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Madeeha Tahir, Muhammad Nawaz Naeem, Maria Javaid, Muhammad Younas, Muhammad Imran*, Naeem Sadiq, and Rabia Safdar

Unsteady flow of fractional Oldroyd-B fluids through rotating annulus

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Abstract: In this paper exact solutions corresponding to the rotational flow of a fractional Oldroyd-B fluid, in an annulus, are determined by applying integral transforms. The fluid starts moving after $t = 0^+$ when pipes start rotating about their axis. The final solutions are presented in the form of usual Bessel and hypergeometric functions, true for initial and boundary conditions. The limiting cases for the solutions for ordinary Oldroyd-B, fractional Maxwell and Maxwell and Newtonian fluids are obtained. Moreover, the solution is obtained for the fluid when one pipe is rotating and the other one is at rest. At the end of this paper some characteristics of fluid motion, the effect of the physical parameters on the flow and a correlation between different fluid models are discussed. Finally, graphical representations confirm the above affirmation.

Keywords: Fractional Oldroyd-B model, fractional differential operator, annulus, rotational flow, velocity field, shear stress, integral transforms

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Madeeha Tahir: Department of Mathematics, Government College University, Faisalabad, Pakistan,
E-mail: dr.madeeha.2017@gmail.com
Muhammad Nawaz Naeem: Department of Mathematics, Government College University, Faisalabad, Pakistan, E-mail: mnawaznaeem@yahoo.com
Maria Javaid: Department of Mathematics, Government College University, Faisalabad, Pakistan, E-mail: maria_9248@yahoo.com
Muhammad Younas: Department of Mathematics, COMSATS Institute of Information Technology, Lahore Campus, Pakistan, E-mail: muhammadyounas@ciitlahore.edu.pk
***Corresponding Author: Muhammad Imran:** Department of Mathematics, Government College University, Faisalabad, Pakistan, E-mail:drmimranchaudhry@gmail.com
Naeem Sadiq: Department of Mathematics, Government College University, Faisalabad, Pakistan, E-mail: naeemsadiq.gc@gmail.com
Rabia Safdar: Department of Mathematics, Government College University, Faisalabad, Pakistan,
E-mail: rabia.safdar1109@gmail.com

1 Introduction

Humans benefit from fluid flow but also need to understand the negative effects in nature, e.g. hurricanes and cyclones. The topographic conditions and the sufficient supply of rain flow is important in the cultivation of crops. Cultivation in deserts can also be achieved with artificial irrigation. Artificial breathing machines, artificial hearts and dialysis systems are designed using fluid dynamics. Hence fluids are vital.

The stress and rate of strain of a fluid is related: fluid is Newtonian if this relationship is linear, if not, the fluid is non-Newtonian. The non-Newtonian fluids, like polymers, plasma, paste and pulps are very important in industry. Therefore, the discussion of non-Newtonian fluids in fluid mechanics is important. All existing fluids cannot be described by the Navier-stocks equations. By categorizing different fluids, we have different mathematical models to represent fluids. Oldroyd-B fluid is one of these rate-type models.

The non-Newtonian fluids possessing shear dependent viscosity are known as viscoelastic fluids. The first rate type fluid model is a Maxwell model that is still in used. Fetecau et. al [1] derived the results of the velocity field and shear stress of an Oldroyd-B fluid on a plate, accelerating constantly. Hayat et. al [2] determined the solutions for MHD flow over an infinite oscillatory plate about an axis that is normal to the oscillatory plate. Some authors determined solutions for a viscoelastic fluid model by reproducing a kernel method (RKM) [3–6]. The unsteady incompressible Oldroyd-B fluid is analyzed by Jamil et. al [7] and special cases are also obtained for Maxwell, second grade and Newtonian fluids. Similarly, Burdujan [8] used this idea on Taylor-Couette flow and calculated the velocity field and shear stress of a rotating fluid between annuls of double cylinders. Numerical solutions of fractional fluids are also under attention [9, 10].

A lot of interest is given to a rotating circular domain corresponding to helical flow for second grade, Maxwell fluids and Oldroyd-B [11–14]. The flow of a polymer in a circular cylinder by considering pulsatile APG was analyzed by Barnes et. al [15, 16]. Davies et. al [17] and Phan-Thien [18] presented the same problem for a White-Metzner fluid. To solve the viscoelastic motion of different fluid models, one needs to learn about fractional calculus [19, 20]. Fractional calculus is very useful in fields of mathematics and physics [21–23]. Furthermore, the interested reader can consult the references [24–30].

In this proposal, we consider the fractional Oldroyd-B fluid model and study the flow due to the circulation of two pipes around its axial. Integral transforms are used to obtain the general solutions and the final result is in the form of hypergeometric functions. There are some limiting cases by applying the limits on physical parameters, i.e $\varsigma \to 1$, $\lambda_r \to 0$ and $\lambda \to 0$, solutions for the ordinary Oldroyd-B, fractional and ordinary Maxwell and Newtonian fluids are discussed respectively.

