

## Research Article

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# Generalized convergence analysis of the fractional order systems

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**Abstract:** The aim of the present work is to generalize the contraction theory for the analysis of the convergence of fractional order systems for both continuous-time and discrete-time systems. Contraction theory is a methodology for assessing the stability of trajectories of a dynamical system with respect to one another. The result of this study is a generalization of the Lyapunov matrix equation and linear eigenvalue analysis. The proposed approach gives a necessary and sufficient condition for exponential and global convergence of nonlinear fractional order systems. The examples elucidate that the theory is very straightforward and exact.

**Keywords:** Contraction theory, Stability, Convergence, Fractional order systems

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## 1 Introduction

Stability analysis is well-known as a base in control theory and many methods have been proposed to check this property. One of the major tools among these methods is Lyapunov theory. In this regard, it is confirmed that determining a Lyapunov function is fundamental in the stability analysis and control design of nonlinear systems (see for example [1, 2]). In the last two decades, Lohmiller and Slotine created a new technique from fluid mechanics and differential geometry known as *contraction theory*, for evaluation of stability [3]. Revisiting the contraction concept could result in introducing the suitable Riemann metrics

[3, 4]. The idea behind this theory is that stability can be assessed through checking the nearby trajectories' convergence, rather than by finding some Lyapunov functions, or by global state transformation using feedback linearization (see [5]). Lohmiller and Slotine used the contraction theory for analyzing the stability of and designing a control system for nonlinear chemical processes [6]. Wang and Slotine used the contraction theory to attain exact and global results for studying the synchronization of two or more coupled systems [7]. Jouffroy and Fossen, using contraction theory, introduced a methodology for analysis of differential nonlinear stability [8]. Pham and colleagues derived a stochastic version of the theory of nonlinear contraction, which provides a bound for the mean square distance between any two trajectories of a stochastically contracting system [9]. Rayguru in his paper designed a novel disturbance observer based dynamic surface controller using contraction framework [10]. The novelty of the proposed approach in the paper of Blocher *et al.* for the learning of robot point-to-point motions, is that they guarantee the stability of a learned dynamical system via Contraction theory [11].

Fractional derivatives and integrals have been gaining more and more interest of scientists due to their extensive applications in different directions of science, social science, engineering and finance [12–20]. In this context, Rostamy *et al.* in order to solve multi-term order fractional differential equations, utilized new matrices based on the Bernstein Polynomials basis, to reduce the equations to a system of algebraic equations [21]. Kumar *et al.* in [22] studied the fractional model of Lienard's equation by constituting a numerical algorithm based on the fractional homotopy analysis transform method. They also, discussed the uniqueness and convergence analysis of the solution of Lienard's equation. Kumar *et al.* presented a time-fractional modified Kawahara equation through a fractional derivative with exponential kernel [23]. In another study, they presented a new non-integer model for convective straight fins with temperature-dependent thermal conductivity associated with Caputo-Fabrizio fractional derivative [24]. They also, presented a new numerical scheme based on a combination of a q-homotopy anal-

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ysis approach and a Laplace transform approach to examine the fractional order Fitzhugh-Nagumo equation which describes the transmission of nerve impulses [25]. Inc *et al.* reduced the time fractional Cahn-Allen and time fractional Klein-Gordon to respective nonlinear ordinary differential equations of fractional order. They solved the reduced fractional ODEs using an explicit power series method and investigated the convergence analysis for the obtained explicit solutions [26]. Kumar *et al.* presented a new fractional extension of a regularized long-wave equation; this is a very important mathematical model in physical sciences, which unfolds the nature of shallow water waves and ion acoustic plasma waves [27].

As mentioned above, *Contraction Theory* and *Fractional Order Systems* (FOSs) are two subjects of much interest in last two decades. In this context, applying the contraction theory, Kamal *et al.* have aimed to design a universally exponentially stable controller for fractional-order systems [28]. Bandyopadhyay and Kamal reconsidered the theory of contraction by substituting the integer order variation of the system state by the fractional-order variation [29]. The major benefit of that methodology is its applicability in the evaluation of the stability of non-differentiable systems and FOSs and the design of a fractional order controller [29]. A *sufficient* condition is acquired by revisiting contraction theory in [29] for the system's exponential convergence.

