Open Phys. 2018; 16:174–182 DE GRUYTER

Research Article

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New type of chaos synchronization in discrete-time systems: the F-M synchronization

https://doi.org/10.1515/phys-2018-0025
Received Nov 21, 2017; accepted Jan 31, 2018

Abstract: In this paper, a new type of synchronization for chaotic (hyperchaotic) maps with different dimensions is proposed. The novel scheme is called $\mathbf{F}-\mathbf{M}$ synchronization, since it combines the inverse generalized synchronization (based on a functional relationship $\mathbf{F}$) with the matrix projective synchronization (based on a matrix $\mathbf{M}$). In particular, the proposed approach enables $\mathbf{F}-\mathbf{M}$ synchronization with index d to be achieved between n-dimensional drive system map and m-dimensional response system map, where the synchronization index d corresponds to the dimension of the synchronization error. The technique, which exploits nonlinear controllers and Lyapunov stability theory, proves to be effective in achieving the $\mathbf{F}-\mathbf{M}$ synchronization not only when the synchronization index d equals n or m, but even if the synchronization index d is larger than the map dimensions n and m. Finally, simulation results are reported, with the aim to illustrate the capabilities of the novel scheme proposed herein.

Keywords: Inverse generalized synchronization, matrix projective synchronization, chaotic systems, discrete-time systems, Lyapunov stability theory

PACS: 05.45.Gg, 05.45.Xt

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1 Introduction

The idea of synchronizing two identical chaotic systems that start from different initial conditions was introduced by Pecora and Carroll [1]. Using a transmitted signal, they showed that synchronization occurs when the Lyapunov exponents for the response subsystem are all negative. Since that pioneering paper, the issue of chaos synchronization has attracted great interest in nonlinear science and engineering [2–5, 34–45]. At the beginning, by modifying the Pecora-Carroll scheme, most of the methods have focused on complete (identical) synchronization [3, 4]. Subsequently, different types of synchronization have been proposed in the literature, for both continuous-time systems and discrete-time systems [7–17]. Among these, projective synchronization provides response system variables that are scaled replicas of the drive system variables [6, 18–21].

A variation of projective synchronization is the so-called full state hybrid projective synchronization [22–25]. In this type of synchronization the scaling factor can be different for each state variable, meaning that the single scaling parameter (originally introduced in Ref. [18]) is replaced by a diagonal scaling matrix. On the other hand, when the scaling matrix is a full matrix, the so-called matrix projective synchronization is achieved [26–29].

Another interesting approach is represented by generalized synchronization, where the drive system and the response system are non-identical dynamical systems [30–32]. This type of synchronization is characterized by the existence of a functional relationship $\mathbf{F}$ between the state of the drive system and the state y of the response system, so that $y = \mathbf{F}(x)$ after a transient time. A variation is represented by the inverse generalized synchronization, where the synchronization condition becomes $x = \mathbf{F}(x)$ after a transient time [33].

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By exploiting previous considerations and by taking into account the concepts of inverse generalized synchronization and matrix synchronization, a new type of synchronization for chaotic maps with different dimensions is proposed in this paper. Specifically, given n-dimensional drive system map and m-dimensional response system map, the novel scheme is called $\mathbf{F} - \mathbf{M}$ synchronization with index d since it combines the inverse generalized synchronization (based on a functional relationship $\mathbf{F}$), with the matrix projective synchronization (based on a matrix $\mathbf{M}$). Note that the synchronization index d corresponds to the dimension of the synchronization error. The technique, which exploits nonlinear controllers and Lyapunov stability theory, proves to be effective in achieving the $\mathbf{F}-\mathbf{M}$ synchronization not only when the synchronization index d equals n or m, but even if the synchronization index d is larger than the map dimensions n and m.

The paper is organized as follows. In Section 2, the $\mathbf{F} - \mathbf{M}$ synchronization with index d is defined. In Section 3, three different theorems are provided, which cover three different $\mathbf{F} - \mathbf{M}$ synchronization cases, with indices $d = m$, $d = n$ and d larger than m and n, respectively. Finally, in order to show the capabilities of the conceived synchronization schemes, Section 4 illustrates the $\mathbf{F} - \mathbf{M}$ synchronization between the two-dimensional Fold map and the three-dimensional generalized Hénon map, when the synchronization indices are $d = 3$, $d = 2$ and $d = 4$, indicating that the method is effective in achieving synchronization even if the synchronization index d is larger than the map dimensions n and m.