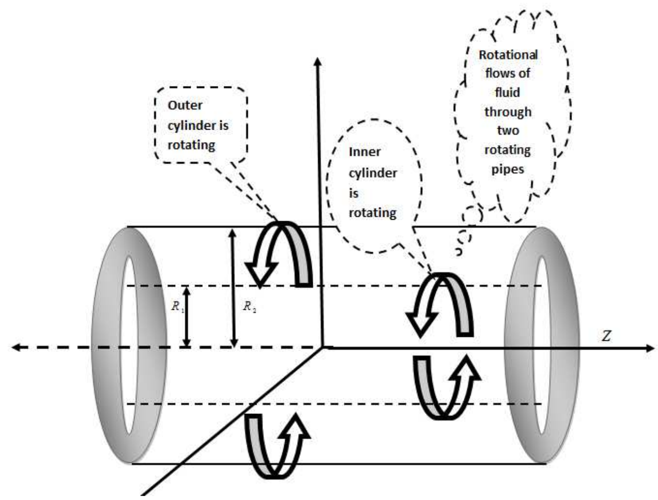

Figure 1: Problem Geometry.

2 Development of the flow

Consider a fractional Oldroyd-B fluid (OBFFD) between two coaxial pipes. The radii of the pipes are R_1 and $R_2(> R_1)$, after time t $=0^+$, both pipes and the fluid between them start rotating along their axis. For the considered problem, the z-axis is along the axis of the pipes, the r-axis is perpendicular to the axis of the pipe and θ is along the tangent to the boundary. Let v_r, v_θ and v_z be the velocity components in the direction of the r, θ and z axes respectively.

The continuity and linear momentum equations for incompressible flow is

$$\text{div}\mathbf{V} = 0, \tag{1}$$

$$\frac{d\mathbf{V}}{dt} = \frac{1}{\rho}\text{div}\mathbf{T} + \mathbf{b}, \tag{2}$$

where $\mathbf{V}$, t, ρ, $\mathbf{b}$, d/dt is velocity, time, density, body force and the material time derivative respectively.

For an incompressible Oldroyd-B fluid [36]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S},$$
$$\mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^T) = \mu[\mathbf{A} + \lambda_r(\dot{\mathbf{A}} - \mathbf{LA} - \mathbf{AL}^T)], \tag{3}$$

where p is the hydrostatic pressure, I denotes the identity tensor, $\mathbf{S}$ is the extra-stress tensor, λ and λ_r are relaxation and retardation times, superposed dot indicates the material time derivative, $\mathbf{L}$ represents the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin Ericsen tensor, μ is the dynamic viscosity of the fluid and the superscript T denotes the transpose operation. For the problem in consideration, the velocity field and the extra stress tensor has the following form [31]

$$\mathbf{V} = \mathbf{V}(r,t) = \omega(r,t)\mathbf{e}_\theta\,, \qquad \mathbf{S} = \mathbf{S}(r,t)\,. \tag{4}$$

For such flows, (1) is true.

The governing equations for a Oldroyd-B fluid, corresponding to the above defined motions are [31]

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\tau(r,t) = \mu(1 + \lambda_r\frac{\partial}{\partial t})\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\omega(r,t)\,, \tag{5}$$

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\frac{\partial w(r,t)}{\partial t} = \nu(1 + \lambda_r\frac{\partial}{\partial t})\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega(r,t)\,, \tag{6}$$

where ν is the kinematic viscosity and $\tau(r,t)$ is the non zero shear stress.

The governing equations for a fractional Oldroyd-B fluid is obtained by using a fractional differential operator D_t^ς, defined by [32, 33]

$$D_t^\varsigma f(t) = \frac{1}{\Gamma(1-\varsigma)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^\varsigma}d\tau; \quad 0 \le \varsigma < 1, \tag{7}$$

in governing equations (5) and (6) we have

$$\left(1 + \lambda^\varsigma D^\varsigma\right)\tau(r,t) = \mu(1 + \lambda_r^\varsigma D^\varsigma)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\omega(r,t)\,, \tag{8}$$

$$\left(1 + \lambda^\varsigma D^\varsigma\right)\frac{\partial w(r,t)}{\partial t} = \nu(1 + \lambda_r^\varsigma D^\varsigma)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega(r,t)\,. \tag{9}$$

3 Initial and boundary conditions

We need three conditions to solve a fractional partial differential equation (9). The relevant initial and boundary conditions are

$$\omega(r,t)|_{t=0}=0,\quad \frac{\partial\omega(r,t)}{\partial t}|_{t=0}=0,$$
$$\tau(r,t)|_{t=0}=0,\quad r\in[R_1,R_2], \tag{10}$$

$$\omega(R_1,t)=R_1\mho_1 t^{\kappa},\quad \omega(R_2,t)=R_2\mho_2 t^{\kappa}\quad \text{for}\ \ t>0, \tag{11}$$

where $\mho_1$, $\mho_2$ and κ are real numbers.