In this article, a generalization of the FOSs convergence analysis is presented. The contraction theory in [29] is extended through the application of a general definition of differential length. The result is a fractional generalization for the Lyapunov matrix equation and linear eigenvalue analysis, providing a *necessary and sufficient* condition for the exponential convergence in a system.

This article is presented in the following way: In Sect. 2, the contraction analysis of FOSs by applying a fractional order infinitesimal variation is described. The generalization of convergence analysis of FOSs is presented in Sect. 3. In Sect. 4, the contraction method for discrete-time FOSs is considered. Numerical examples are given to illustrate the theory.

## 2 Contraction analysis of FOSs by fractional order infinitesimal variation

Stability analysis using differential approximation, is well-known as a base in control theory. The advantage of the

contraction method is that, it yields *global* and *exact* results of stability analysis for nonlinear systems. This section presents a summary of the basic results of Bandyopadhyay and Kamal [29], to which reference can be made for more details.

Consider an autonomous dynamical system

$$\dot{x} = f(x(t)) \quad x(0) = x_0 \quad (1)$$

where  $f$  is a nonlinear vector field and  $x(t)$  is an  $n$ -dimensional state vector. The dynamic system (1) can be stated in the fractional derivative form

$$\dot{x} = {}^{RL}D_t^1 x = f(x(t))$$

where  ${}^{RL}D_t^1$  indicates the Riemann-Liouville (R-L) fractional derivative, which for the order  $\alpha \in \mathbb{R}^+$ ,  $m - 1 < \alpha < m$ ,  $m \in \mathbb{N}$ , is defined as [29]:

$$\begin{aligned} {}^{RL}D_t^\alpha f(t) &= D^m D^{-(m-\alpha)} f(t) \\ &= \frac{d^m}{dt^m} \left[ \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau \right]. \end{aligned} \quad (2)$$

For simplicity, we omit the left superscript  $RL$  and subscript  $t_0$  and assume  $D_t^\alpha$  to be the Riemann-Liouville  $\alpha$ -order fractional derivative operator that exists and is continuous. It is clear that for the system (1),  $\delta \dot{x} = \delta f(x(t)) = \frac{\partial f}{\partial x} \delta x$ , the fractional virtual dynamics is represented by

$$\delta^\alpha D_t^1 x = \delta^\alpha f(x(t)) = D_x^\alpha f(x(t)) (D_x^\alpha x)^{-1} \delta^\alpha x, \quad (3)$$

where  $\delta^\alpha x$  and  $\delta^\alpha \dot{x}$  are termed as virtual displacement and virtual velocity respectively [29]. Now, consider an FOS:

$$D_t^\alpha x(t) = f(x(t)). \quad (4)$$

Taking  $D_t^{1-\alpha}$  in both sides of the dynamical system (4), one can write (see [29] Page 195, [30] Page 29):

$$D_t^1 x(t) = \dot{x}(t) = D_t^{1-\alpha} f(x(t)),$$

now by (3) we have

$$\delta^\alpha D_t^1 x = D_x^\alpha D_t^{1-\alpha} f(x(t)) (D_x^\alpha x)^{-1} \delta^\alpha x. \quad (5)$$

**Theorem 1.** *If the matrix  $D_x^\alpha D_t^{1-\alpha} f(x(t)) (D_x^\alpha x)^{-1}$  is uniformly negative definite (UND), all the solution trajectories of the system (4) converge exponentially to a single trajectory, discounting the initial conditions.*

*Proof.* The time derivative of squared distance between the two neighboring trajectories will be

$$D_t^1 \left( (\delta^\alpha x)^T \delta^\alpha x \right) \quad (6)$$

$$\begin{aligned}
&= 2(\delta^\alpha x)^T D_x^\alpha D_t^{1-\alpha} f(x(t)) (D_x^\alpha x)^{-1} \delta^\alpha x \\
&\leq 2\lambda_m(x, t) (\delta^\alpha x)^T \delta^\alpha x
\end{aligned}$$

where  $\lambda_m(x, t)$  is the largest eigenvalue of the symmetric part of

$$D_x^\alpha D_t^{1-\alpha} f(x(t)) (D_x^\alpha x)^{-1}.$$

Therefore, from (6) one can find

$$\|\delta^\alpha x\| \leq \|\delta^\alpha x_0\| e^{\int_0^t \lambda_m(x, t) dt}. \quad (7)$$