2 F – M synchronization with index *d*

The drive and the response chaotic systems considered herein are in the following forms

$$X(k+1) = AX(k) + f\big(X(k)\big) \tag{1}$$

$$Y(k+1) = BY(k) + g\big(Y(k)\big) + U, \tag{2}$$

where $X(k) \in \mathbb{R}^n$ and $Y(k) \in \mathbb{R}^m$ are state vectors of the drive and slave systems, respectively, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ are the linear parts of the drive system and response system, respectively, $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and $g\colon \mathbb{R}^m \to \mathbb{R}^m$ are the nonlinear parts of the drive system and response system, respectively, and $U \in \mathbb{R}^m$ is a vector controller.

Before introducing the new concept of $\mathbf{F} - \mathbf{M}$ synchronization for the drive system (1) and the response system (2), the definitions of matrix projective synchronization (MPS) and inverse generalized synchronization (IGS) are provided.

Definition 2.1. *Matrix projective synchronization is said to be achieved between the n-dimensional drive system $X(k)$ and m-dimensional response system $Y(k)$ if there exists a controller $U = (u_i)_{1\le i\le m}$ and an $m\times n$ matrix $\mathbf{M}$ such that the synchronization error*

$$e(k) = Y(k) - \mathbf{M}X(k) \tag{3}$$

satisfies the condition $\lim_{k\to\infty} \|e(k)\| = 0$.

Definition 2.2. *Inverse generalized synchronization is said to be achieved between the n-dimensional drive system $X(k)$ and m-dimensional response system $Y(k)$ if there exists a controller $U = (u_i)_{1\le i\le m}$ and a map $\mathbf{F}\colon \mathbb{R}^m \to \mathbb{R}^n$ such that the synchronization error*

$$e(k) = \mathbf{F}\big(Y(k)\big) - X(k) \tag{4}$$

satisfies the condition $\lim_{k\to\infty} \|e(k)\| = 0$.

Next, we introduce a new type of synchronization.

Definition 2.3. *The n-dimensional drive system* (1) *and m-dimensional response system* (2) *with state vectors $X(k)$ and $Y(k)$, respectively, achieve $\mathbf{F} - \mathbf{M}$ synchronization if there exists a controller $U = (u_i)_{1\le i\le m}$, a map $\mathbf{F}\colon \mathbb{R}^m \to \mathbb{R}^d$ and a $d\times n$ matrix $\mathbf{M}$ such that the synchronization error*

$$e(k) = \mathbf{F}\big(Y(k)\big) - \mathbf{M}X(k) \tag{5}$$

satisfies the condition $\lim_{k\to\infty} \|e(k)\| = 0$. *The constant d is called the synchronization index and represents the dimension of the $\mathbf{F} - \mathbf{M}$ synchronization.*

3 F – M synchronization criterions

In this section, different cases of synchronization are analyzed. The first case is when the synchronization index d equals m, *i.e.*, the dimension of the response system. The second case is when the synchronization index d equals n, *i.e.*, the dimension of the drive system. Finally, the case when the synchronization index d is larger than both m and n is analyzed in details.

3.1 Case: $d = m$

Theorem 3.1. *Given the drive system* (1), *the response system* (2), *an invertible function $\mathbf{F}\colon \mathbb{R}^m \to \mathbb{R}^m$ and a matrix*

$\mathbf{M} \in \mathbb{R}^{m\times n}$, $\mathbf{F}-\mathbf{M}$ *synchronization with index m is achieved if the controller U in* (2) *is given by*

$$U = -BY(k) - g\big(Y(k)\big) + \mathbf{F}^{-1}(-R_1), \qquad (6)$$

where $\mathbf{F}^{-1}$ *is the inverse function of* $\mathbf{F}$ *and*

$$R_1 = (L_1 - B)e(k) - \mathbf{M}\bigg(AX(k) + f\big(X(k)\big)\bigg), \qquad (7)$$

provided that $L_1 \in \mathbb{R}^{m\times m}$ *is chosen such that the eigenvalues of the matrix* $(B - L_1)$ *are placed strictly inside the unit disk.*