3.1 Analytic solutions for the velocity field

Using the Laplace transform Eqs. (9) and (11) implies that

$$\left(q+\lambda^{\varsigma}q^{\varsigma+1}\right)\overline{\omega}(r,q)$$
$$=\nu\left(1+\lambda_r^{\varsigma}q^{\varsigma}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r}\right)\overline{\omega}(r,q);\quad r\in(R_1,R_2), \tag{12}$$

$$\overline{\omega}(R_1,q)=\frac{R_1\mho_1}{q^{\kappa+1}},\ \ \overline{\omega}(R_2,q)=\frac{R_2\mho_2}{q^{\kappa+1}}, \tag{13}$$

where the Laplace transform of the function $\omega(r,t)$ is $\overline{\omega}(r,q)$. The finite Hankel transform [34] of the function $\overline{\omega}(r,q)$, can be expressed as

$$\overline{\omega}_H(r_n,q)=\int_{R_1}^{R_2} r\overline{\omega}(r,q)A(r,r_n)dr,\quad n=1,2,3,\ldots \tag{14}$$

where

$$A(r,r_n)=J_1(rr_n)Y_1(R_2r_n)-J_1(R_2r_n)Y_1(rr_n), \tag{15}$$

r_n are the positive roots of the transcendental equation $A(R_1,r)=0$, and $J_p(\cdot)$ is Bessel functions of the first kind and $Y_p(\cdot)$ is Bessel functions of second kind of order p.

Taking the Hankel transform of the differential equation (12) and keeping in mind the boundary condition (13) and following identities

$$\frac{d}{dr}A(r,r_n)=r_n\left[J_0(rr_n)Y_1(R_2r_n)-J_1(R_2r_n)Y_0(rr_n)\right]$$
$$-\frac{1}{r}A(r,r_n), \tag{16}$$

$$J_0(z)Y_1(z)-J_1(z)Y_0(z)=-\frac{2}{\pi z}, \tag{17}$$

we get

$$\overline{\omega}_H(r_n,q)=2\nu\frac{(1+\lambda_r^{\varsigma}q^{\varsigma})[R_2\mho_2J_1(R_1r_n)-R_1\mho_1J_1(R_2r_n)]}{\pi J_1(R_1r_n)}$$
$$\frac{1}{q^{\kappa+1}(q+\nu r_n^2+\lambda^{\varsigma}q^{\varsigma+1}+\nu\lambda_r^{\varsigma}r_n^2q^{\varsigma})}. \tag{18}$$

Writing the above equations in a suitable form

$$\overline{\omega}_H(r_n,q)=\frac{2[R_2\mho_2J_1(R_1r_n)-R_1\mho_1J_1(R_2r_n)]}{\pi r_n^2J_1(R_1r_n)}$$
$$\left[\frac{1}{q^{\kappa+1}}-\frac{1+\lambda^{\varsigma}q^{\varsigma}}{q^{\kappa}(q+\nu r_n^2+\lambda^{\varsigma}q^{\varsigma+1}+\nu\lambda_r^{\varsigma}r_n^2q^{\varsigma})}\right], \tag{19}$$

The inverse Hankel transform of $\overline{\omega}_H(r_n,q)$ [34]

$$\overline{\omega}(r,q)=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2J_1^2(R_1r_n)A(r,r_n)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\overline{\omega}_H(r_n,q). \tag{20}$$

Using the inverse Hankel transform on Eq. (19), and the identity

$$\frac{R_2(r^2-R_1^2)}{(R_2^2-R_1^2)r}=\pi\sum_{n=1}^{\infty}\frac{J_1^2(R_1r_n)A(r,r_n)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)},$$

we obtain

$$\overline{\omega}(r,q)=\frac{\mho_1R_1^2(R_2^2-r^2)+\mho_2R_2^2(r^2-R_1^2)}{r(R_2^2-R_1^2)}\frac{1}{q^{\kappa+1}}$$
$$-\pi\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)A(r,r_n)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\frac{1+\lambda^{\varsigma}q^{\varsigma}}{q\kappa}$$
$$\times\frac{\left(\mho_2R_2J_1(R_1r_n)-\mho_1R_1J_1(R_2r_n)\right)}{(q+\nu r_n^2+\lambda^{\varsigma}q^{\varsigma+1}+\nu\lambda_r^{\varsigma}r_n^2q^{\varsigma})}. \tag{21}$$

Using the identity

$$\frac{1}{q+\nu r_n^2+\lambda^{\varsigma}q^{\varsigma+1}+\nu\lambda_r^{\varsigma}r_n^2q^{\varsigma}}=\frac{1}{\lambda^{\varsigma}}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}$$
$$\times\left(-\frac{\nu r_n^2}{\lambda^{\varsigma}}\right)^k\lambda_r^{\varsigma m}\frac{q^{\varsigma m-k-1}}{\left(q^{\varsigma}+\frac{1}{\lambda^{\varsigma}}\right)^{k+1}}, \tag{22}$$

