Research Article

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A modified Fermi-Walker derivative for inextensible flows of binormal spherical image

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Abstract: Fermi-Walker derivative and biharmonic particle play an important role in skillful applications. We obtain a new characterization on binormal spherical indicatrix by using the Fermi-Walker derivative and parallelism in space. We suggest that an inextensible flow is the necessary and sufficient condition for this particle. Finally, we give some characterizations for a non-rotating frame of this binormal spherical indicatrix.

Keywords: Fermi-Walker derivative-parallelism, Energy, Bienergy, Heisenberg group, Faraday tensor

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1 Introduction

In the literature there are many studies on Fermi-Walker transport and Fermi-Walker derivative. Simple description for the construction of Fermi-Walker transported frames out of an arbitrary set of tetrad fields was presented in Ref. [1–3].

Recently, a research on Fermi-Walker transports has been expanded to Minkowski spacetime in Ref. [4, 5]. Frenet-Serret equations constructed by Synge on world lines are a strong instrument for studying motion of non-zero rest mass for test particles in an assumed gravitational field [6–11]. Also, Frenet-Serret equations have been generalized from non-null to null trajectories in a spacetime by using a new formalism with Fermi-Walker transport in Ref. [4].

In Ref. [5, 10–16], some curves corresponding to their flows has been investigated. A new characterization of inextensible flows for curves with Fermi-Walker derivative and its parallelism on the 3-dimensional space has also been constructed. More precisely, they have constructed new figures as illustrations of the moving charged particle in electromagnetic field. Flows of curves of a given curve are also widely studied in Ref. [17, 18]. Some characterizations of curves and surfaces are given in Ref. [19–29].

In Ref. [2], Fermi-Walker derivative, Fermi–Walker parallelism, non-rotating frame, Fermi-Walker terms with Darboux vector are given in Minkowski 3- dimensional space. In Ref. [15], flows of biharmonic particles on a new spacetime are defined by using Bianchi type-I (B-I) cosmological model. A geometrical description of timelike biharmonic particle in spacetime is also given. In Ref. [17], a new method for inextensible flows of timelike curves in a conformally flat, quasi conformally flat and conformally symmetric 4-dimensional LP-Sasakian manifold is developed.

The structure of the paper is as follows. First, we construct a new characterization for inextensible flows of binormal spherical indicatrix and Fermi-Walker parallelism by using Fermi-Walker derivative in space. Finally, we give some characterizations for non-rotating frame of binormal spherical image.

2 Preliminaries

In this section, we study relationship between the Fermi-Walker derivative and the Frenet fields of curves. Moreover, we obtain some characterizations and an example of the curve.

Fermi transport and derivative have the following theories.

Fermi-Walker transport is defined by

$$\nabla_{\mathbf{T}}^{FW}\mathbf{V} = \nabla_{\mathbf{T}}\mathbf{V} - \mathbf{T}\left\langle \mathbf{V}, \nabla_{\mathbf{T}}\mathbf{T}\right\rangle + \nabla_{\mathbf{T}}\mathbf{T}\left\langle \mathbf{V}, \mathbf{T}\right\rangle = 0. \tag{1}$$

$\nabla_{\mathbf{T}}^{FW}\mathbf{V}$ is called Fermi-Walker derivative of $\mathbf{V}$ with respect to $\mathbf{T}$ along the curve in space.

With this definition the following features are satisfied [24]:

1. If the curve is a geodesic, then the Fermi-Walker transport is identical to parallel transport: if $\nabla_{\mathbf{T}}\mathbf{T} = 0$, then $\nabla_{\mathbf{T}}^{FW}\mathbf{V} = \nabla_{\mathbf{T}}\mathbf{V}$.
2. $\nabla_{\mathbf{T}}^{FW}\mathbf{T} = 0$, that is, the tangent to the curve is always Fermi-Walker transported.

