost perfect description of the charged lepton masses. The observed tiny deviations from the ($k_L = 1$, $\delta_L = 2/9$) case could then be attributed to some higher order corrections of the underlying scheme.

3 Modification of parametrization

A peculiar feature of Koide's and Brannen-Rosen's observations is that if instead of the charged lepton pole masses one takes their running masses as evaluated at a higher mass scale μ, the left-hand sides of Eqs (1) and (5) correspond to k_L and δ_L deviating more from their 'perfect' values of 1 and $2/9$ (at the mass scale of M_Z the deviations are 0.2 % and 0.5% respectively [7, 8]). This suggests that a deeper understanding of the observed regularity should be sought at the low (and not high) energy scale. Another peculiarity is the actual value of the phase parameter δ_L which is $2/9$ *radians* (and not a simple fraction times π that might be expected). This value of δ_L suggests a less geometric and more algebraic origin of the regularities observed in the lepton spectrum. One may then wonder if replacing the spread parameter k_f by its logarithm, *i.e.*

$$\lambda_f = \ln(k_f) \tag{7}$$

- which introduces a completely parallel treatment of the spread (λ_f) and pattern (δ_f) parameters (via the dependence of the r.h.s. of (4) on a *single* complex argument $\lambda_f + i\delta_f$) - could provide a welcome translation to the algebraic level suggested by the peculiar value of δ_L. After all, the 'simple' value of $k_L = 1$ corresponds to the equally 'simple' value for λ_L, *i.e.* 0. Obviously, the replacement of k_f by λ_f may be productive only if it leads to a novel observation elsewhere. Now, it should be noted that the phenomenological analysis performed in [4] suggested the set of values $\delta_L = 2/9$, $\delta_D = 4/27$, $\delta_U = 2/27$, and $\delta_\nu = 0$ for charged leptons, quarks ($f = D, U$), and neutrinos. With $\lambda_L = 0$ belonging to this set, it seems that a replacement of k_L by λ_L (a counterpart of δ_f's) might be a good idea. We need to investigate if other values of λ_f also belong to this set. In order to analyse this point we turn to quarks where the values of k_f are known to deviate significantly from 1 if the running quark masses are used. Indeed, at the mass scale $\mu = M_Z$ one obtains: for the down quarks $k_D \approx 1.1$, and for the up quarks $k_U \approx 1.3$ [7, 9].

4 Analysis

Equations (1) and (5) work well for lepton pole masses (and not for running masses) to such a high degree of accuracy that it seems appropriate to consider the counterparts of these masses in the quark sector as well. This suggests that the pole masses should be used for the heavy (b, c, t) quarks and the low-energy scale masses should be utilised for the light (u, d, s) quarks.

Furthermore, we observe that Eqs (1) and (5) depend on mass ratios ($z_1 = m_1/m_3$, $z_2 = m_2/m_3$) alone. In order to study the quark sector in detail it is therefore sufficient to estimate the relevant mass ratios in the light and heavy sectors as well as the relative mass scale of the two sectors. Now, the relative mass ratios can be estimated independently in the light and heavy sectors.

In the light quark sector the results of lattice calculations (supplemented with the phenomenological studies of isospin breaking effects) give (in the $\overline{MS}$ scheme at the renormalization scale $\mu = 2$ GeV) [10]:

$$m_u/m_d \approx 0.46(5) \quad \text{and} \quad m_s/m_d \approx 19.9(2). \tag{8}$$

These numbers agree very well with Weinberg's old low-energy estimates based on the ratios of pion and kaon masses [11]:

$$m_u/m_d \approx 0.56, \quad m_s/m_d \approx 20.2. \tag{9}$$

We may therefore safely assume that $m_u/m_d = 0.50 \pm 0.05$ and that $m_s/m_d = 20.0$.

In the sector of heavy quarks the perturbatively calculated pole masses m_q are related to the running $\overline{m}_q(\mu)$ masses via [10]:

$$m_q = \overline{m}_q(\overline{m}_q) \left[1 + \frac{4\overline{\alpha}_s(\overline{m}_q)}{3\pi} + \dots\right], \tag{10}$$

where the dots symbolize substantial higher order corrections [10]. As these corrections are quite uncertain we treat the relevant independent ratios of heavy quark pole masses (*i.e.* m_c/m_t, m_b/m_t) in two ways.

First, we approximate these ratios by $m_c(m_c)/m_t(m_t)$ and $m_b(m_b)/m_t(m_t)$, where $m_q(m_q)$ are the central values of the estimates given in [12] (in GeV):

$$m_c(m_c) = 1.29^{+0.05}_{-0.11}, \tag{11}$$
$$m_b(m_b) = 4.19^{+0.18}_{-0.16} \quad \text{and} \quad m_t(m_t) = 163.3^{+1.1}_{-1.}.$$

With $m_c(m_c)/m_t(m_t)$ and $m_b(m_b)/m_t(m_t)$ being pure numbers and m_t, at around $160 - 170$ GeV, setting the absolute scale of heavy quark masses, the relative scale of the light and heavy quark masses may be parametrized by m_s. The choice of m_s fixes then the values of the four mass

ratios z_1 and z_2 (in the up and down quark sectors). In a previous paper [4] it was argued that the pattern of phases δ_f is particularly simple for $m_s \approx 160\ MeV$ (*i.e.* not for the value of m_s of the order of $90 - 100\ MeV$ which is appropriate for $\mu = 2\ GeV$). It is therefore interesting to see how the values of λ_D and λ_U change when m_s varies from 90 to $160\ MeV$ or so. The upper part of Table 1 shows the relevant m_s-dependence of λ_D and λ_U obtained using (in the estimates of m_c/m_t, m_s/m_b, etc.) the values given in Eq. (11).

Table 1: Values of λ_D and λ_U

m_s (MeV)	90	100	130	160	170
λ_D	+0.094	+0.081	+0.046	+0.016	+0.007
λ_U	+0.214	+0.213	+0.212	+0.212	+0.212
λ_D	+0.112	+0.100	+0.068	+0.039	+0.031
λ_U	+0.194	+0.194	+0.193	+0.193	+0.192

Second, instead of employing Eq. (11), we directly use the estimates of pole masses given in [12] (*i.e.* $(m_c, m_b, m_t) = (1.84, 4.92, 172.9)$) (in GeV). The corresponding variations of λ_D and λ_U are presented in the lower part of Table 1. We view the differences between the upper and lower parts of Table 1 as providing an estimate of the errors involved.

We observe that for $m_s \approx 160 - 170$ MeV the values of $\lambda_{U,D}$ in the quark sector are not far from $\lambda_L = 0$ and $\delta_L = 2/9$. Deviations from these values may be tentatively assigned (as in the case of charged leptons) to some higher order corrections (which, on account of strong interactions being involved, could be larger than in the lepton case). Note that the regularities in question (for $\lambda_{U,D}$ here and for $\delta_{U,D}$ in [4]) are observed at approximately the same value of $m_s \approx 160$ MeV.

5 Conclusions

In conclusion it seems that there is a numerical hint that the doubly special character of Koide's parametrization can also be seen in the quark sector (with both λ_D and λ_U belonging to the set $(0, 2/27, 4/27, 2/9)$) (provided m_s is taken to be around 160 MeV).

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