Eq. (21) can be further simplified to give

$$\overline{\omega}(r,q)=\frac{\mho_1R_1^2(R_2^2-r^2)+\mho_2R_2^2(r^2-R_1^2)}{r(R_2^2-R_1^2)}\frac{1}{q^{\kappa+1}}-\frac{\pi}{\lambda^{\varsigma}}$$
$$\times\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)A(r,r_n)\left(\mho_2R_2J_1(R_1r_n)-\mho_1R_1J_1(R_2r_n)\right)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}$$
$$\times\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\left(-\frac{\nu r_n^2}{\lambda^{\varsigma}}\right)^k\lambda_r^{\varsigma m}$$
$$\times\frac{q^{\varsigma m-k-\kappa-1}+\lambda q^{\varsigma+\varsigma m-k-\kappa-1}}{\left(q^{\varsigma}+\frac{1}{\lambda^{\varsigma}}\right)^{k+1}}. \tag{23}$$

Now using inverse Laplace transform on Eq. (23) and taking into account (A_1) [35] from the Appendix, we find that

$$\omega(r,t)=\frac{\mho_1R_1^2(R_2^2-r^2)+\mho_2R_2^2(r^2-R_1^2)}{r(R_2^2-R_1^2)}t^{\kappa}-\frac{\pi}{\lambda^{\varsigma}}$$

$$\sum_{n=1}^{\infty} \frac{J_1(R_1 r_n) A(r, r_n) \left(\mho_2 R_2 J_1(R_1 r_n) - \mho_1 R_1 J_1(R_2 r_n)\right)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)}$$
$$\sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} \left(-\frac{\nu r_n^2}{\lambda^\varsigma}\right)^k \lambda_r^{\varsigma m}$$
$$\times \left[G_{\varsigma, \varsigma m-k-\kappa-1, k+1}\left(-\lambda^{-1}, t\right)\right.$$
$$\left. + \lambda G_{\varsigma, \varsigma+\varsigma m-k-\kappa-1, k+1}\left(-\lambda^{-1}, t\right)\right]. \tag{24}$$

3.2 Calculation of the shear stress

Using the Laplace transform, Eq. (5) implies that

$$\overline{\tau}(r, q) = \mu \frac{1 + \lambda_r^\varsigma q^\varsigma}{1 + \lambda^\varsigma q^\varsigma} \left(\frac{\partial}{\partial r} - \frac{1}{r}\right) \overline{\omega}(r, q). \tag{25}$$

Using Eqs. (21) in the above equation we get

$$\overline{\tau}(r, q) = \mu \frac{1 + \lambda_r^\varsigma q^\varsigma}{1 + \lambda^\varsigma q^\varsigma} \frac{2 R_1^2 R_2^2}{(R_2^2 - R_1^2) r^2} \frac{\mho_2 - \mho_1}{q^{\kappa+1}}$$
$$+ \frac{\mu \pi}{\lambda^\varsigma} \sum_{n=1}^{\infty} \frac{\left[\frac{2}{r} A(r, r_n) - r_n A^*(r, r_n)\right] J_1(R_1 r_n)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)}$$
$$\times \left(\mho_2 R_2 J_1(R_1 r_n) - \mho_1 R_1 J_1(R_2 r_n)\right) \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{k!}{m!(k-m)!}$$
$$\times \left(-\frac{\nu r_n^2}{\lambda^\varsigma}\right)^k \lambda_r^{\varsigma m} \frac{(q^{\varsigma m-k-\kappa-1} + \lambda_r^\varsigma q^{\varsigma+\varsigma m-k-\kappa-1})}{\left(q^\varsigma + \frac{1}{\lambda^\varsigma}\right)^{k+1}}\Bigg], \tag{26}$$

where

$$A^*(r, r_n) = J_1(r r_n) Y_0(R_2 r_n) - J_0(R_2 r_n) Y_1(r r_n). \tag{27}$$

Now, using the inverse Laplace transform on Eq. (26), finally the shear stress

$$\tau(r, t) = \frac{2\mu R_1^2 R_2^2 (\mho_2 - \mho_1)}{\lambda^\varsigma (R_2^2 - R_1^2) r^2} \left[R_{\varsigma, -\kappa-1}\left(-\lambda^{-\varsigma}, t\right)\right.$$
$$\left. + \lambda_r^\varsigma R_{\varsigma, \varsigma-\kappa-1}\left(-\lambda^{-\varsigma}, t\right)\right] + \frac{\mu \pi}{\lambda^\varsigma}$$
$$\times \sum_{n=1}^{\infty} \frac{\left[\frac{2}{r} A(r, r_n) - r_n A^*(r, r_n)\right] J_1(R_1 r_n)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} (\mho_2 R_2 J_1(R_1 r_n)$$
$$- \mho_1 R_1 J_1(R_2 r_n)) \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} \left(-\frac{\nu r_n^2}{\lambda^\varsigma}\right)^k$$
$$\times \lambda_r^{\varsigma m} \left[G_{\varsigma, \varsigma m-k-\kappa-1, k+1}\left(-\lambda^{-\varsigma}, t\right)\right.$$
$$\left. + \lambda_r^\varsigma G_{\varsigma, \varsigma+\varsigma m-k-\kappa-1, k+1}\left(-\lambda^{-\varsigma}, t\right)\right]. \tag{28}$$

