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Research Article

Tanki Motsepa, Taha Aziz*, Aeeman Fatima, and Chaudry Masood Khalique

Algebraic aspects of evolution partial differential equation arising in the study of constant elasticity of variance model from financial mathematics

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Abstract: The optimal investment-consumption problem under the constant elasticity of variance (CEV) model is investigated from the perspective of Lie group analysis. The Lie symmetry group of the evolution partial differential equation describing the CEV model is derived. The Lie point symmetries are then used to obtain an exact solution of the governing model satisfying a standard terminal condition. Finally, we construct conservation laws of the underlying equation using the general theorem on conservation laws.

Keywords: CEV model; Terminal condition; Lie point symmetries; Group-invariant solution; Conservation laws

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1 Introduction

An important problem in mathematics of finance is the mathematical modelling of optimal investment-consumption decisions under uncertainty conditions. The investment-consumption problem [1] has been extensively investigated in many works with various modifications and extensions. Cox [2] and Cox and Ross [3] have derived the well-known constant elasticity of variance (CEV) option pricing model and Schroder [4] afterwards broadened the model by stating the CEV option pricing formula in respect of noncentral Chi-square distribution. The CEV model is mostly used to investigate the option and asset pricing formula, as was investigated by Beckers [5], Davydov and Linetsky [6], Emanuel and Macbeth [7] and recently by Hsu et al. [8]. Here, we reconsider the CEV model [9]. Literature survey witness that there have been few studies [10–17] recently reported where its solutions are presented. Nonetheless, to the best of our knowledge, no work has been published on the closed-form solutions of the CEV model, which is the aim of the present work.

The classical Lie symmetry theory was discovered by the Norwegian mathematician Marius Sophus Lie (1842-1899) in the nineteenth century. This theory systematically unites the widely known ad hoc methods to find exact solutions for differential equations. After many years of discovery, the Lie's theory was popularized by Ovsiannikov in Novosibirsk, Russia in the middle of twentieth century and by Birkhoff and Olver in the West. Lie's theory is one of the most effective tools to find exact analytical solutions of nonlinear partial differential equations and is established on the analysis of the invariance under one-parameter group of point transformations. See for example, the references [18–24].

Recently Lie's theory has been applied to partial differential equations (PDEs) of mathematical finance. One of the earliest study on the subject is [25], where the classical Black-Scholes equation was discussed. The bond-pricing equation via Lie group approach was investigated in [26, 27]. The invariant analysis of some well-known financial mathematics models was presented [28–31]. More recently the Lie's theory has been applied to various PDEs of financial mathematics. See for example, Motsepa et al. [32], Lekalakala et al. [33], Nteumagne and Moitsheki [34], Bozhkov and Dimas [35], Caister et al. [36, 37], Naicker et al. [38], Lo [39], Taylor and Glasgow [40], Wang et al. [41] and Pooe et al. [42], are few important studies to mention.

In this paper, we discuss the optimal investment-consumption problem under the CEV model

$$\frac{\partial F}{\partial t} + \left[\frac{\lambda\alpha - \delta}{1-\lambda} + \frac{\lambda}{2(\lambda-1)^2}\left(\frac{\beta-\alpha}{K}\right)^2 x\right] F \qquad (1)$$

***Corresponding Author: Taha Aziz:** School of Computer, Statistical and Mathematical Sciences, North-West University, Potchefstroom Campus, Private Bag X 6001, Potchefstroom 2531, South Africa; Email: tahaaziz77@yahoo.com; Tel.: +27-18-2852494
Tanki Motsepa, Aeeman Fatima, Chaudry Masood Khalique: International Institute for Symmetry Analysis and Mathematical Modeling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag *X* 2046, Mmabatho 2735, South Africa

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$$+\left[\frac{2\lambda}{(\lambda-1)}\gamma(\beta-\alpha)-2\gamma\beta\right]x\frac{\partial F}{\partial x}$$
$$+\gamma(2\gamma+1)K^2\frac{\partial F}{\partial x}+2\gamma^2K^2x\frac{\partial^2 F}{\partial x^2}=0$$

with the terminal condition

$$F(T,x)=1,\quad t\in[0,T] \tag{2}$$

from the viewpoint of Lie symmetry approach. The application of Lie's theory, in general, reduces the partial differential equation in two independent variables to an ordinary differential equation and provides us with the group-invariant solution. Some nontrivial conservation laws are also constructed for the CEV model by employing a general theorem proved in [43].

