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Research Article

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On the conservation laws and solutions of a (2+1) dimensional KdV-mKdV equation of mathematical physics

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Abstract: In this paper, we study a (2+1) dimensional KdV-mKdV equation, which models many physical phenomena of mathematical physics. This equation has two integral terms in it. By an appropriate substitution, we convert this equation into two partial differential equations, which do not have integral terms and are equivalent to the original equation. We then work with the system of two equations and obtain its exact travelling wave solutions in form of Jacobi elliptic functions. Furthermore, we employ the multiplier method to construct conservation laws for the system. Finally, we revert the results obtained into the original variables of the (2+1) dimensional KdV-mKdV equation.

Keywords: (2+1) dimensional KdV-mKdV equation; travelling wave solutions; Jacobi elliptic functions; conservation laws; multiplier method

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1 Introduction

In the last few decades nonlinear partial differential equations (NPDEs) have been vastly studied by researchers as these equations model many physical phenomena of the real world. This is apparent from many research papers which have appeared in the literature. One of the main interest of the researchers is to look for closed form solutions of NPDEs. Nonetheless, this is not a simple task. Despite of this fact, many researchers have developed novel approaches to solve NPDEs. These approaches include the inverse scattering transform method [1], the Hirota's bilinear method [2], Bäcklund transformation [3], the Darboux transformation [4], the simplest equation method [5, 6], the (G'/G)-expansion method [7, 8], the reduction mKdV equation method [9], the F–expansion method [10], the sine-cosine method [11], the extended tanh method [12], the exp-function expansion method [13–15], the multiple exp-function method [16, 17] the Lie symmetry method [18–21], etc.

In this paper we study a NLPDE known as (2+1) dimensional KdV-mKdV equation [22]

$$\begin{aligned} &u_t + u_{xxy} + 4uu_y - 4u^2u_y + 2u_x\partial_x^{-1}u_y \\ &- 2u_x\partial_x^{-1}(2uu_y) = 0, \end{aligned} \tag{1}$$

which models many physical phenomena of mathematical physics. The (1+1) dimensional KdV-mKdV equation describes the wave propagation of bounded particle with a harmonic force [23]. In particularly, it describes the propagation of ion acoustic waves of small amplitude without Landau damping in plasma physics [24]. The propagation of thermal pulse through single crystal of sodium fluoride in solid physics can also be explained by this equation [25, 26].

In the study of differential equations (DEs) conservation laws play a very important role. Also, in exploring the existence, uniqueness and stability of solutions to DEs, researchers have employed conservation laws [27–29]. Conservation laws can be engaged in determining exact solutions of NLPDEs [30, 31].

To study the (2+1) dimensional KdV-mKdV (1), we start by letting $v = \partial_x^{-1}(u_y - 2uu_y) = \int(u_y - 2uu_y)dx$. This transforms (1) to a system of two partial differential equations (PDEs)

$$\begin{aligned} &u_t + u_{xxy} + 4uu_y - 4u^2u_y + 2u_xv = 0, \\ &v_x - u_y + 2uu_y = 0. \end{aligned} \tag{2}$$

The outline of the paper is as follows. In Section 2, firstly, we use the travelling wave variable to reduce (2) to

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two ordinary differential equations (ODEs) and then obtain its exact explicit solutions. We compute conservation laws of (1) with the help of multiplier method in Section 3. Finally, in Section 4 we give concluding remarks.

2 Exact solutions of (2)

We first use the travelling wave variable $z = x + by + ct$ to reduce the PDEs (2) to two ordinary differential equations (ODEs). Thus by letting

$$u(x,y,t) = U(z),\ \ v(x,y,t) = V(z), \quad \text{where } z = x + by + ct, \tag{3}$$

the system (2) transforms to

$$bU'''(z) + 4b(1 - U(z))U(z)U'(z) + 2V(z)U'(z) + cU'(z) = 0, \tag{4a}$$

$$2bU(z)U'(z) - bU'(z) + V'(z) = 0, \tag{4b}$$

which is a system of nonlinear ODEs. We now decouple the above system by solving the second equation for V. Integrating Eq. (4b) with respect to z, we obtain