4 Limiting cases

1. Letting $\varsigma \to 1$ into Eqs. (24) and (28) the velocity field

$$\omega(r, t) = \frac{\mho_1 R_1^2 (R_2^2 - r^2) + \mho_2 R_2^2 (r^2 - R_1^2)}{r(R_2^2 - R_1^2)} t^\kappa - \frac{\pi}{\lambda}$$
$$\times \sum_{n=1}^{\infty} \frac{J_1^2(R_1 r_n) A(r, r_n) \left(\mho_2 R_2 J_1(R_1 r_n) - \mho_1 R_1 J_1(R_2 r_n)\right)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)}$$
$$\times \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} \left(-\frac{\nu r_n^2}{\lambda}\right)^k$$
$$\times \lambda_r^m \left[G_{1, m-k-\kappa-1, k+1}\left(-\lambda^{-1}, t\right)\right.$$
$$\left. + \lambda G_{1, 1+m-k-\kappa-1, k+1}\left(-\lambda^{-1}, t\right)\right], \tag{29}$$

and the shear stress

$$\tau(r, t) = \frac{2\mu R_1^2 R_2^2 (\mho_2 - \mho_1)}{\lambda (R_2^2 - R_1^2) r^2} \left[R_{1, -\kappa-1}\left(-\lambda^{-1}, t\right) + \lambda_r\right.$$
$$\left. \times R_{1, -\kappa}\left(-\lambda^{-1}, t\right)\right] + \frac{\mu \pi}{\lambda}$$
$$\times \sum_{n=1}^{\infty} \frac{\left[\frac{2}{r} A(r, r_n) - r_n A^*(r, r_n)\right] J_1(R_1 r_n)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)}$$
$$\times \left(\mho_2 R_2 J_1(R_1 r_n) - \mho_1 R_1 J_1(R_2 r_n)\right) \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{k!}{m!(k-m)!}$$
$$\times \left(-\frac{\nu r_n^2}{\lambda}\right)^k \lambda_r^m \left[G_{1, m-k-\kappa-1, k+1}\left(-\lambda^{-1}, t\right)\right.$$
$$\left. + \lambda_r G_{1, 1+m-k-\kappa-1, k+1}\left(-\lambda^{-1}, t\right)\right], \tag{30}$$

for the ordinary Oldroyd-B fluid.

2. Making $\lambda_r \to 0$ in Eqs. (24) and (28), the velocity field

$$\omega(r, t) = \frac{\mho_1 R_1^2 (R_2^2 - r^2) + \mho_2 R_2^2 (r^2 - R_1^2)}{r(R_2^2 - R_1^2)} t^\kappa$$
$$- \frac{\pi}{\lambda^\varsigma} \sum_{n=1}^{\infty} \frac{J_1(R_1 r_n) A(r, r_n)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} \left\{\mho_2 R_2 J_1(R_1 r_n)\right.$$
$$\left. - \mho_1 R_1 J_1(R_2 r_n)\right\} \sum_{k=0}^{\infty} \left(\frac{-\nu r_n^2}{\lambda^\varsigma}\right)^k$$
$$\times \left\{G_{\varsigma, -k-\kappa-1, k+1}\left(-\lambda^{-\varsigma}, t\right)\right.$$
$$\left. + \lambda G_{\varsigma, \varsigma-k-\kappa-1, k+1}\left(-\lambda^{-\varsigma}, t\right)\right\}, \tag{31}$$

and the shear stress

$$\tau(r, t) = \frac{2\mu R_1^2 R_2^2 (\mho_2 - \mho_1)}{\lambda r^2 (R_2^2 - R_1^2)} R_{\varsigma, -2}\left(-\lambda^{-1}, t\right)$$
$$+ \frac{\pi \mu}{\lambda^\varsigma} \sum_{n=1}^{\infty} \frac{J_1(R_1 r_n) \left(\frac{2}{r} A(r, r_n) - r_n A^*(r, r_n)\right)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} \times$$
$$\times \left\{\mho_2 R_2 J_1(R_1 r_n) - \mho_1 R_1 J_1(R_2 r_n)\right\}$$
$$\times \sum_{k=0}^{\infty} \left(\frac{-\nu r_n^2}{\lambda^\varsigma}\right)^k G_{\varsigma, -k-2, k+1}\left(-\lambda^{-\varsigma}, t\right), \tag{32}$$

corresponding to a fractional Maxwell fluid.

3. Making $\varsigma \to 1$ in Eqs. (31) and (32), the velocity field is

$$\omega(r, t) = \frac{\mho_1 R_1^2 (R_2^2 - r^2) + \mho_2 R_2^2 (r^2 - R_1^2)}{r(R_2^2 - R_1^2)} t^\kappa$$

$$-\frac{\pi}{\lambda\varsigma}\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)A(r,r_n)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\{\mho_2R_2J_1(R_1r_n)-\mho_1R_1J_1(R_2r_n)\}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\times\left\{G_{1,-k-\kappa-1,k+1}\left(-\lambda^{-1},t\right)+\lambda G_{1,1-k-\kappa-1,k+1}\left(-\lambda^{-1},t\right)\right\}, \quad (33)$$

and the shear stress is

$$\tau(r,t)=\frac{2\mu R_1^2R_2^2(\mho_2-\mho_1)}{\lambda r^2(R_2^2-R_1^2)}R_{\varsigma,-2}\left(-\lambda^{-1},t\right)+\frac{\pi\mu}{\lambda\varsigma}\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)\left(\frac{2}{r}A(r,r_n)-r_n\,A^{*}(r,r_n)\right)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\times\{\mho_2R_2J_1(R_1r_n)-\mho_1R_1J_1(R_2r_n)\}\times\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^kG_{1,-k-2,k+1}\left(-\lambda^{-1},t\right), \quad (34)$$

for the ordinary Maxwell fluid.