2 Lie symmetries of (1)

In this section, we compute the Lie symmetries admitted by the model equation (1) and utilised them to obtain the closed-form group-invariant solution of the PDE (1) satisfying the terminal condition (2).

For detailed description on Lie symmetry method and its applications to various disciplines the reader is referred to Refs [18–24]. However, in this section we give detailed calculations on finding Lie point symmetries of (1).

In order to facilitate the calculations when employing the Lie group approach, we rewrite the PDE (1) in the form

$$\frac{\partial F}{\partial t}+ex\frac{\partial^2 F}{\partial x^2}+(cx+d)\frac{\partial F}{\partial x}+(a+bx)F=0, \tag{3}$$

where

$$a=\frac{\lambda\alpha-\delta}{1-\lambda},\quad b=\frac{\lambda(\beta-\alpha)^2}{2(\lambda-1)^2K^2},\quad c=2\left(\frac{\lambda\gamma(\beta-\alpha)}{(\lambda-1)}-\gamma\beta\right),$$
$$d=\gamma(2\gamma+1)K^2,\quad e=2\gamma^2K^2.$$

The operator

$$X=\tau(t,x,F)\frac{\partial}{\partial t}+\xi(t,x,F)\frac{\partial}{\partial x}+\eta(t,x,F)\frac{\partial}{\partial F}$$

is a Lie symmetry of PDE (3) if and only if

$$X^{[2]}[F_t+exF_{xx}+(cx+d)F_x+(a+bx)F]\,|_{(3)}=0, \tag{4}$$

where $X^{[2]}$ denotes the second prolongation of X which is defined by

$$X^{[2]}=X+\zeta^t\frac{\partial}{\partial F_t}+\zeta^x\frac{\partial}{\partial F_x}+\zeta^{xx}\frac{\partial}{\partial F_{xx}} \tag{5}$$

with

$$\zeta^t=D_t\eta-F_tD_t\tau-F_xD_t\xi, \tag{6}$$
$$\zeta^x=D_x\eta-F_tD_x\tau-F_xD_x\xi,$$
$$\zeta^{xx}=D_x\zeta^x-F_{tx}D_x\tau-F_{xx}D_x\xi$$

and the total derivative operators

$$D_t=\frac{\partial}{\partial t}+F_t\frac{\partial}{\partial F}+F_{tt}\frac{\partial}{\partial F_t}+F_{tx}\frac{\partial}{\partial F_x}+\cdots, \tag{7}$$
$$D_x=\frac{\partial}{\partial x}+F_x\frac{\partial}{\partial F}+F_{xx}\frac{\partial}{\partial F_x}+F_{tx}\frac{\partial}{\partial F_t}+\cdots.$$

Expanding the symmetry condition (4) with the help of (5)-(7) and then splitting on the derivatives of F leads to

$$\tau_F=0,$$
$$\xi_F=0,$$
$$\eta_{FF}=0,$$
$$\tau_x=0,$$
$$\xi+x\tau_t-2x\xi_x=0,$$
$$(a+bx)\eta+bF\xi+\tau_tF(a+bx)-\eta_FF(a+bx)$$
$$+\eta_x(cx+d)+ex\eta_{xx}+\eta_t=0,$$
$$c\xi+\tau_t(cx+d)-\xi_x(cx+d)+2ex\eta_{xF}-ex\xi_{xx}-\xi_t=0,$$