If  $\lambda_m(x, t)$  is strictly uniformly negative then any infinitesimal length  $\|\delta^\alpha x\|$  converges exponentially to zero as  $t$  tends to infinity. Thus all the solution trajectories of the system (4) converge exponentially to a single trajectory, irrespective of the initial conditions.  $\square$

Note that by a matrix  $A$  being UND, we mean that

$$\exists \beta > 0 \text{ such that } \frac{1}{2} (A + A^T) \leq -\beta I < 0,$$

by convention, the above can be written as  $A \leq -\beta I < 0$ .

**Example 1.** Consider the following FOS:

$$D_t^\alpha x = u. \quad (8)$$

In order to design a controller  $u$  that is able to stabilize the system (8), the convergence condition of Theorem 1, must be established. Therefore,

$$D_x^\alpha D_t^{1-\alpha} (u) (D_x^\alpha x)^{-1}$$

must be UND in the whole state space. There are many possible values of  $u$  which satisfies the above convergence condition. One can choose

$$u = D_t^{\alpha-1} D_x^{-\alpha} (-kx^{1-\alpha}),$$

where  $k > 0$ . Substituting the value of  $u$  in the convergence condition, one can get [29, 30]:

$$\begin{aligned}
&D_x^\alpha D_t^{1-\alpha} (u) (D_x^\alpha x)^{-1} \\
&= D_x^\alpha D_t^{1-\alpha} (D_t^{\alpha-1} D_x^{-\alpha} (-kx^{1-\alpha})) (D_x^\alpha x)^{-1} \\
&= \frac{-kx^{1-\alpha}}{x^{1-\alpha}} = -k\Gamma(2-\alpha) < 0
\end{aligned}$$

Therefore, the proposed controller stabilizes the system (8).

This example shows the advantage of the contraction theory for analyzing the stability of fractional order systems and designing the fractional order controller.

**Remark 1.** Consider the linear time-invariant (LTI) FOS:

$$D_t^\alpha x = Ax. \quad (9)$$

Applying  $D_t^{1-\alpha}$  to both sides, one can write [29, 30]:

$$\dot{x} = D_t^{1-\alpha} Ax.$$

Consider two neighboring trajectories of the above equation and the virtual displacement  $\delta x$  between them. This leads to the following:

$$\delta \dot{x} = \frac{\partial D_t^{1-\alpha} Ax}{\partial x} \delta x.$$

The rate of change of the squared distance  $(\delta x)^T \delta x$  between these two trajectories is given by

$$\frac{d(\delta x)^T \delta x}{dt} = 2(\delta x)^T \delta \dot{x} = 2(\delta x)^T \left[ \frac{\partial D_t^{1-\alpha} Ax}{\partial x} \right] \delta x.$$

Consider the following R-L derivative:

$$D_t^{1-\alpha} Ax = \frac{1}{\Gamma(1-(1-\alpha))} \frac{d}{dt} \int_0^t \frac{Ax(\tau)}{(t-\tau)^{1-\alpha}} d\tau,$$

thus we have

$$\frac{\partial D_t^{1-\alpha} Ax}{\partial x} = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t \frac{A}{(t-\tau)^{1-\alpha}} d\tau = \frac{At^{\alpha-1}}{\Gamma(\alpha)}. \quad (10)$$

So the system (9) is stable if  $\frac{At^{\alpha-1}}{\Gamma(\alpha)}$  is UND. Since  $t^{\alpha-1}$  and  $\Gamma(\alpha)$  are positive numbers, thus, one concludes that (9) is stable if  $A$  is UND.

We can see that if  $\alpha = 1$ , then  $\frac{At^{\alpha-1}}{\Gamma(\alpha)} = A$ , which is the Jacobian for an integer order LTI system.

**Remark 2.** An application of the contraction method is its use in studying the synchronization of a given nonlinear system coupled with a contracting virtual system in order to conclude the convergence and stability of the original system rather than through finding a Lyapunov function.