Using Appendix A_2, the expressions (33) and (34) in the simplified form for $\kappa=1$, in terms of expositional functions, are

$$\omega_M(r,t)=\frac{\mho_1R_1^2(R_2^2-r^2)+\mho_2R_2^2(r^2-R_1^2)}{r(R_2^2-R_1^2)}t-\frac{\pi}{\nu}\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)A(r,r_n)}{r_n^2[J_1^2(R_1r_n)-J_1^2(R_2r_n)]}\{\mho_2R_2J_1(R_1r_n)-\mho_1R_1J_1(R_2r_n)\}\left\{1-\lambda\frac{q_{1n}^2e^{q_{2n}t}-q_{2n}^2e^{q_{1n}t}}{q_{2n}-q_{1n}}\right\}, \quad (35)$$

and

$$\tau_M(r,t)=\frac{2\mu R_1^2R_2^2(\mho_2-\mho_1)}{r^2(R_2^2-R_1^2)}\left\{t-\lambda\left(1-e^{-t/\lambda}\right)\right\}+\pi\rho\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)\left(\frac{2}{r}A(r,r_n)-r_nA^{*}(r,r_n)\right)}{r_n^2[J_1^2(R_1r_n)-J_1^2(R_2r_n)]}\times\{\mho_2R_2J_1(R_1r_n)-\mho_1R_1J_1(R_2r_n)\}\times\left\{1+\frac{q_{1n}e^{q_{2n}t}-q_{2n}e^{q_{1n}t}}{q_{2n}-q_{1n}}\right\}. \quad (36)$$

4. By now letting $\lambda\to 0$ in Eqs. (35) and (36) or $\varsigma\to 1$ and $\lambda\to 0$ in Eqs. (31) and (32) and using $\lim_{\lambda\to 0}\frac{1}{\lambda^k}G_{1,b,k}\left(\frac{-1}{\lambda},t\right)=\frac{t^{-b-1}}{\Gamma(-b)}$; $b<0$, the velocity field is

$$\omega_N(r,t)=\frac{\mho_1R_1^2(R_2^2-r^2)+\mho_2R_2^2(r^2-R_1^2)}{r(R_2^2-R_1^2)}t-\frac{\pi}{\nu}\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)A(r,r_n)}{r_n^2[J_1^2(R_1r_n)-J_1^2(R_2r_n)]}\{\mho_2R_2J_1(R_1r_n)-\mho_1R_1J_1(R_2r_n)\}\left\{1-e^{-\nu r_n^2t}\right\}, \quad (37)$$

and the shear stress is

$$\tau_N(r,t)=\frac{2\mu R_1^2R_2^2(\mho_2-\mho_1)}{r^2(R_2^2-R_1^2)}t+\pi\rho\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)\left(\frac{2}{r}A(r,r_n)-r_nA^{*}(r,r_n)\right)}{r_n^2[J_1^2(R_1r_n)-J_1^2(R_2r_n)]}\times\{\mho_2R_2J_1(R_1r_n)-\mho_1R_1J_1(R_2r_n)\}\left\{1-e^{-\nu r_n^2t}\right\}, \quad (38)$$

corresponding to a Newtonian fluid.

5 Flow through a circular pipe

The finite Hankel transform of the function $f(r,t)$ is

$$f_H(r_n')=\int_0^{R_2}r\,\omega(r)J_1(rr_n')dr, \quad (39)$$

and the inverse Hankel transform of $f_H(r_n')$ is

$$\omega(r)=\frac{2}{R_2^2}\sum_{n=1}^{\infty}\frac{J_1(rr_n')}{J_2^2(R_2r_n')}f_H(r_n'). \quad (40)$$

where $J_1(R_2r)=0$ as r_n are the positive roots of the equation.

By making $J_1(R_2r_n)=0$, and using the identities $J_0(z)+J_2(z)=\frac{2}{z}J_1(z)$ in Eq. (15) we get

$$A(r,r_n)=\frac{2J_1(rr_n)}{\pi R_2r_nJ_2(R_2r_n)}, \quad (41)$$

Writing the Integral

$$\int_{R_1}^{R_2}rf(r,t)A(r,r_n)dr=\int_0^{R_2}rf(r,t)A(r,r_n)dr-\int_0^{R_1}rf(r,t)A(r,r_n)dr. \quad (42)$$