which are linear PDEs in τ, ξ and η. Solving the above system gives

$$\tau=C_1+\frac{2}{\theta}C_3\exp(\theta t)-\frac{2}{\theta}C_4\exp(-\theta t),$$
$$\xi=2C_3x\exp(\theta t)+2C_4x\exp(-\theta t),$$
$$\eta=C_2F-\frac{1}{e\theta}C_3F\exp(\theta t)\left(2ae+d\theta+c\theta x-\theta^2x-cd\right)$$
$$-\frac{1}{e\theta}C_4F\exp(-\theta t)\left(-2ae+cx\theta+cd+d\theta+\theta^2x\right),$$

where $\theta=\sqrt{c^2-4be}$ and $C_1,\cdots,C_4$ are arbitrary constants. Since τ, ξ and η contain four constants, we conclude that the Lie algebra of infinitesimal symmetries of the PDE (3) is spanned by the four vector fields

$$X_1=\frac{\partial}{\partial t},$$
$$X_2=\frac{2}{\theta}\exp(\theta t)\frac{\partial}{\partial t}+2x\exp(\theta t)\frac{\partial}{\partial x}$$
$$-\frac{1}{e\theta}F\exp(\theta t)\left(2ae+d\theta+c\theta x-\theta^2x-cd\right)\frac{\partial}{\partial F},$$
$$X_3=-\frac{2}{\theta}\exp(-\theta t)\frac{\partial}{\partial t}+2x\exp(-\theta t)\frac{\partial}{\partial x}$$
$$-\frac{1}{e\theta}F\exp(-\theta t)\left(-2ae+cx\theta+cd+d\theta+\theta^2x\right)\frac{\partial}{\partial F},$$
$$X_4=F\frac{\partial}{\partial F}.$$

3 Group invariant solution

Many researchers have developed various analytical methods for solving partial differential equations, such as inverse scattering transform [44], the Bäcklund transformation [45], the Hirota bilinear method [46], the Painlevé analysis method [47], the Bell polynomials [48], the homoclinic breather limit method [49].

We now obtain the closed-form group-invariant solution for the PDE (3) by making use of the Lie symmetry algebra calculated in the previous section. Firstly we calculate symmetry Lie algebra admitted by (3) that satisfies the terminal condition (2) [26, 28].

We consider the linear combination of the Lie point symmetries, namely

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4, \tag{8}$$

and use the terminal conditions

$$X(t-T)\,|_{t=T,F=1} = 0 \tag{9}$$

and

$$X(F-1)\,|_{t=T,F=1} = 0. \tag{10}$$

The condition (9) yields

$$a_1 + \frac{2a_2}{\theta}\exp(\theta T) - \frac{2a_3}{\theta}\exp(-\theta T) = 0, \tag{11}$$

while condition (10) gives

$$a_4 + a_2\left(\frac{cd}{e\theta} - \frac{2a}{\theta} - \frac{d}{e} + \frac{c^2x}{e\theta} - \frac{4bx}{\theta} - \frac{cx}{e}\right)\exp(\theta T) \tag{12}$$
$$+ a_3\left(\frac{2a}{\theta} - \frac{cd}{e\theta} - \frac{d}{e} - \frac{c^2x}{e\theta} + \frac{4bx}{\theta} - \frac{cx}{e}\right)\exp(-\theta T) = 0.$$

Splitting Eq. (12) on powers of x yields

$$a_4 + a_2\left(\frac{cd}{e\theta} - \frac{2a}{\theta} - \frac{d}{e}\right)\exp(\theta T) \tag{13}$$
$$+ a_3\left(\frac{2a}{\theta} - \frac{cd}{e\theta} - \frac{d}{e}\right)\exp(-\theta T) = 0,$$

$$a_2\left(\frac{c^2}{e\theta} - \frac{4b}{\theta} - \frac{c}{e}\right)\exp(\theta T) \tag{14}$$
$$+ a_3\left(-\frac{c^2}{e\theta} + \frac{4b}{\theta} - \frac{c}{e}\right)\exp(-\theta T) = 0.$$

Solving equations (11), (13) and (14), we obtain

$$a_1 = \frac{a_2c(\theta-c)}{be\theta}\exp(\theta T), \tag{15}$$
$$a_3 = \frac{a_2(2be + c(\theta - c))}{2be}\exp(2\theta T),$$
$$a_4 = -\frac{a_2(\theta-c)(ac-2bd)}{be\theta}\exp(\theta T).$$