Proof. According to definition 2.3, the error system between the drive system (1) and the response system (2) can be derived as follows:

$$\begin{aligned} e(k+1) = {} & (B - L_1)e(k) \\ & + \mathbf{F}\bigg(BY(k) + g\big(y(k)\big) + U\bigg) + R_1. \end{aligned} \qquad (8)$$

By substituting the control law (6) along with (7), the error system (8) reduces to

$$e(k+1) = (B - L_1)e(k). \qquad (9)$$

If the eigenvalues of the matrix $(B - L_1)$ are placed strictly inside the unit disk, then, from the asymptotic stability theory of linear discrete-time systems, it follows that all the solutions of the error system (9) go to zero as $k \to \infty$. Therefore, the systems (1) and (2) are globally $\mathbf{F}-\mathbf{M}$ synchronized with index m. □

3.2 Case: $d = n$

In this case, the synchronization index d is taken as the dimension of the drive system n.

Theorem 3.2. $\mathbf{F}-\mathbf{M}$ *synchronization between the drive system* (1) *and the response system* (2) *will be achieved if the controller U is selected as follows:*

$$U = -BY(k) - g\big(Y(k)\big) + \mathbf{F}^{-1}(-R_2), \qquad (10)$$

where $\mathbf{F}^{-1}: \mathbb{R}^n \to \mathbb{R}^m$ *is the inverse function of* $\mathbf{F}: \mathbb{R}^m \to \mathbb{R}^n$ *and*

$$R_2 = (L_2 - A)e(k) - \mathbf{M}\bigg(AX(k) + f\big(X(k)\big)\bigg), \qquad (11)$$

provided that $\big((A - L_2)^T(A - L_2) - I\big)$ *is a negative definite matrix and* L_2 *and* $\mathbf{M}$ *two matrices of dimension* $(n \times n)$.

Proof. The error system between the drive system (1) and the response system (2) given in (5) can be rewritten as follows:

$$e(k+1) = (A - L_2)e(k) + \mathbf{F}\bigg(BY(k) + g\big(Y(k)\big) + U\bigg) + R_2. \qquad (12)$$

By substituting the control law (10) into (12) along with (11), the error system becomes

$$e(k+1) = (A - L_2)e(k). \qquad (13)$$

By constructing the candidate Lyapunov function in the form $V\big(e(k)\big) = e^{\mathrm{T}}(k)e(k)$, it follows that

$$\begin{aligned} \Delta V\big(e(k)\big) &= e^{\mathrm{T}}(k+1)e(k+1) - e^{\mathrm{T}}(k)e(k) \\ &= e^{\mathrm{T}}(k)(A - L_2)^{\mathrm{T}}(A - L_2)e(k) - e^{\mathrm{T}}(k)e(k) \\ &= e^{\mathrm{T}}(k)\big[(A - L_2)^{\mathrm{T}}(A - L_2) - I\big]e(k). \end{aligned}$$

Since by assumption, $\big((A - L_2)^T(A - L_2) - I\big)$ is a negative definite matrix, it follows that $\Delta V\big(e(k)\big) < 0$. Thus, from the Lyapunov stability theory, the zero solution of the error system (13) is globally asymptotically stable, *i.e.*, $\lim_{k\to\infty} \big\|e(k)\big\| = 0$. Consequently, systems (1) and (2) are globally $\mathbf{F}-\mathbf{M}$ synchronized with index n. □

3.3 Case: $d \neq n, m$

In this case, the synchronization index d assumes a value different from that of both n and m.

Theorem 3.3. $\mathbf{F}-\mathbf{M}$ *synchronization between the drive system* (1) *and the response system* (2) *will occur if the controller U is selected as:*

$$U = -BY(k) - g\big(Y(k)\big) + \mathbf{F}^{-1}(-R_3), \qquad (14)$$

where R_3 *is given by*

$$R_3 = -L_3e(k) - \mathbf{M}\bigg(AX(k) + f\big(X(k)\big)\bigg), \qquad (15)$$

with $L_3 = diag(l_1, l_2, \ldots, l_d)$, $\mathbf{F}: \mathbb{R}^m \to \mathbb{R}^d$, $\mathbf{F}^{-1}: \mathbb{R}^d \to \mathbb{R}^m$ *and* $\mathbf{M} \in \mathbb{R}^{d\times n}$, *provided that all the elements* l_i *satisfy the condition*

$$0 < |l_i| < 1, \qquad i = 1, 2, \ldots, d. \qquad (16)$$

Proof. The error system between systems (1) and (2) can be written as:

$$e(k+1) = L_3e(k) + \mathbf{F}\bigg(BY(k) + g\big(Y(k)\big) + U\bigg) + R_3. \qquad (17)$$