Introducing Eq. (41) and taking $R_1=0$, the above integral takes the form

$$\int_{R_1}^{R_2}rf(r,t)A(r,r_n)dr=\frac{2}{\pi R_2r_nJ_2(R_2r_n)}\int_0^{R_2}rf(r,t)J_1(rr_n)dr, \quad (43)$$

as a result from (24) and (28) we recovered the expansions for the velocity field

$$\omega(r,t)=\mho_2rt-\frac{2\mho_2}{\lambda}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_nJ_2(R_2r_n)}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}$$

$$\times\left(-\frac{\nu r_n^2}{\lambda}\right)^k \lambda_r^m \left[G_{\varsigma,\varsigma m-k-2,k+1}\left(-\lambda^{-1},\ t\right)\right.$$
$$\left.+\lambda G_{\varsigma,\varsigma+\varsigma m-k-2,k+1}\left(-\lambda^{-1},\ t\right)\right], \tag{44}$$

and

$$\tau(r,t)=\frac{2\mu\pi\mho_2}{\lambda}\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{J_2(R_2r_n)}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}$$
$$\times\left(-\frac{\nu r_n^2}{\lambda}\right)^k \lambda_r^m \left[G_{\varsigma,\varsigma m-k-2,\,k+1}\left(-\lambda^{-1},t\right)\right.$$
$$\left.+\lambda_r G_{\varsigma,\varsigma+\varsigma m-k-2,\,k+1}\left(-\lambda^{-1},t\right)\right], \tag{45}$$

through a cylinder already obtained by Kamran *et al.* [31] in Equation (21) and (27) for $p = 1$.

6 Conclusions and results

In this article, exact solutions for a fractional Oldroyd-B fluid between two rotating pipes are determined. Fluid motion is produced due to rotation of the pipes around their axis with time dependent angular velocities. The solutions determined by the use of integral transforms and presented in terms of Bessel functions and hypergeometric functions which are free of integrals. The final result satisfies the initial and boundary conditions. In the limiting cases, the corresponding results for Oldroyd-B fluid, Maxwell and Newtonian fluid, are obtained from general results. Moreover, the solution for the fluid through a circular pipe is obtained as a special case.

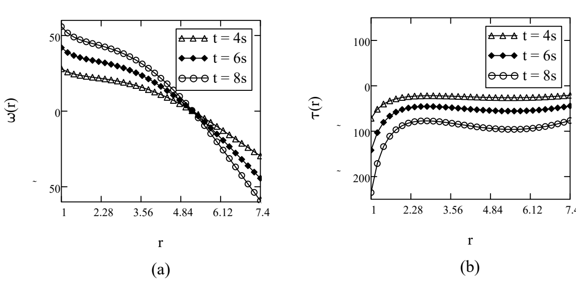

Figure 2: Profiles of the velocity field $\omega(r, t)$ given by Eqs. (24) and (28) for $R_1 = 1$, $R_2 = 7.4$, $\mho_1 = 1$, $\mho_2 = -1$, $\lambda = 4$, $\lambda_r = 1$, $\nu = 0.002$, $\varsigma = 0.1$ and different values of time

.

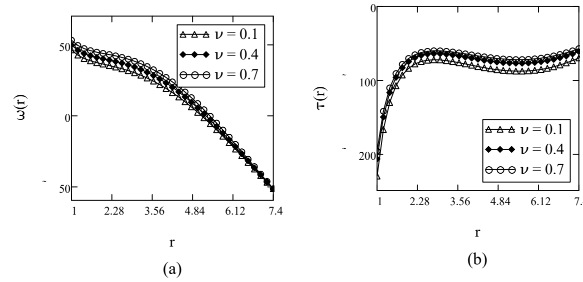

Figure 3: Profiles of the velocity field $\omega(r, t)$ given by Eqs. (24) and (28) for $t = 7$, $R_1 = 1$, $R_2 = 7.4$, $\mho_1 = 1$, $\mho_2 = -1$, $\lambda = 20$, $\lambda_r = 5$, $\varsigma = 0.9$ and different values of ν

.

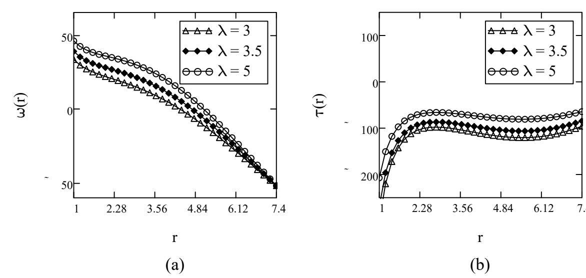

Figure 4: Profiles of the velocity field $\omega(r, t)$ given by Eqs. (24) and (28) for $t = 7$, $R_1 = 1$, $R_2 = 7.4$, $\mho_1 = 1$, $\mho_2 = -1$, $\nu = 0.2$, $\lambda_r = 1$, $\varsigma = 0.5$ and different values of λ

.

The velocity profiles and shear stress are illustrated in Figures 2 — 7 for different values of parameters. Fig. 2 for the effect of t, Fig. 3 for the effect of ν, Fig. 4 for the effect of λ, Fig. 5 for the effect of λ_r and Fig. 6 shows the effect of the fractional parameter ς on the fluid motion. Clearly, the velocity as well as shear stress are increasing as a function of t and ν, The effect is qualitatively the same for fractional parameter ς and λ, more exactly the velocity $\omega(r, t)$ is a decreasing function with regards to ς.