Substituting the values of a_1, a_3 and a_4 from Eq. (15) into Eq. (8) and solving $X(F) = 0$, we obtain the two invariants

$$J_1 = \frac{x\exp(\theta t)}{(c+\theta)\exp(2\theta t) - 2c\exp(\theta(t+T)) + (c-\theta)\exp(2\theta T)},$$

$$J_2 = F\left((c+\theta)\exp(\theta t) + (\theta - c)\exp(\theta T)\right)^{d/e} \times \exp\Bigg(at + \frac{4bex\exp(2\theta t) + x(c-\theta)(c+\theta)\exp(2\theta T)}{2e\left((c+\theta)\exp(2\theta t) - 2c\exp(\theta(t+T)) + (c-\theta)\exp(2\theta T)\right)} - \frac{cdt}{2e} - \frac{d\theta t}{2e}\Bigg).$$

Thus, the group-invariant solution of the PDE (3) is given by $J_2 = G(J_1)$, which yields

$$F(t,x) = G(z)\left\{(c+\theta)\exp(\theta t) + (\theta-c)\exp(\theta T)\right\}^{-\frac{d}{e}} \tag{16}$$
$$\times\exp\left(\frac{4bex\exp(2\theta t) + x(c-\theta)(c+\theta)\exp(2\theta T)}{2e\left(-(c+\theta)\exp(2\theta t) + 2c\exp(\theta(t+T)) + (\theta-c)\exp(2\theta T)\right)} + \frac{cdt}{2e} + \frac{d\theta t}{2e} - at\right),$$

where $z = J_1$. Now substituting Eq. (16) into PDE (3) yields a second-order ODE for $G(z)$, viz.,

$$ezG''(z) - 2\exp(\theta T)\left\{2bdG(z) + c^2zG'(z)\right\} + dG'(z) \tag{17}$$
$$- 8bzG(z)\exp(2\theta T)\left(2be - c^2\right) = 0.$$

Solving the reduced ODE (17) for $G(z)$, we obtain

$$G(z) = C_1\exp(4bz\exp(\theta T))$$
$$- C_2 2^{\frac{d}{e}-1}z^{1-\frac{d}{e}}\exp(4bz\exp(\theta T))\left(-\frac{\theta^2 z\exp(\theta T)}{e}\right)^{\frac{d}{e}-1}$$
$$\times\Gamma\left[1 - \frac{d}{e}, -\frac{2\exp(\theta T)z\theta^2}{e}\right],$$

where C_1 and C_2 are constants and $\Gamma[.,.]$ is the incomplete Gamma function [50, 51]. The terminal condition dictates that we take $C_2 = 0$, hence the above solution for $G(z)$ takes the form

$$G(z) = C_1\exp(4bz\exp(\theta T)).$$

Substituting the value of G into Eq. (16), the solution for $F(x,t)$ is written as

$$F(t,x) = C_1\exp(4bz\exp(\theta T))\left\{(c+\theta)\exp(\theta t) \right. \tag{18}$$
$$\left. + (\theta-c)\exp(\theta T)\right\}^{-\frac{d}{e}} \times \exp\Bigg[-at$$
$$+ \frac{4bex\exp(2\theta t) + x(c-\theta)(c+\theta)\exp(2\theta T)}{2e\left(-(c+\theta)\exp(2\theta t) + 2c\exp(\theta(t+T)) + (\theta-c)\exp(2\theta T)\right)}$$
$$+ \frac{cdt}{2e} + \frac{d\theta t}{2e}\Bigg].$$

Finally, making use of the terminal condition $F(x,T) = 1$ in Eq. (18), we obtain

$$C_1 = \left((\theta-c)\exp(\theta T) + (c+\theta)\exp(2\theta T)\right)^{d/e}\exp\left(aT - \frac{cdT}{2e} - \frac{d\theta T}{2e}\right).$$