By substituting the control law (14), along with (15), the error system becomes

$$e(k+1) = L_3 e(k). \tag{18}$$

We consider the quadratic Lyapunov function $V(e(k)) = \sum_{i=1}^{d} e_i^2(k)$. It follows that

$$\begin{aligned}\Delta V(e(k)) &= V(e(k+1)) - V(e(k)) \\ &= \sum_{i=1}^{d} e_i^2(k+1) - \sum_{i=1}^{d} e_i^2(k) = \sum_{i=1}^{d} (l_i^2 - 1) e_i^2(k).\end{aligned}$$

By condition (16), $\Delta V(e(k)) < 0$. Thus, from the Lyapunov stability theory, it follows that $\lim_{k\to\infty} e_i(k) = 0$ for $i = 1, 2, \ldots, d$ and hence, systems (1) and (2) are globally $\mathbf{F}-\mathbf{M}$ synchronized with dimension $d \neq n, m$. □

4 Synchronization examples using the 2-D Fold map and the 3-D generalized Hénon map

In this section, we validate the theoretical results illustrated above. The Fold map is considered as the drive system and the controlled three-dimensional generalized Hénon map is taken as the response system. The Fold map can be described as follows:

$$\begin{cases} x_1(k+1) = x_2(k) + a x_1(k) \\ x_2(k+1) = b + x_1^2(k). \end{cases} \tag{19}$$

When $a = -0.1$ and $b = -1.7$, the chaotic attractor of the Fold map with the initial valuables $x_1(0) = -0.5$ and $x_2(0) = -0.3$ is displayed in Figure 1 (a) [34].
Note that the linear part and nonlinear part of the map (19) are given by

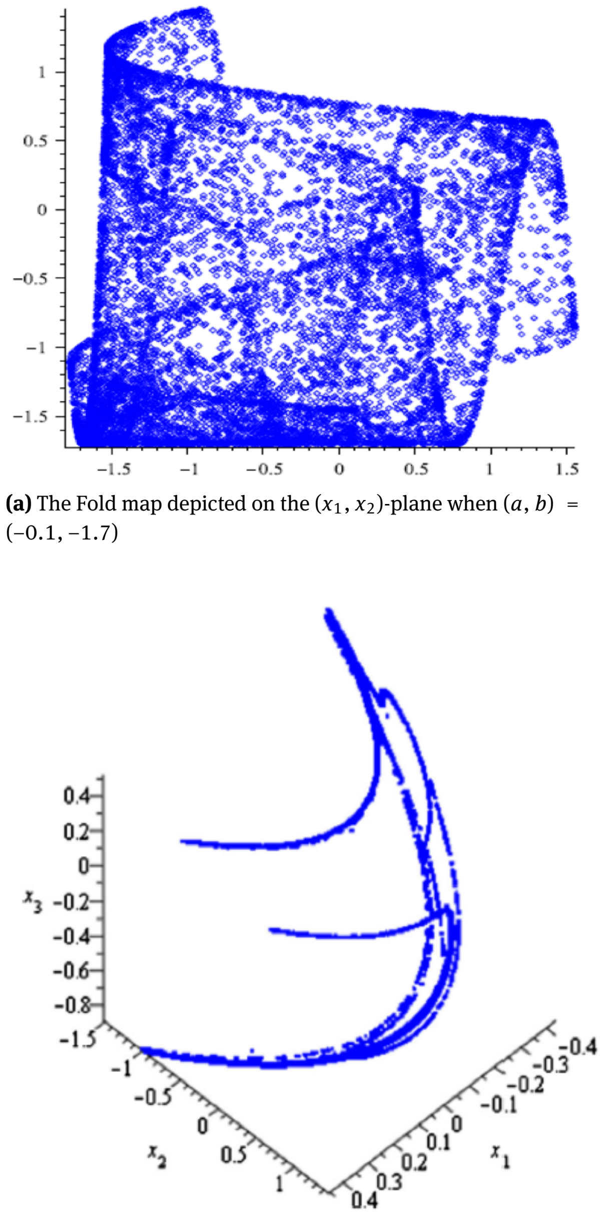
$$A = \begin{bmatrix} a & 1 \\ 0 & 0 \end{bmatrix} \text{ and } f(X(k)) = \begin{bmatrix} 0 \\ b + x_1^2(k) \end{bmatrix}.$$