$$V(z) = -bU(z)^2 + bU(z) + c_1, \tag{5}$$

where c_1 is an arbitrary constant of integration. The substitution of the value of V from (5) into Eq. (4a) gives the third-order nonlinear ODE

$$bU'''(z) - 6bU(z)^2U'(z) + 6bU(z)U'(z) + cU'(z) + 2c_1U'(z) = 0. \tag{6}$$

This equation can be solved in the following manner. Integrating (6) with respect to z yields

$$bU'' - 2bU^3 + 3bU^2 + (c + 2c_1)U + c_2 = 0, \tag{7}$$

where c_2 is an arbitrary constant of integration. We multiply (7) by the integrating factor U' and obtain

$$bU'U'' - 2bU^3U' + 3bU^2U' + (c + 2c_1)UU' + c_2U' = 0. \tag{8}$$

Now integration of (8) with respect to z gives us

$$U'^2 - U^4 + 2U^3 + \frac{U^2}{b}(c + 2c_1) + \frac{2c_2U}{b} + \frac{2c_3}{b} = 0, \tag{9}$$

where c_3 is a constant. The solution of (9) can be presented in form of the Jacobi elliptic function [32] and is a bit cumbersome to write here. However, by imposing the asymptotic boundary conditions $U, U', U'' \to 0$ for $|\ z\ | \to \infty$, we obtain $c_2 = 0$ and $c_3 = 0$ and one can write a special solution of (9) given by

$$U(z) = -\frac{4(c-k)^2e^{\theta}}{4b^2\sqrt{k-c} + 4b(c-k)\left(\sqrt{k-c} + e^{\theta}\right) + (k-c)^{3/2}e^{2\theta}},$$

where $k = -2c_1$, $\theta = \sqrt{(k-c)/b}\,(z+l)$ and l is a constant. Thus, the exact solutions of (1) can be presented in the form of the Jacobi elliptic functions and its special solution is

$$u(t,x,y) = -\frac{4(c-k)^2e^{\theta}}{4b^2\sqrt{k-c} + 4b(c-k)\left(\sqrt{k-c} + e^{\theta}\right) + (k-c)^{3/2}e^{2\theta}}, \tag{10}$$

where

$$\theta = \sqrt{\frac{k-c}{b}}\,(x + by + ct + l)\,.$$

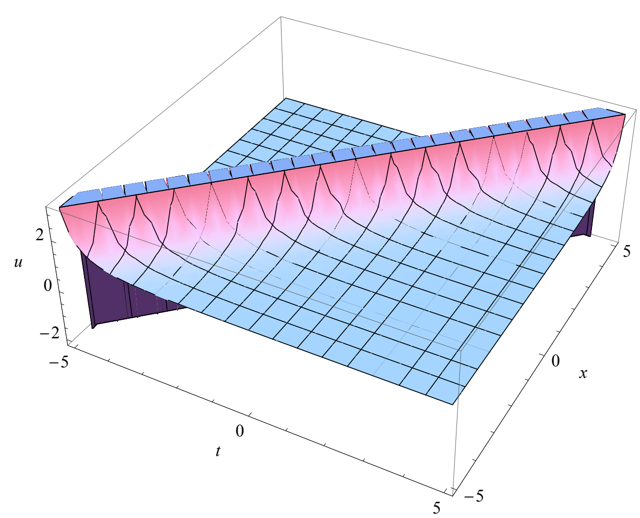

Figure 1: Profile of the solution (10)

3 Conservation laws

We now derive the conservation laws for (1). For this purpose the multiplier method will be employed [33]. But first we recall some basic definitions and results that we use later in the section.