**Example 2.** Consider the following nonlinear FOS:

$$D_t^\alpha x_1 = -x_1 + x_1 x_2$$

$$D_t^\alpha x_2 = -x_1^2 - x_2$$

or, in the matrix equation form:

$$\begin{bmatrix} D_t^\alpha x_1 \\ D_t^\alpha x_2 \end{bmatrix} = \begin{bmatrix} -1 & x_1 \\ -x_1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

One can consider the following virtual  $y$ -system:

$$\begin{bmatrix} D_t^\alpha y_1 \\ D_t^\alpha y_2 \end{bmatrix} = \begin{bmatrix} -1 & x_1 \\ -x_1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Since, the matrix

$$\begin{bmatrix} -1 & x_1 \\ -x_1 & -1 \end{bmatrix}$$

is UND, according to Remark 1, the virtual system is contracting with two particular solutions, namely

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Since the virtual system is contracting, all the trajectories and especially two particular solutions converge to each other. Therefore, the arbitrary  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  tends to  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$  and the original  $x$ -system is stable.

One can see that this stability analysis is intuitive and much simpler than finding Lyapunov function.

**Definition 1.** Given the FOS  $D_t^\alpha x(t) = f(x(t))$ , a region  $\Omega \subseteq \mathbb{R}^n$  of the state space is entitled as a contraction (semi-contraction) region, if the matrix  $D_x^\alpha D_t^{1-\alpha} f(x(t)) (D_x^\alpha x)^{-1}$  is UND (uniformly negative semi-definite) in that region.

### 3 Generalization of the convergence analysis of FOS

An extension of Theorem 1 can be deduced by the use of a broad description of differential length. The result can be considered as a generalization for the fractional type of Lyapunov matrix equation and linear eigenvalue analysis. Furthermore, it gives a necessary and sufficient condition for exponential convergence.

#### 3.1 General definition of length

If the coordinate system of  $x$  is transformed to the coordinate system of  $z$ , and the vectors  $\delta^\alpha x$  and  $\delta^\alpha z$  are respectively virtual displacements between two neighboring trajectories of  $x$  and  $z$  coordinate systems, then  $\delta^\alpha z$ , using the coordinate transformation can be expressed as

$$\delta^\alpha z = \Theta \delta^\alpha x(t) \quad (11)$$

where  $\Theta(x, t)$  is an invertible square matrix. Therefore, a generalization of the squared length is as follows:

$$(\delta^\alpha z)^T \delta^\alpha z = (\delta^\alpha x(t))^T M(x, t) \delta^\alpha x(t) \quad (12)$$

where  $M(x, t) = \Theta^T \Theta$  should be a symmetric, uniformly positive definite (UPD) and continuously differentiable

metric (in other words, a Riemannian metric). If these conditions hold for  $M$ , and  $\delta^\alpha z$  converge exponentially to 0, then  $\delta^\alpha x$  converges exponentially to 0.

The time derivatives of the left and right hand sides of equation (12) lead to a generalization of the linear eigenvalue analysis (Section 3.1.1), and a generalized fractional form of the Lyapunov equation (Section 3.1.2), respectively.

##### 3.1.1 Generalized eigenvalue analysis

**Theorem 2.** For the system  $D_t^\alpha x(t) = f(x(t))$ , if the matrix

$$F = (\dot{\Theta} + \Theta D_x^\alpha (D_t^{1-\alpha} f(x)) (D_x^\alpha x)^{-1}) \Theta^{-1}$$

is UND, where  $\Theta$  is defined in (11), then all system trajectories converge globally to a single trajectory exponentially regardless of the initial conditions, and the rate of global exponential convergence is equal to the largest eigenvalues of the symmetric part of  $F$ .