Finally, Fig. 7 is the comparison of $\omega(r, t)$ corresponding to the motion of the Newtonian, fractional and ordinary Maxwell, and fractional and ordinary Oldroyd-B for the same values of time and for the same material and fractional parameters. From this comparison, the Oldroyd-B fluid is the slowest and the Newtonian fluid is the fastest on the whole flow domain. SI units are used for all figures, and the roots $r_n = \frac{n\pi}{(R_2-R_1)}$.

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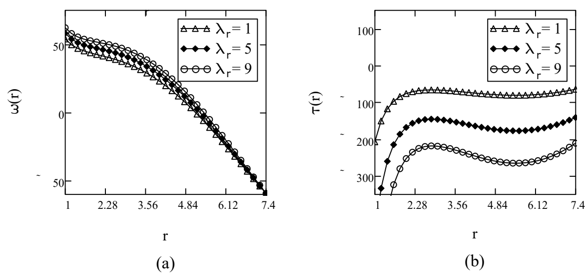

Figure 5: Profiles of the velocity field $\omega(r, t)$ given by Eqs. (24) and (28) for $t = 8,\ R_1 = 1,\ R_2 = 7.4,\ \mho_1 = 1,\ \mho_2 = -1,\ \nu = 0.2,\ \lambda = 10,\ \varsigma = 0.5$ and different values of λ_r

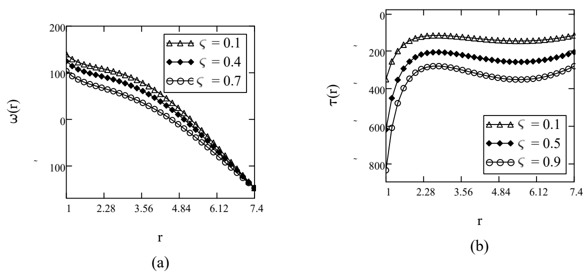

Figure 6: Profiles of the velocity field $\omega(r, t)$ given by Eqs. (24) and (28) for $t = 20,\ R_1 = 1,\ R_2 = 7.4,\ \mho_1 = 1,\ \mho_2 = -1,\ \nu = 0.001,\ \lambda = 4,\ \lambda_r = 1$ and different values of ς

Appendix

$$L^{-1}\left\{\frac{q^b}{(q^a - d)^c}\right\} = G_{a,b,c}(d, t) = \sum_{j=0}^{\infty} \frac{d^j\, \Gamma(c + j)}{\Gamma(c)\Gamma(j + 1)} \times \frac{t^{(c+j)a-b-1}}{\Gamma[(c + j)a - b]},\ \mathrm{Re}(ac - b) > 0,\ \left|\frac{d}{q^a}\right| < 1, \qquad (A_1)$$

$$\sum_{k=0}^{\infty} \left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{1,-k-1,k+1}\left(-\lambda^{-1}, t\right) = \frac{e^{q_{2n}t} - e^{q_{1n}t}}{q_{2n} - q_{1n}}, \qquad (A_2)$$

$$\sum_{k=0}^{\infty} \left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{1,-k-2,k+1}\left(-\lambda^{-1}, t\right) \times = \frac{\lambda}{\nu r_n^2}\left(1 + \frac{q_{1n}e^{q_{2n}t} - q_{2n}e^{q_{1n}t}}{q_{2n} - q_{1n}}\right), \qquad (A_3)$$

$$R_{1,-2}\left(-\lambda^{-1}, t\right) = \lambda t - \lambda^2\left(1 - e^{-t/\lambda}\right); \quad q_{1n},\ q_{2n} = \frac{-1 \pm \sqrt{1 - 4\nu\lambda r_n^2}}{2\lambda}. \qquad (A_4)$$

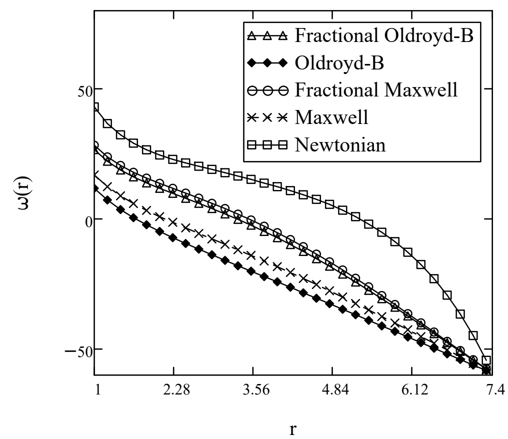

Figure 7: Profiles of the velocity field $\omega(r, t)$ corresponding to the Newtonian, Maxwell, Fractional Maxwell, Oldroyd-B and fractional Oldroyd-B fluids, for $t = 8,\ R_1 = 1,\ R_2 = 7.4,\ \mho_1 = 1,\ \mho_2 = -1,\ \nu = 0.001,\ \lambda = 3.6,\ \lambda_r = 3$ and $\varsigma = 0.8$

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