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3. If $\mathbf{V}$ and $\mathbf{W}$ are Fermi-Walker transported vector fields, their inner product remains constant along the curve

$$\frac{d}{ds}\left\langle \mathbf{V}(s), \mathbf{W}(s)\right\rangle = \langle \nabla_{\mathbf{T}}\mathbf{V}, \mathbf{W}\rangle + \langle \mathbf{V}, \nabla_{\mathbf{T}}\mathbf{W}\rangle , \tag{2}$$

$$\nabla_{\mathbf{T}}^{FW}\mathbf{V} = 0 = \nabla_{\mathbf{T}}\mathbf{V} - \mathbf{T}\left\langle \mathbf{V}, \nabla_{\mathbf{T}}\mathbf{T}\right\rangle + \nabla_{\mathbf{T}}\mathbf{T}\left\langle \mathbf{V}, \mathbf{T}\right\rangle . \tag{3}$$

If we write the equation (3) at equation (2)

$$\begin{aligned}\frac{d}{ds}\left\langle \mathbf{V}(s), \mathbf{W}(s)\right\rangle &= \langle \mathbf{T}\left\langle \mathbf{V}, \nabla_{\mathbf{T}}\mathbf{T}\right\rangle - \nabla_{\mathbf{T}}\mathbf{T}\left\langle \mathbf{V}, \mathbf{T}\right\rangle , \mathbf{W}\rangle \\ &+ \langle \mathbf{V}, \mathbf{T}\left\langle \mathbf{W}, \nabla_{\mathbf{T}}\mathbf{T}\right\rangle - \nabla_{\mathbf{T}}\mathbf{T}\left\langle \mathbf{W}, \mathbf{T}\right\rangle\rangle \\ &= \langle \mathbf{T}, \mathbf{W}\rangle\left\langle \mathbf{V}, \nabla_{\mathbf{T}}\mathbf{T}\right\rangle - \langle \nabla_{\mathbf{T}}\mathbf{T}, \mathbf{W}\rangle\left\langle \mathbf{V}, \mathbf{T}\right\rangle \\ &+ \langle \mathbf{V}, \mathbf{T}\rangle\left\langle \mathbf{W}, \nabla_{\mathbf{T}}\mathbf{T}\right\rangle - \langle \mathbf{V}, \nabla_{\mathbf{T}}\mathbf{T}\rangle\left\langle \mathbf{W}, \mathbf{T}\right\rangle \\ &= 0.\end{aligned} \tag{4}$$

is attained, [25].

Physical sense of above attribute is that $\mathbf{V}$ is orthogonal to $\mathbf{T}$ along curve. Thus, horizontal change in it along the curve can only stem from rotation of the vector in a plane perpendicular to $\mathbf{T}$. Property of Fermi-Walker transport $\nabla_{\mathbf{T}}^{FW}\mathbf{V} = 0$ intends that vector is moved without any rotation. This is fundamental to detect the action of gyroscopes when moved with accelerated observers.

3 Construction of Fermi-Walker derivative for inextensible flows of curves

Lemma 3.1. Let $\alpha : I \subset R \to \mathbb{M}$ be a curve in space and $\mathbf{V}$ be a vector field along the curve α. For a map $\Gamma : I \times (-\varepsilon, \varepsilon) \to \mathbb{M}$, putting $\Gamma(s, 0) = \alpha(s)$, $(\frac{\partial \Gamma}{\partial s}(s, t)) = \mathbf{V}(s)$.

Therefore, the following functions can be obtained:

i Speed function $v(s, t) = \left\| \frac{\partial \Gamma}{\partial s}(s, t)\right\|$,

ii Curvature function $\kappa(s, t)$ of $\alpha_t(s)$,

iii Torsion function $\tau(s, t)$ of $\alpha_t(s)$.

The variations of those functions at $t = 0$ are

$$\mathbf{V}(v) = \left(\frac{\partial v}{\partial t}(s, t)\right)\Bigg|_{t=0} = g(\nabla_{\mathbf{T}}\mathbf{V}, \mathbf{T})v, \tag{5}$$

$$\begin{aligned}\mathbf{V}(\kappa) = \left(\frac{\partial \kappa}{\partial t}(s, t)\right)\Bigg|_{t=0} &= g(\nabla_{\mathbf{T}}^{2}\mathbf{V}, \mathbf{N}) - 2\kappa g(\nabla_{\mathbf{T}}\mathbf{V}, \mathbf{T}) \\ &+ g(R(\mathbf{V}, \mathbf{T})\mathbf{T}, \mathbf{N}),\end{aligned} \tag{6}$$

$$\begin{aligned}\mathbf{V}(\tau) = \left(\frac{\partial \tau}{\partial t}(s, t)\right)\Bigg|_{t=0} &= \left[\frac{1}{\kappa}g(\nabla_{\mathbf{T}}^{2}\mathbf{V} + R(\mathbf{V}, \mathbf{T})\mathbf{T}, \mathbf{B}\right]_{s} \\ &+ \kappa(\nabla_{\mathbf{T}}\mathbf{V}, \mathbf{B}) + \tau g(\nabla_{\mathbf{T}}\mathbf{V}, \mathbf{T}) + g(R(\mathbf{V}, \mathbf{T})\mathbf{N}, \mathbf{B}),\end{aligned} \tag{7}$$

where R is the curvature tensor.