*Proof.* Using the property of variation that  $D_t^1 \delta^\alpha x = \delta^\alpha D_t^1 x$ , and considering (5), by getting derivatives from both sides of  $\delta^\alpha z = \Theta \delta^\alpha x$ , we have

$$\begin{aligned} D_t^1 \delta^\alpha z &= \dot{\Theta} \delta^\alpha x + \Theta D_t^1 \delta^\alpha x & (13) \\ &= \dot{\Theta} \delta^\alpha x + \Theta \delta^\alpha \dot{x} \\ &= \dot{\Theta} \delta^\alpha x + \Theta D_x^\alpha (D_t^{1-\alpha} f(x)) (D_x^\alpha x)^{-1} \delta^\alpha x \\ &= (\dot{\Theta} + \Theta D_x^\alpha (D_t^{1-\alpha} f(x)) (D_x^\alpha x)^{-1}) \delta^\alpha x \\ &= (\dot{\Theta} + \Theta D_x^\alpha (D_t^{1-\alpha} f(x)) (D_x^\alpha x)^{-1}) \Theta^{-1} \delta^\alpha z \\ &= F \delta^\alpha z \end{aligned}$$

where

$$F = (\dot{\Theta} + \Theta D_x^\alpha (D_t^{1-\alpha} f(x)) (D_x^\alpha x)^{-1}) \Theta^{-1}. \quad (14)$$

Hence, by (13) the rate of change in squared length, which quantifies the contraction rate of the volume, is represented as

$$D_t^1 ((\delta^\alpha z)^T \delta^\alpha z) = 2 \delta^\alpha z^T D_t^1 \delta^\alpha z = 2 \delta^\alpha z^T F \delta^\alpha z \quad (15)$$

So, as in the proof of Theorem 1,  $\delta^\alpha z$  and thus  $\delta^\alpha x$ , converge to 0 in regions where  $F$  is UND.  $\square$

In (14),  $F$  is called the *generalized Jacobian* for FOS (4).

##### 3.1.2 Metric analysis

**Theorem 3.** The system  $D_t^\alpha x(t) = f(x(t))$  is contracting if  $\exists \beta_M > 0$ , such that

$$(D_x^\alpha (D_t^{1-\alpha} f(x)) (D_x^\alpha x)^{-1})^T M + \dot{M} \quad (16)$$

$$+ M \left( D_x^\alpha \left( D_t^{1-\alpha} f(x) \right) (D_x^\alpha x)^{-1} \right) \\ \leq -\beta_M M.$$

*Proof.* Remember that by (12),  $(\delta^\alpha z)^T \delta^\alpha z = (\delta^\alpha x(t))^T M(x, t) \delta^\alpha x(t)$ . The rate of change of the right hand side by using (5) is:

$$\begin{aligned} \frac{d}{dt} ((\delta^\alpha x(t))^T M(x, t) \delta^\alpha x(t)) \\ = (\delta^\alpha x(t))^T \left( \left( D_x^\alpha \left( D_t^{1-\alpha} f(x) \right) (D_x^\alpha x)^{-1} \right)^T M \right. \\ \left. + \dot{M} + M \left( D_x^\alpha \left( D_t^{1-\alpha} f(x) \right) (D_x^\alpha x)^{-1} \right) \right) \delta^\alpha x(t) \\ = (\delta^\alpha x(t))^T \Phi \delta^\alpha x(t), \end{aligned} \quad (17)$$

where

$$\begin{aligned} \Phi = \left( D_x^\alpha \left( D_t^{1-\alpha} f(x) \right) (D_x^\alpha x)^{-1} \right)^T M + \dot{M} \\ + M \left( D_x^\alpha \left( D_t^{1-\alpha} f(x) \right) (D_x^\alpha x)^{-1} \right). \end{aligned} \quad (18)$$

So that exponential convergence to a single trajectory can be concluded in an area identified by  $\Phi \leq -\beta_M M$  (where  $\beta_M$  is a positive constant).  $\square$

From Theorems 2 and 3, one can conclude that the region identified by  $\Phi \leq -\beta_M M$ , is the region for which  $F$  in (14) is UND.

### 3.2 Generalized contraction analysis

The above subsection results in the subsequent generalized definition, superseding Definition 1.

**Definition 2.** Given the FOS  $D_t^\alpha x(t) = f(x(t))$ , a region of the state space is called a contraction (semi-contraction) region in accordance with a UPD metric  $M(x, t) = \Theta^T \Theta$ , if equivalently  $F$  or  $\Phi$  are UND (semi-definite) in that region.