Therefore, the solution of (1+1) evolution PDE (3) satisfying the terminal condition is given by

$$F(t,x) = \left(\frac{2\theta\exp(\theta T)}{(c+\theta)\exp(\theta t)+(\theta-c)\exp(\theta T)}\right)^{\frac{d}{e}} \times \exp\left[\frac{(t-T)(d(c+\theta)-2ae)}{2e} - \frac{2bx\left(\exp(\theta t)-\exp(\theta T)\right)}{c\left(\exp(\theta t)-\exp(\theta T)\right)+\theta\left(\exp(\theta t)+\exp(\theta T)\right)}\right].$$

4 Conservation laws

In this section we derive conservation laws for the (1+1) evolution partial differential equation (3). In classical physics, conservations laws are physical quantities which describe the conservation of energy, mass, linear momentum, angular momentum, and electric charge. One particularly important result concerning conservation laws is the celebrated Noether theorem, which gives us a sophisticated and useful way of constructing conservation laws when a Noether point symmetry connected to a Lagrangian is known for the corresponding Euler-Lagrange equation.

Recently, a theorem due to Ibragimov was proved, which gives a method to construct conservation laws irrespective of the existence of a Lagrangian. We start here by stating the definition of an adjoint equation.

Definition 1 Let

$$E(x,t,f,f_t,f_x,f_{xx}) = 0 \tag{19}$$

be a second-order PDE with x, t as independent variables and f a dependent variable. Then, its adjoint equation is [43]

$$E^{*}(x,t,f_t,f_x,f_{xx},g,g_t,g_x,g_{xx}) = 0, \tag{20}$$

where

$$E^{\star} = \frac{\delta(gE)}{\delta f},$$

with

$$\frac{\delta}{\delta f} = \frac{\partial}{\partial f} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}\ldots D_{i_s}\frac{\partial}{\partial f_{i_1\ldots i_s}}$$

denoting the Euler–Lagrange operator and g is a new dependent variable. Here

$$D_i = \frac{\partial}{\partial x^i} + f_i\frac{\partial}{\partial f} + f_{ij}\frac{\partial}{\partial f_j} + \ldots$$

are the total derivative operators.

The 2-tuple vector $T = (T^t, T^x)$, is a conserved vector of (19) if T^t and T^x satisfy

$$D_t T^t + D_x T^x|_{(19)} = 0. \tag{21}$$

We now state the following theorem.

Theorem 1 [43] Any Lie point, Lie–Bäcklund or non-local symmetry

$$X = \xi^i(x,f,f_{(1)},\ldots)\frac{\partial}{\partial x^i} + \eta(x,f,f_{(1)},\ldots)\frac{\partial}{\partial f},$$

of Eq. (19) gives a conservation law $D_i(T^i) = 0$ for the system (19)–(20). The conserved vector is given by

$$T^i = \xi^i\mathcal{L} + w\left[\frac{\partial\mathcal{L}}{\partial f_i} - D_j\left(\frac{\partial\mathcal{L}}{\partial f_{ij}}\right)\right] + D_j(w)\left[\frac{\partial\mathcal{L}}{\partial f_{ij}}\right],$$

where w and $\mathcal{L}$ are determined as follows:

$$w = \eta - \xi^j f_j, \quad \mathcal{L} = gE(x,f,f_{(1)},\ldots,f_{(s)}).$$

We now apply the above theorem to our problem. The adjoint equation to the equation (3) is

$$w(a+bx-c) - w_x(cx+d-2e) + exw_{xx} - w_t = 0, \tag{22}$$

and hence the second-order Lagrangian of the equation (3) and its adjoint equation (22) is written as

$$\mathcal{L} = w[F_t + exF_{xx} + (cx+d)F_x + (a+bx)F].$$

We now apply the above theorem to each Lie point symmetry of Eq. (3). We start with $X_1 = \partial/\partial t$. Corresponding to symmetry X_1 we obtain the conserved vector with components

$$T_1^t = w\left((a+bx)F + F_x(cx+d) + exF_{xx} + F_t\right) - F_t w,$$
$$T_1^x = -F_t\left(w(cx+d-e) + exw_x\right) - exwF_{tx}.$$