Second, a flow of $\alpha_t(s)$ may be represented as

$$\frac{\partial \alpha_t}{\partial t} = \beta_1\mathbf{T} + \beta_2\mathbf{N} + \beta_3\mathbf{B}, \tag{8}$$

where $\beta_1, \beta_2, \beta_3$ are smooth functions.

On the other hand, Körpınar–Turhan obtained flows of binormal spherical images of curves in space [7].

Now, we investigate conditions of Frenet vectors.

Theorem 3.2. *If ϕ is binormal spherical indicatrix of α, then*

$$\nabla_t\mathbf{T}^{\phi} = -\nabla_t\mathbf{N} = \left(\beta_1\kappa - \beta_3\tau + \frac{\partial \beta_2}{\partial s}\right)\mathbf{T} - \mathcal{C}\mathbf{B}, \tag{9}$$

$$\begin{aligned}\nabla_t\mathbf{N}^{\phi} &= \left[\frac{\partial}{\partial t}\frac{\kappa}{\beta} + \left(\beta_2\tau + \frac{\partial \beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right]\mathbf{T} \\ &+ \left[\mathcal{C}\left[\frac{\tau}{\beta}\right] + \left(\beta_1\kappa - \beta_3\tau + \frac{\partial \beta_2}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]\mathbf{N} \\ &+ \left[\left(\beta_2\tau + \frac{\partial \beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right] - \frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right]\right]\mathbf{B},\end{aligned} \tag{10}$$

$$\begin{aligned}\nabla_t\mathbf{B}^{\phi} &= \left[\frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right] - \left(\beta_2\tau + \frac{\partial \beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]\mathbf{T} \\ &+ \left[\left(\beta_1\kappa - \beta_3\tau + \frac{\partial \beta_2}{\partial s}\right)\left[\frac{\tau}{\beta}\right] - \mathcal{C}\left[\frac{\kappa}{\beta}\right]\right]\mathbf{N} \\ &+ \left[\frac{\partial}{\partial t}\left[\frac{\kappa}{\beta}\right] + \left(\beta_2\tau + \frac{\partial \beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right]\mathbf{B},\end{aligned} \tag{11}$$

where

$$\beta = \sqrt{\kappa^2 + \tau^2}, \mathcal{C} = \langle \nabla_t\mathbf{N}, \mathbf{B}\rangle . \tag{12}$$

Theorem 3.3 (Main Theorem).

$$\begin{aligned}\nabla_t\nabla_s^{FW}\mathbf{X} &= \nabla_s^{FW}\nabla_t\mathbf{X} + \nabla_t\kappa^{\phi}(\mathbf{B}^{\phi}\wedge\mathbf{X}) \\ &+ \kappa^{\phi}(\nabla_t\mathbf{B}^{\phi}\wedge\mathbf{X}).\end{aligned} \tag{13}$$

Proof. By using the definition of Fermi-Walker transport we have the above equality. This completes the proof.

Theorem 3.4.

$$\begin{aligned}\nabla_s^{FW}\nabla_t\mathbf{T}^{\phi} &= \frac{\partial}{\partial s}\left(\beta_1\kappa - \beta_3\tau + \frac{\partial \beta_2}{\partial s}\right)\mathbf{T} \\ &+ \left[\kappa\left(\beta_1\kappa - \beta_3\tau + \frac{\partial \beta_2}{\partial s}\right) + \mathcal{C}\tau\right. \\ &\left. - \kappa^{\phi}\left(\frac{\tau}{\beta}\mathcal{C} + \frac{\kappa}{\beta}\left(\beta_1\kappa - \beta_3\tau + \frac{\partial \beta_2}{\partial s}\right)\right)\right]\mathbf{N} - \frac{\partial \mathcal{C}}{\partial s}\mathbf{B}.\end{aligned} \tag{14}$$

Proof. From Serret-Frenet formulas and Fermi-Walker derivative, we have

$$\nabla_s^{FW}\mathbf{X} = \nabla_s\mathbf{X} - \kappa^{\phi}(\mathbf{B}^{\phi}\wedge\mathbf{X}) \tag{3.11}$$