The result of generalized convergence can be expressed, as follows:

**Corollary 1.** Consider the FOS  $D_t^\alpha x(t) = f(x(t))$  and a ball of constant radius with respect to the metric  $M(x, t)$ , positioned at a given trajectory and confined at all times in a contraction region. Any trajectory, which starts in that ball remains in that ball and converges exponentially to the given trajectory. In addition, if the whole state space is a contraction region, global exponential convergence to the given trajectory is guaranteed.

*Proof.* Immediate from Theorems 2 and 3 and Definition 2.  $\square$

The converse of Theorem 2 is also valid.

**Theorem 4.** Any exponentially convergent FOS is contracting in respect of an appropriate metric.

*Proof.* Suppose that the system (4) is exponentially convergent, which implies that there exist  $\beta > 0$  and  $k \geq 1$ , such that along any system trajectory  $x(t)$  and for any  $t \geq 0$ ,

$$(\delta^\alpha x)^T \delta^\alpha x \leq k(\delta^\alpha x_0)^T \delta^\alpha x_0 e^{-\beta t}.$$

Suppose the metric  $M(x(t), t)$  satisfies the following Lyapunov form ordinary differential equation:

$$\begin{aligned} \dot{M} = -\beta_M M - \left( D_x^\alpha \left( D_t^{1-\alpha} f(x) \right) (D_x^\alpha x)^{-1} \right)^T M \\ - M \left( D_x^\alpha \left( D_t^{1-\alpha} f(x) \right) (D_x^\alpha x)^{-1} \right) \\ M(t=0) = kI, \end{aligned} \quad (19)$$

by substituting (19) in (17), and solving the differential equation (17), it is easy to find that

$$k(\delta^\alpha x_0)^T \delta^\alpha x_0 e^{-\beta t} = \delta^\alpha x^T M \delta^\alpha x$$

Now by the assumption that  $(\delta^\alpha x)^T \delta^\alpha x \leq k(\delta^\alpha x_0)^T \delta^\alpha x_0 e^{-\beta t}$ , we find that

$$(\delta^\alpha x)^T \delta^\alpha x \leq \delta^\alpha x^T M \delta^\alpha x \quad (20)$$

Since (20) holds for any  $\delta^\alpha x$ , this concludes that  $M$  is UPD. Therefore, with respect to an appropriate metric, any exponentially convergent system is contracting.  $\square$

Theorems 2 and 4 correspond to necessary and sufficient conditions for the exponential convergence of FOS.

**Example 3.** Consider the autonomous differential equation

$$D_t^\alpha x = -k \text{sign}(x) = f(x) \quad (21)$$

where  $k > 0$  and  $\text{sign}(x)$  is defined as

$$\text{sign}(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \quad (22)$$

Consider the differential coordinate transformation  $\delta^\alpha z = \Theta \delta^\alpha x$  (where  $\Theta$  is constant). To check the stability of system (21), we calculate the value of  $F$  in (14) as:

$$\begin{aligned} F &= \left( \dot{\Theta} + \Theta D_x^\alpha \left( D_t^{1-\alpha} f(x) \right) (D_x^\alpha x)^{-1} \right) \Theta^{-1} \\ &= \left( \dot{\Theta} + \Theta D_x^\alpha \left( D_t^{1-\alpha} (-k \text{sign}(x)) \right) (D_x^\alpha x)^{-1} \right) \Theta^{-1} \\ &= \left( \dot{\Theta} + \Theta \left( -\frac{k \text{sign}(x) t^{\alpha-1} x^{-\alpha} (\Gamma(2-\alpha))}{(\Gamma(1-\alpha))(\Gamma(\alpha)) x^{1-\alpha}} \right) \right) \Theta^{-1} \end{aligned}$$



$$\begin{aligned}
&= \Theta \left( -\frac{k^{\frac{x}{|x|}} t^{\alpha-1} (\Gamma(2-\alpha))}{(\Gamma(1-\alpha))(\Gamma(\alpha))x} \right) \Theta^{-1} \\
&= \Theta \left( -\frac{kt^{\alpha-1} (\Gamma(2-\alpha))}{(\Gamma(1-\alpha))(\Gamma(\alpha))|x|} \right) \Theta^{-1} < 0
\end{aligned}$$

since  $-\frac{kt^{\alpha-1} (\Gamma(2-\alpha))}{(\Gamma(1-\alpha))(\Gamma(\alpha))|x|} < 0$ . Therefore, any two trajectories of the FOS (21) converge to each other.