The controlled three-dimensional generalized Hénon map can be described as

$$\begin{cases} y_1(k+1) = -\beta y_2(k) + u_1 \\ y_2(k+1) = \quad y_3(k) + 1 - \alpha y_2^2(k) + u_2 \\ y_3(k+1) = \quad \beta y_2(k) + y_1(k) + u_3, \end{cases} \tag{20}$$

where $U = [u_1, u_2, u_3]^T$ is the vector controller. The 3D generalized Hénon map, *i.e.*, system (20) with $U = [0, 0, 0]^T$, is chaotic when $\alpha = 1.07$ and $\beta = 0.3$ [35] as shown in Figure 1 (b). Similarly, the linear part and nonlinear part of system (20) are given by

$$B = \begin{bmatrix} 0 & -\beta & 0 \\ 0 & 0 & 1 \\ 1 & \beta & 0 \end{bmatrix} \text{ and } g(Y(k)) = \begin{bmatrix} 0 \\ 1 - \alpha y_2^2(k) \\ 0 \end{bmatrix}.$$



(a) The Fold map depicted on the (x_1, x_2)-plane when $(a, b) = (-0.1, -1.7)$

(b) The 3D generalized Hénon map when $\alpha = 1.07$ and $\beta = 0.3$

Figure 1: The chaotic attractors of the drive and response systems

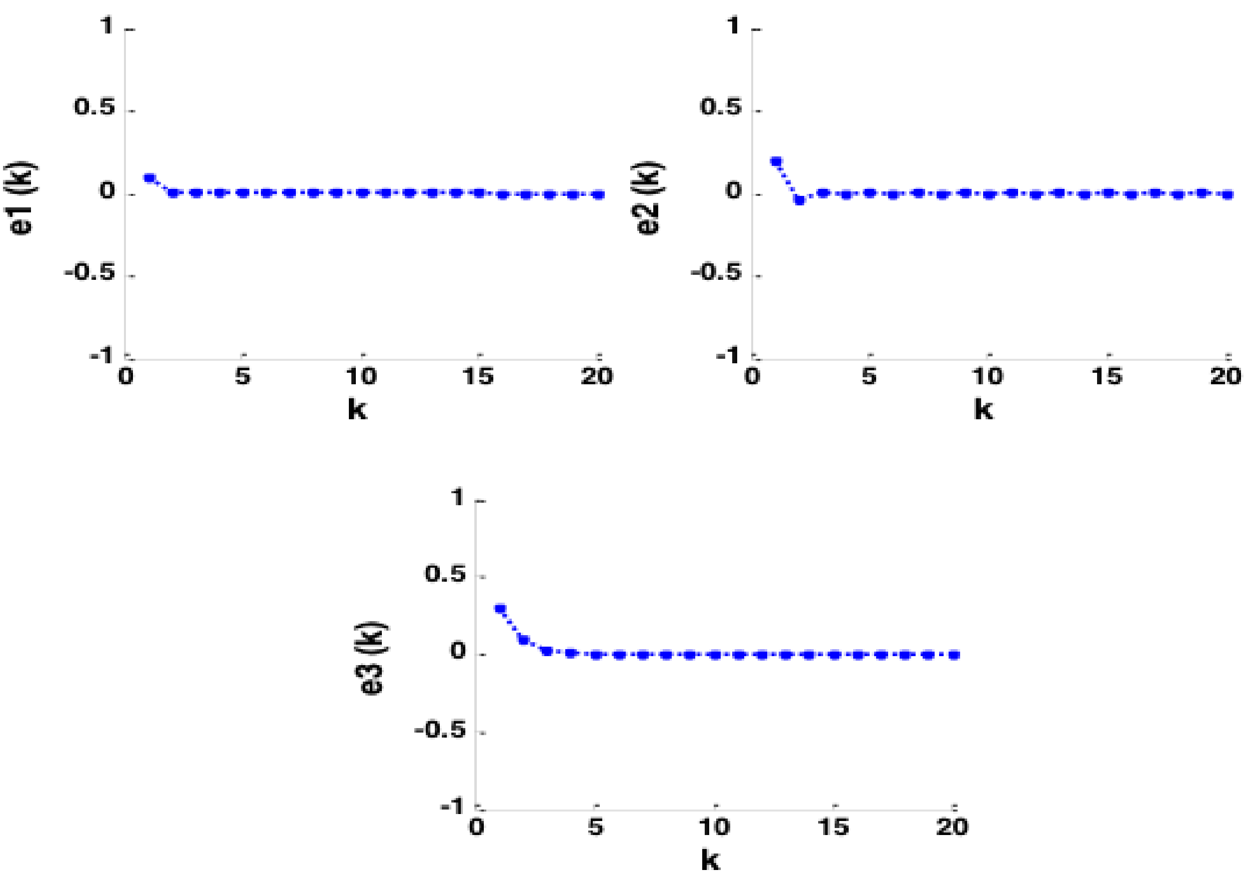


Figure 2: The error system of the **F** – **M** synchronization with index $d = 3$ as a function of k

4.1 F – M synchronization with index $d = 3$

In this case, the map **F** and the scaling matrix **M** are selected as follows:

$$\mathbf{F}\left(y_1(k), y_2(k), y_3(k)\right) = \left[y_1(k) + y_2(k), y_2^2(k), y_3(k)\right]^T, \quad (21)$$

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}. \quad (22)$$