Using flow of $\mathbf{T}^{\phi}$, we obtain

$$\nabla_s^{FW}\nabla_t\mathbf{T}^{\phi} = \nabla_s\nabla_t\mathbf{T}^{\phi} - \kappa^{\phi}(\mathbf{B}\wedge\nabla_t\mathbf{T}^{\phi}). \tag{3.12}$$

Since the above equation, we get

$$\begin{aligned}\nabla_s\nabla_t\mathbf{T}^{\phi} &= \frac{\partial}{\partial s}\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\mathbf{T} - \frac{\partial\mathfrak{C}}{\partial s}\mathbf{B} \\ &+ \left[\kappa\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right) + \mathfrak{C}\tau\right]\mathbf{N} - \kappa\left(\mathbf{B}\wedge\nabla_t\mathbf{T}^{\phi}\right).\end{aligned} \tag{15}$$

By using the properties of cross product we can easily write that

$$\begin{aligned}\mathbf{B}^{\phi}\wedge\nabla_t\mathbf{T}^{\phi} &= \left(\frac{\tau}{\beta}\mathbf{T}\right. \\ &\left.+\frac{\kappa}{\beta}\mathbf{B}\right)\wedge\left(\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\mathbf{T} - \mathfrak{C}\mathbf{B}\right) \\ &= \left(\frac{\tau}{\beta}\mathfrak{C} + \frac{\kappa}{\beta}\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\right)\mathbf{B}\wedge\mathbf{T} \\ &= \left(\frac{\tau}{\beta}\mathfrak{C}+\frac{\kappa}{\beta}\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\right)\mathbf{N}.\end{aligned} \tag{16}$$

Then it is easy to obtain that

$$\begin{aligned}\nabla_s^{FW}\nabla_t\mathbf{T}^{\phi} &= \frac{\partial}{\partial s}\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\mathbf{T} \\ &+ \left[\kappa\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right) + \mathfrak{C}\tau\right. \\ &\left. - \kappa^{\phi}\left(\frac{\tau}{\beta}\mathfrak{C} + \frac{\kappa}{\beta}\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\right)\right]\mathbf{N} - \frac{\partial\mathfrak{C}}{\partial s}\mathbf{B},\end{aligned} \tag{17}$$

which completes the proof.

Theorem 3.5.

$$\begin{aligned}\nabla_s^{FW}\nabla_t\mathbf{N}^{\phi} &= \left[\frac{\partial}{\partial s}\left[\frac{\partial}{\partial t}\left[\frac{\kappa}{\beta}\right] + \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right]\right. \\ &- \kappa\left[\mathfrak{C}\left[\frac{\tau}{\beta}\right] + \left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right] \\ &\left.- \frac{\kappa\kappa^{\phi}}{\beta}\left[\mathfrak{C}\left[\frac{\tau}{\beta}\right] + \left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]\right]\mathbf{T} \\ &+ \left[\kappa\left[\frac{\partial}{\partial t}\left[\frac{\kappa}{\beta}\right] + \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right]\right. \\ &+ \frac{\partial}{\partial s}\left[\mathfrak{C}\left[\frac{\tau}{\beta}\right] + \left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right] \\ &- \tau\left[\left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right] - \frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right]\right] \\ &+ \left[\frac{\kappa\kappa^{\phi}}{\beta}\left[\frac{\partial}{\partial t}\frac{\kappa}{\beta} + \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right]\right. \\ &\left.\left.- \frac{\tau}{\beta}\left[\left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right] - \frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right]\right]\right]\right]\mathbf{N} \\ &+ \left[\frac{\partial}{\partial s}\left[\left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right] - \frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right]\right]\right. \\ &+ \tau\left[\mathfrak{C}\left[\frac{\tau}{\beta}\right] + \left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right] \\ &\left.+ \frac{\kappa^{\phi}\tau}{\beta}\left[\mathfrak{C}\left[\frac{\tau}{\beta}\right] + \left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]\right]\mathbf{B}.\end{aligned} \tag{18}$$

Proof. This can be verified using an argument similar to above theorem.