As we can see in this Example, one of the main advantages of using the fractional order variation in the contraction theory is that it also works for analyzing the stability of non-differentiable systems.

## 4 Contraction method for discrete-time FOS

Consider the following integer order discrete-time system:

$$x(k+1) = f(x(k)) \quad (23)$$

where  $f(\cdot)$  is a smooth nonlinear vector function. The discrete-time fractional-order system (DFOS) can be represented as follows, which for more details the reader is referred to [31]:

$$x(k+1) = f(x(k)) + (\alpha - 1)x(k) + \sum_{p=1}^k C_p x(k-p) \quad (24)$$

where  $C_p = (-1)^p \binom{\alpha}{p+1}$ . Since all the absolute values of the coefficients decrease as the iteration  $p$  increases for each fractional-order  $\alpha$  [31], in order to decrease the calculations and the needed space for each computing, we can use a finite truncation to approximate a DFOS as follows:

$$x(k+1) = f(x(k)) + (\alpha - 1)x(k) + \sum_{p=1}^L C_p x(k-p) \quad (25)$$

where  $L$  denotes the truncation length and it is selected appropriately according to a practical problem.

**Theorem 5.** *Exponential convergence of system (25) is guaranteed if*

$$\left( \frac{\partial g_k}{\partial x(k)} \right)^T \frac{\partial g_k}{\partial x(k)} - I \quad (26)$$

be UND, where

$$g_k = f(x(k)) + (\alpha - 1)x(k) + \sum_{p=1}^L C_p x(k-p). \quad (27)$$

*Proof.* The associated virtual dynamics of (25) is

$$\delta x(k+1) = \frac{\partial g_k}{\partial x(k)} \delta x(k) \quad (28)$$

so that the virtual length dynamics is

$$\begin{aligned}
\delta x^T(k+1) \delta x(k+1) &= \\
\delta x^T(k) \left( \frac{\partial g_k}{\partial x(k)} \right)^T \frac{\partial g_k}{\partial x(k)} \delta x(k),
\end{aligned} \quad (29)$$

therefore, the rate of change of the left hand side is

$$\begin{aligned}
&\delta x^T(k+1) \delta x(k+1) - \delta x^T(k) \delta x(k) \\
&= \delta x^T(k) \left( \frac{\partial g_k}{\partial x(k)} \right)^T \frac{\partial g_k}{\partial x(k)} \delta x(k) - \delta x^T(k) \delta x(k) \\
&= \delta x^T(k) \left[ \left( \frac{\partial g_k}{\partial x(k)} \right)^T \frac{\partial g_k}{\partial x(k)} - I \right] \delta x(k)
\end{aligned}$$

thus, trajectories will exponentially converge to a single trajectory, if

$$\left( \frac{\partial g_k}{\partial x(k)} \right)^T \frac{\partial g_k}{\partial x(k)} - I \quad (30)$$

be UND.  $\square$

**Corollary 2.** *For the linear DFOS*

$$x(k+1) = Ax(k) + (\alpha - 1)x(k) + \sum_{p=1}^L C_p x(k-p) \quad (31)$$

trajectories will exponentially converge to a single trajectory, if

$$B^T B - I$$

be UND, where

$$B = A + (\alpha - 1)I.$$

*Proof.* We have

$$\begin{aligned}
g_k &= Ax(k) + (\alpha - 1)x(k) + \sum_{p=1}^L C_p x(k-p) \\
&= (A + (\alpha - 1)I)x(k) + \sum_{p=1}^L C_p x(k-p)
\end{aligned} \quad (32)$$

therefore,

$$\frac{\partial g_k}{\partial x(k)} = A + (\alpha - 1)I,$$

which concludes the proof.  $\square$

In the discrete-time version, using the generalized virtual displacement

$$\delta z(k) = \Theta_k(x(k), k) \delta x(k) \quad (33)$$

and by relation (28) we have:

$$\begin{aligned}
&\delta z^T(k+1) \delta z(k+1) \\
&= \delta x^T(k) \left( \frac{\partial g_k}{\partial x(k)} \right)^T \Theta_{k+1}^T \Theta_{k+1} \frac{\partial g_k}{\partial x(k)} \delta x(k)
\end{aligned} \quad (34)$$

$$= \delta z^T(k) F_k^T F_k \delta z(k)$$

where

$$F_k = \Theta_{k+1} \frac{\partial g_k}{\partial x(k)} \Theta_k^{-1} \quad (35)$$

is the *fractional order discrete-time* generalized Jacobian. Now, we can provide the following generalized definition of a contraction region for DFOS.