Then, the error system is defined as

$$\begin{cases} e_1(k) = y_1(k) + y_2(k) - x_1(k) + x_2(k) \\ e_2(k) = y_2^2(k) - 2x_2(k) \\ e_3(k) = y_3(k) - 3x_1(k). \end{cases} \quad (23)$$

According to Theorem 3.1, if the controller U is selected as in (6), then, the matrix

$$L_1 = \begin{bmatrix} 0.1 & -\beta & 0 \\ 0 & -0.2 & 1 \\ 1 & \beta & 0.3 \end{bmatrix} \quad (24)$$

yields error system

$$\begin{cases} e_1(k+1) = 0.1e_1(k) \\ e_2(k+1) = -0.2e_2(k) \\ e_3(k+1) = 0.3e_3(k), \end{cases} \quad (25)$$

with eigenvalues placed strictly inside the unit disk. As a consequence, **F** – **M** synchronization with index $d = 3$ is achieved between the Fold map and the 3D generalized Hénon map as shown in Figure 2.

4.2 F – M synchronization with index $d = 2$

In this case, the map **F** and the scaling matrix **M** are chosen as follows

$$\mathbf{F}\left(y_1(k), y_2(k), y_3(k)\right) = \left[y_1(k) + y_3(k), y_2(k), y_3(k)\right]^T, \quad (26)$$

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \quad (27)$$

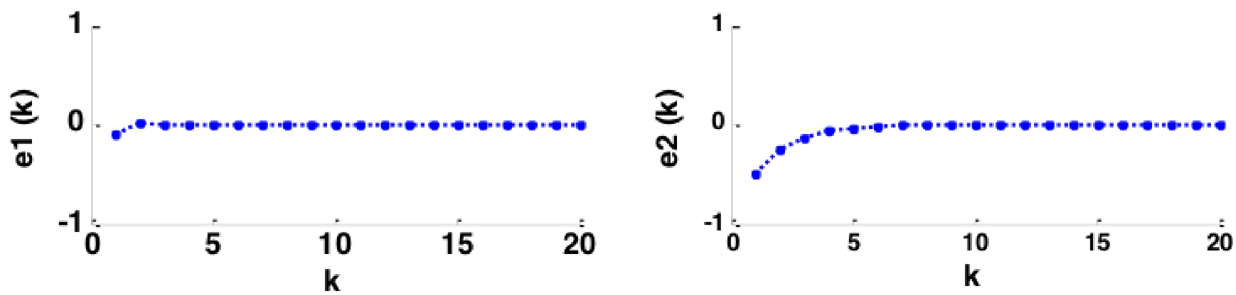


Figure 3: The error system of the **F** – **M** synchronization with index $d = 2$ as a function of k

By considering (5), the error system is written as

$$\begin{cases} e_1(k) = y_1(k) + y_3(k) - x_1(k) + x_3(k) \\ e_2(k) = y_2(k)y_3(k) - x_2(k). \end{cases} \quad (28)$$