Theorem 3.6.

$$\begin{aligned}\nabla_s^{FW}\nabla_t\mathbf{B}^{\phi} &= \left[\frac{\partial}{\partial s}\left[\frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right] - \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]\right. \\ &- \kappa\left[\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\tau}{\beta}\right] - \mathfrak{C}\left[\frac{\kappa}{\beta}\right]\right] \\ &\left.- \frac{\kappa\kappa^{\phi}}{\beta}\left[\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\tau}{\beta}\right] - \mathfrak{C}\left[\frac{\kappa}{\beta}\right]\right]\right]\mathbf{T} \\ &+ \left[\kappa\left[\frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right] - \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]\right. \\ &+ \frac{\partial}{\partial s}\left[\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\tau}{\beta}\right] - \mathfrak{C}\left[\frac{\kappa}{\beta}\right]\right] \\ &- \tau\left[\frac{\partial}{\partial t}\left[\frac{\kappa}{\beta}\right] + \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right] \\ &+ \kappa^{\phi}\left[\frac{\kappa}{\beta}\left[\frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right] - \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]\right. \\ &\left.\left.- \frac{\tau}{\beta}\left[\frac{\partial}{\partial t}\left[\frac{\kappa}{\beta}\right] + \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right]\right]\right]\mathbf{N} \\ &+ \left[\frac{\partial}{\partial s}\left[\frac{\partial}{\partial t}\left[\frac{\kappa}{\beta}\right] + \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right]\right. \\ &+ \tau\left[\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\tau}{\beta}\right] - \mathfrak{C}\left[\frac{\kappa}{\beta}\right]\right] \\ &\left.+ \frac{\tau\kappa^{\phi}}{\beta}\left[\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\tau}{\beta}\right] - \mathfrak{C}\left[\frac{\kappa}{\beta}\right]\right]\right]\mathbf{B}.\end{aligned} \tag{19}$$

Proof. This can be verified using an argument similar to above theorem.

Using above theorems, we get the following corollaries by straight-forward computations.

Corollary 3.7. *If $\nabla_t \mathbf{T}^\phi$ along the curve, parallel to the Fermi–Walker terms, then*

$$\frac{\partial}{\partial s}\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right) = 0, \tag{20}$$

$$\Big[\kappa\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right) \tag{21}$$
$$+\mathcal{C}\tau - \kappa^\phi\left(\frac{\tau}{\beta}\mathcal{C} + \frac{\kappa}{\beta}\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\right)\Big] = 0,$$

$$\frac{\partial\mathcal{C}}{\partial s} = 0. \tag{22}$$

Corollary 3.8. *If $\nabla_t \mathbf{N}^\phi$ along the curve, parallel to the Fermi–Walker terms, then*

$$\Big[\frac{\partial}{\partial s}\left[\frac{\partial}{\partial t}\left[\frac{\kappa}{\beta}\right] + \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right] \tag{23}$$
$$-\kappa\left[\mathcal{C}\left[\frac{\tau}{\beta}\right] + \left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]$$
$$-\frac{\kappa\kappa^\phi}{\beta}\left[\mathcal{C}\left[\frac{\tau}{\beta}\right] + \left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]\Big] = 0,$$

$$\Big[\kappa\left[\frac{\partial}{\partial t}\left[\frac{\kappa}{\beta}\right] + \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right]$$
$$+\frac{\partial}{\partial s}\left[\mathcal{C}\left[\frac{\tau}{\beta}\right] + \left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]$$
$$-\tau\left[\left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right] - \frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right]\right]$$
$$+\Big[\frac{\kappa\kappa^\phi}{\beta}\left[\frac{\partial}{\partial t}\frac{\kappa}{\beta} + \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right]$$
$$-\frac{\tau}{\beta}\left[\left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right] - \frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right]\right]\Big]\Big] = 0,$$

$$\Big[\frac{\partial}{\partial s}\left[\left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right] - \frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right]\right]$$
$$+\tau\left[\mathcal{C}\left[\frac{\tau}{\beta}\right] + \left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]$$
$$+\frac{\tau\kappa^\phi}{\beta}\left[\mathcal{C}\left[\frac{\tau}{\beta}\right] + \left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]\Big] = 0.$$

Corollary 3.9. *If $\nabla_t \mathbf{B}^\phi$ along the curve, parallel to the Fermi–Walker terms, then*

$$\Big[\frac{\partial}{\partial s}\left[\frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right] - \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right] \tag{24}$$
$$-\kappa\left[\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\tau}{\beta}\right] - \mathcal{C}\left[\frac{\kappa}{\beta}\right]\right]$$
$$-\frac{\kappa\kappa^\phi}{\beta}\left[\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\tau}{\beta}\right] - \mathcal{C}\left[\frac{\kappa}{\beta}\right]\right]\Big] = 0,$$