**Definition 3.** Given the DFOS:  $x(k+1) = g_k(x(k))$ , with  $g_k$  given in (27), a region of the state space is recognized as a contraction region in respect of a UPD metric  $M_k(x(k), k) = \Theta_k^T \Theta_k$ , if in that region

$$\exists \beta > 0, \quad F_k^T F_k - I \leq -\beta I < 0$$

where  $F_k = \Theta_{k+1} \frac{\partial g_k}{\partial x(k)} \Theta_k^{-1}$ .

**Remark 3.** Corollary 1 can be immediately changed to the discrete version.

**Example 4.** Consider the fractional order discrete-time Logistic system [31]:

$$x(k+1) = \mu x(k)(1-x(k)) + (\alpha-1)x(k) + \sum_{p=1}^L C_p x(k-p). \quad (36)$$

By putting  $x(k+1) = x(k)$ , the fixed point (equilibrium point) will be

$$x^* = \frac{2 - \mu - \alpha \pm \sqrt{(\mu + \alpha - 2)^2 + 4\mu c}}{-2\mu}$$

where

$$c = \sum_{p=1}^L C_p x(k-p)$$

Let

$$g_k = \mu x(k)(1-x(k)) + (\alpha-1)x(k) + \sum_{p=1}^L C_p x(k-p)$$

therefore,

$$\frac{\partial g_k}{\partial x(k)} = \mu + \alpha - 1 - 2\mu x(k)$$

and the fractional order discrete-time Logistic system will be convergent if

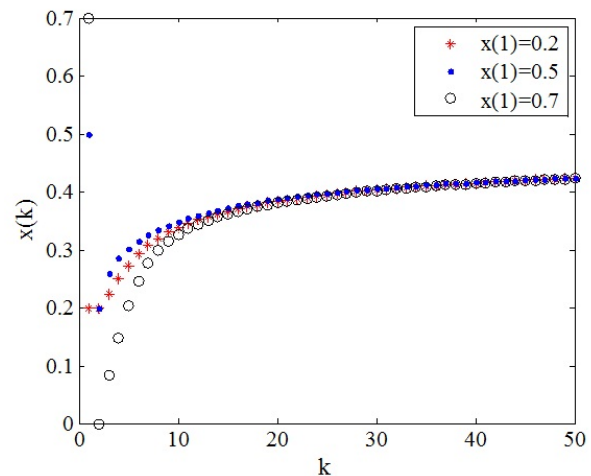
$$\left( \frac{\partial g_k}{\partial x(k)} \right)^T \frac{\partial g_k}{\partial x(k)} - I < 0.$$

Numerically, choosing  $\alpha = 0.4$ ,  $\mu = 2$  and  $L = 50$ , the fixed point will be  $x^* = 0.4228$  and

$$\left( \frac{\partial g_k}{\partial x(k)} \right)^2 - 1 = -0.9151 < 0$$

therefore, the system will converge to  $x = 0.4228$  for each arbitrary initial point (see Figure 1).

This example, shows the simplicity of contraction theory for analyzing the convergence of DFOSs.



**Figure 1:** Convergence of fractional order discrete-time Logistic system for three arbitrary initial points 0.2, 0.5 and 0.7.

## Conclusion

In this article, applying a broad description of differential length, we have generalized the fractional order contraction theory. The proposed approach is useful for analyzing the stability of non-differentiable systems and also FOSs; furthermore, it leads to *necessary and sufficient* conditions for exponential convergence of an FOS. The theory was also stated for the case of discrete-time FOS. Numerical examples illustrate the proposed method.

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