Let

$$F_\alpha(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, \ldots, m, \tag{11}$$

be a kth-order system of PDEs with $x = (x^1, x^2, \ldots, x^n)$ and $u = (u^1, u^2, \ldots, u^m)$ the n independent and

m dependent variables, respectively. The subscripts in $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ denote all first, second, ..., kth-order partial derivatives and $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_jD_i(u^\alpha), \ldots$ respectively. The differential operator D_i denotes the *total derivative operator* with respect to x^i defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \quad i = 1, \ldots, n. \tag{12}$$

The operator

$$E_u \equiv \frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\ldots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \tag{13}$$

$$\alpha = 1, \ldots, m,$$

is called the *Euler-Lagrange operator*.

The n–tuple $T = (T^1, T^2, \ldots, T^n)$, with $T^k \in \mathcal{A}$, $k = 1, \ldots, n$, where $\mathcal{A}$ is the space of differential functions, is a *conserved vector* of (11) if T^k satisfies

$$D_k T^k = 0 \tag{14}$$

for all solutions of (11).

The functions $Q_\alpha(x, u, u_{(1)}, \ldots)$ are *multipliers* yielding a conservation law of the Euler-Lagrange equation if

$$Q_\alpha F_\alpha = D_i T^i \tag{15}$$

holds identically. For our equation we assume multipliers of the zeroth-order,
that is, $Q_\alpha = Q_\alpha(t, x, y, u, v)$. The multiplier Q_α is a solution of the determining equation

$$E_u(Q_\alpha F_\alpha) = 0. \tag{16}$$

The expansion of the determining equation provides us with an over-determining system of PDEs, whose solution gives us the multipliers. The conservation laws are then obtained using the homotopy formula [33, 34].

3.1 Conservation laws of (1)

In this subsection we derive the conservation laws for (1) using the multiplier approach. For the coupled system (2), we look for the zeroth-order multipliers of the form, $Q_1 = Q_1(t, x, y, u, v)$ and $Q_2 = Q_2(t, x, y, u, v)$ that are given by

$$\begin{aligned} Q_1 &= u^2 f_1(t) - u f_1(t) - \frac{1}{2} y f_1'(t) + f_2(t) + 2u f_3(y-t) \\ &\quad - f_3(y-t), \\ Q_2 &= u f_1(t) - \frac{1}{2} f_1(t) + f_3(y-t), \end{aligned}$$

where f_1, f_2, depend of t and f_3 is a function of $y-t$. We can now write down the associated nonlocal conserved vector of (1) as:

$$\begin{aligned} T^t &= \frac{1}{2}u^2 f_1(t) - \frac{1}{2} u f_1(t) + u f_3(y-t), \\ T^x &= \left\{(u^2 - u)\int (u_y - 2uu_y)dx \right. \\ &\quad \left. - \frac{1}{2}u_x u_y + \frac{1}{2} u\, u_{xy} - \frac{1}{2}u_{xy}\right\} f_1(t) \\ &\quad + \left(f_2(t) - \frac{1}{2} y f_1'(t)\right)\int (u_y - 2uu_y)\,dx + u_{xy} f_3(y-t) \\ &\quad + (2u-1) f_3(y-t)\int (u_y - 2uu_y)dx, \\ T^y &= -\frac{1}{2}u\left(u^3 + u - u_{xx} - 2u^2\right) f_1(t) - \frac{1}{2}u(uy - y) f_1'(t) \\ &\quad + u(u-1) f_2(t) + u f_3(y-t). \end{aligned}$$

Remark 2. Since the functions f_1, f_2 and f_3 appearing in the multipliers are arbitrary, as a result (1) has infinitely many nonlocal conservation laws.

4 Conclusions

The (2+1) dimensional KdV-mKdV (1) was investigated from the point of view of solutions and conservation laws. We introduced a new variable v and wrote the (1) as a system of two PDEs, which did not have integral terms. The travelling wave variable was then utilized to reduce the system of PDEs to two nonlinear ODEs. The resulting system of ODEs was decoupled and solved directly. As a result we obtained travelling wave solutions of (1) in the form of Jacobi elliptic functions. These newly obtained solutions are very important in explaining physical situations of some real world problems that are related to the equation. Furthermore, conservation laws of (1) were also computed using the multiplier method. This resulted in infinitely many nonlocal conserved vectors. The significance of conservation laws was mentioned in Section 1 of the paper.

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