According to Theorem 3.2, by taking the controller U as in (10)-(11) and by selecting the following matrix

$$L_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad (29)$$

it can be readily shown that the matrix $\left((A-L_2)^T(A-L_2)-I\right)$ is a negative definite matrix. As a result, the zero solution of the error system

$$\begin{cases} e_1(k+1) = -0.1e_1(k) \\ e_2(k+1) = \ \ 0.5e_2(k), \end{cases} \quad (30)$$

is globally asymptotically stable. According to Theorem 2, this means that the Fold map and the 3D generalized Hénon system are **F** – **M** synchronized with index $d = 2$. Simulation results are shown in Figure 3.

4.3 F – M synchronization with index $d = 4$

According to the definition 2.3, it is clear that the dimension d of the **F** – **M** synchronization can be larger than the dimensions of both the drive system map and the response system map. In order to show this nice property, which represents a new result in the field of chaos synchronization, the value $d = 4$ is now chosen, *i.e.*, a value larger than the dimensions of both the Fold map and the 3D generalized Hénon map. In particular, the **F** – **M** synchronization error is now defined as:

$$\left[e_1(k), e_2(k), e_3(k), e_4(k)\right]^T = \mathbf{F}\left(y_1(k), y_2(k), y_3(k)\right) - \mathbf{M}\left[x_1(k), x_2(k)\right]^T, \quad (31)$$

where

$$\mathbf{F}\left(y_1(k), y_2(k), y_3(k)\right) = \left[y_1(k), y_1(k) - y_3(k), y_2(k)y_3(k), y_3^2(k)\right]^T, \quad (32)$$

$$\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 3 \\ 2 & 1 \end{bmatrix}. \quad (33)$$

According to the Theorem 3.3, by taking the controller U as in (14)-(15) and by selecting the diagonal matrix L_3 as

$$L_3 = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & -0.4 \end{bmatrix}, \quad (34)$$

where all the four diagonal control parameters satisfy the condition (16), the following error system is obtained:

$$\begin{cases} e_1(k+1) = \ \ 0.1e_1(k) \\ e_2(k+1) = -0.2e_2(k) \\ e_3(k+1) = \ \ 0.3e_2(k) \\ e_4(k+1) = -0.4e_2(k). \end{cases} \quad (35)$$

According to Theorem 3.3, since the zero solution of system (35) is globally asymptotically stable, *i.e.*, the Fold system and the 3D generalized Hénon map are **F** – **M** synchronized with index $d = 4$. The results of the numerical simulations are plotted in Figure 4.

5 Discussion

This section will carry out some comparisons between the proposed method and similar papers published in the literature, with the aim to highlight the differences between the available approaches. In particular, attention is focused on two interesting synchronization methods based on fuzzy rules [35, 36]. For example, in [35] the author illustrates a H^∞ synchronization method based on the T–S fuzzy model and the delayed feedback control. In particular, in [35] the closed loop error system is