The symmetry X_2 provides us with the conserved vector whose components are

$$T_2^t = -\frac{w\exp(t\theta)}{e\theta}\Big[F\left(d\theta + cx\theta + 2bex - c^2x - cd\right) - 2e\left\{F_x\left(x(c-\theta)+d\right) + exF_{xx}\right\}\Big],$$

$$\begin{aligned}T_2^x = \frac{\exp(t\theta)}{e\theta}\Big[&e2exw_x\left(xF_x\theta + F_t\right)\\ &- wxF_x\left(2ae + d\theta + cx\theta + 4bex - c^2x - cd\right)\\ &+ 2F_t\left(-x\theta + cx + d - e\right)\\ &+ 2exF_{tx} + Fwc\left(-2ex(a+2bx) - d\left(2x\theta + e\right) + d^2\right)\\ &+ 2ae\left(x\theta - d + e\right) - d^2\theta + c^2x\left(2d - x\theta\right) + de\theta\\ &+ 2bex^2\theta - 4bdex + c^3x^2\\ &+ exw_x\left(2ae + d\theta + cx\theta + 4bex - c^2x - cd\right)\Big],\end{aligned}$$

where $\theta = \sqrt{c^2 - 4be}$. Likewise, the conserved vectors associated with the symmetries X_3 and X_4 are given by

$$T_3^t = -\frac{w\exp(-\theta t)}{e\theta}\Big[F\left\{c\left(x\theta + d\right) + d\theta - 2bex + c^2x\right\}$$

$$+ 2e\left\{F_x\left(x\left(\theta + c\right) + d\right) + exF_{xx}\right\}\Big],$$

$$T_3^y = \frac{\exp(-\theta t)}{e\theta}\Big[2ew\left(F_t\left(x\left(\theta + c\right) + d - e\right) + eFF_{tx}\right) - xF_x$$
$$- 2e(a + 2bx) + c\left(x\theta + d\right) + d\theta + c^2 x$$
$$+ 2exw_x\left(xF_x\theta - F_t\right) + 2Fwcex(a + 2bx) + d\left(e - 2x\theta\right)$$
$$- d^2 + 2ae\left(x\theta + d - e\right) - d^2\theta - c^2 x\left(x\theta + 2d\right) + de\theta$$
$$+ 2bex^2\theta + 4bdex - c^3 x^2$$
$$+ exw_x\left(-2e(a + 2bx) + c\left(x\theta + d\right) + d\theta + c^2 x\right)\Big],$$

$$T_4^t = wF,$$
$$T_4^y = F\left(w(cx + d - e) - exw_x\right) + exwF_x,$$

respectively.

5 Concluding remarks

The evolution (1+1) PDE (1) describing the optimal investment-consumption problem under the CEV model [9] satisfied the classical Black–Scholes–Merton equation with boundary condition which differ from those often used in the most common cases. It is well-known that the evolution (1+1) PDE (1) is related to the heat equation via the equivalence transformations and thus its general solution can be obtained. However, in this paper, for the first time, we have solved the PDE (1) subject to the terminal condition (2) by utilizing the Lie group method. This demonstrates the usefulness of Lie's theory. We found a four-dimensional Lie symmetry algebra for evolution PDE (1). Using the nontrivial Lie point symmetry operator, we have shown that the governing PDE can be transformed into a second-order variable coefficient ODE. The reduced ODE is solved to obtain a new exact closed-form solution of the CEV model which also satisfy the terminal condition. Thus for the first time with the application of Lie's theory closed-form solution of (1) is derived. Finally, we constructed conservation laws corresponding to the four Lie point symmetries by employing a general theorem on conservation laws. This is the first time that the evolution PDE (1) for optimal investment-consumption problem has been considered from the view point of group theoretical approach and the conservation laws have been derived in the literature.

Conflict of Interests: The authors declare that there is no conflict of interests regarding the publication of this paper.

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