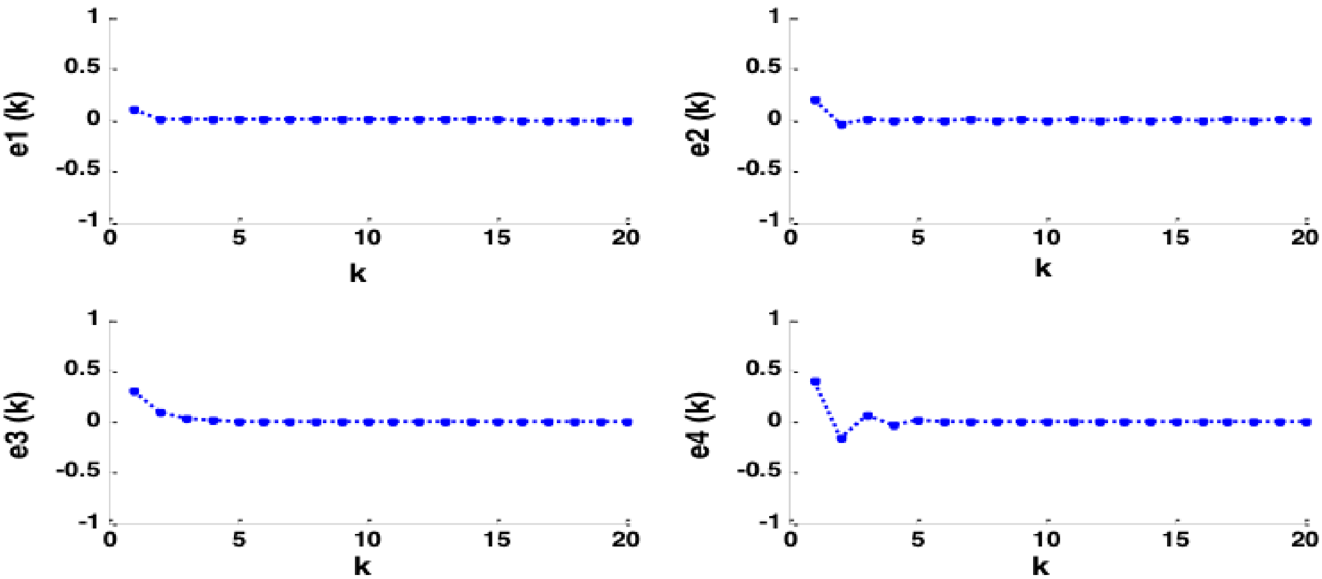


Figure 4: The error system of the **F** – **M** synchronization with index $d = 4$ as a function of k

asymptotically synchronized using an existence criterion for the controller. The approach is mainly based on the Lyapunov–Krasovskii method and on the linear matrix inequality (LMI). A similarity between the approach in [35] and the proposed one lies in the adoption of Lyapunov-based methods in order to guarantee the stability of the error system. However, a remarkable difference between the two methods is that the one in [35] has been designed for continuous-time systems, while the proposed one holds for discrete-time systems. Referring to [36], the author proposes a receding horizon control technique for achieving H^∞ synchronization in chaotic systems with external disturbance. The approach, which is based on a new set of matrix inequality conditions, is applied to synchronize the chaotic Lorenz system. Even though the method in [36] is very interesting, it has been developed for continuous-time systems, while the proposed approach has been conceived for discrete-time systems. By summarizing, we believe that the approaches in [35, 36] can be considered interesting tools for synchronizing continuous-time chaotic systems, while the proposed method is valuable for synchronizing discrete-time chaotic systems with multiple choices of synchronization indices d (*i.e.*, not only when the synchronization index d equals n or m, but even if the synchronization index d is larger than the map dimensions n and m).

Finally, we would make some comments on future developments of the present work. In particular, we are conscious that circuit implementations of synchronization schemes are an important issue. For this reason, we are currently working on the hardware implementation of the **F**-**M** synchronization method, starting from the Grassi – Miller map [37]. Namely, we would remark that a co-author (Grassi) has already implemented that map in [37], so we are preparing a forthcoming paper where all the details related to the circuit implementation of the conceived synchronization scheme will be provided.

6 Conclusion

In this paper, a new type of chaos synchronization, called **F** – **M** synchronization with index d, has been proposed. The novelty relies on the fact that the approach combines two different synchronization types, the inverse generalized synchronization (based on a functional relationship **F**) and the matrix projective synchronization (based on a matrix **M**). The technique exploits nonlinear controllers and Lyapunov stability theory in order to synchronize n-dimensional drive system maps and m-dimensional response system maps. The approach has proved to be effective in achieving synchronized dynamics not only when the synchronization index d equals n or m, but even if the synchronization index d is larger than the map dimensions n and m which is an interesting result. Finally, simulation results involving the Fold map and the 3D generalized Hénon map are provided, with the aim to highlight the capabilities of the presented new scheme. As a concluding remark, we would like to highlight that the basic idea of the present paper, the combination of two different synchronization types in order to create a novel synchronization scheme, can be further generalized. This can be achieved by considering two different synchronization types as "building blocks" to obtain several new synchronization schemes using the technique developed in this paper. Consequently, the approach illustrated here can be considered as a "methodology" to create new synchronization schemes starting from two well-established synchronization types.

Acknowledgement: The authors acknowledge Prof. Guan-Rong Chen, Department of Electronic Engineering, City University of Hong Kong for suggesting many helpful references. The author Xiong Wang was supported by the National Natural Science Foundation of China (No. 61601306) and Shenzhen Overseas High Level Talent Peacock Project Fund (No. 20150215145C).

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