$$\Big[\kappa\left[\frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right] - \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]$$
$$+\frac{\partial}{\partial s}\left[\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\tau}{\beta}\right] - \mathcal{C}\left[\frac{\kappa}{\beta}\right]\right]$$
$$-\tau\left[\frac{\partial}{\partial t}\left[\frac{\kappa}{\beta}\right] + \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right]$$
$$+\kappa^\phi\Big[\frac{\kappa}{\beta}\left[\frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right] - \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]$$
$$-\frac{\tau}{\beta}\left[\frac{\partial}{\partial t}\left[\frac{\kappa}{\beta}\right] + \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right]\Big]\Big] = 0,$$

$$+\Big[\frac{\partial}{\partial s}\left[\frac{\partial}{\partial t}\left[\frac{\kappa}{\beta}\right] + \left(\beta_2\tau + \frac{\partial\beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right]$$
$$+\tau\left[\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\tau}{\beta}\right] - \mathcal{C}\left[\frac{\kappa}{\beta}\right]\right]$$
$$+\frac{\tau\kappa^\phi}{\beta}\left[\left(\beta_1\kappa - \beta_3\tau + \frac{\partial\beta_2}{\partial s}\right)\left[\frac{\tau}{\beta}\right] - \mathcal{C}\left[\frac{\kappa}{\beta}\right]\right]\Big] = 0.$$

4 Applications to electrodynamics

In this section, we construct the Fermi–Walker derivative in the motion of a charged particle under the action of only electric or magnetic fields.

The equation of motion of a charged particle of mass m and electric charge q under the electric field $\mathcal{E}$ and magnetic field $\mathcal{B}$ is given by the Lorentz equation. In Gaussian system of units, we have [25]:

$$m\frac{dv}{ds} = q\mathcal{E} + qv \times \mathcal{B}. \tag{25}$$

Case I. Only a magnetic induction $\mathcal{B}$ (no electric field $\mathcal{E}$), the equation of motion is

$$m\frac{d\mathbf{T}^\phi}{ds} = q\mathbf{T}^\phi \times \mathcal{B}. \tag{26}$$

From above equation and Frenet frame, we easily choose

$$\mathcal{B} = -\frac{\kappa^\phi m}{q}\mathbf{B}^\phi. \tag{27}$$

Then,

$$\mathcal{B} = -\frac{\kappa^\phi m}{q}\left(\frac{\tau}{\beta}\mathbf{T} + \frac{\kappa}{\beta}\mathbf{B}\right). \tag{28}$$

Therefore, we can write

$$\nabla_s^{FW}\mathcal{B} = -\frac{\partial}{\partial s}\left(\frac{\kappa^\phi m}{q}\frac{\tau}{\beta}\right)\mathbf{T} - \frac{\kappa^\phi m}{q}\frac{\tau}{\beta}\kappa\mathbf{N} \tag{29}$$
$$+\left[\frac{\kappa^\phi m}{q}\frac{\kappa\tau}{\beta} - \frac{\partial}{\partial s}\left(\frac{\kappa^\phi m}{q}\frac{\kappa}{\beta}\right)\right]\mathbf{B}.$$

which implies that

$$\frac{\partial}{\partial s}\left(\frac{\kappa^{\phi} m}{q}\frac{\tau}{\beta}\right)=0, \tag{30}$$

$$\frac{\kappa^{\phi} m}{q}\frac{\tau}{\beta}\kappa=0, \tag{31}$$

$$\frac{\kappa^{\phi} m}{q}\frac{\kappa\tau}{\beta}=\frac{\partial}{\partial s}\left(\frac{\kappa^{\phi} m}{q}\frac{\kappa}{\beta}\right). \tag{32}$$

On the other hand, we obtain

$$\begin{aligned}\nabla_t \mathcal{B} = &\Bigg[-\frac{\partial}{\partial t}\left(\frac{\kappa^{\phi} m}{q}\right)\frac{\tau}{\beta} \\ &-\frac{\kappa^{\phi} m}{q}\left[\frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right]-\left(\beta_2\tau+\frac{\partial\beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]\Bigg]\mathbf{T} \\ &-\frac{\kappa^{\phi} m}{q}\left[\left(\beta_1\kappa-\beta_3\tau+\frac{\partial\beta_2}{\partial s}\right)\left[\frac{\tau}{\beta}\right]-\mathcal{C}\left[\frac{\kappa}{\beta}\right]\right]\mathbf{N} \\ &+\left[\frac{\kappa}{\beta}-\frac{\kappa^{\phi} m}{q}\left[\frac{\partial}{\partial t}\left[\frac{\kappa}{\beta}\right]+\left(\beta_2\tau+\frac{\partial\beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right]\right]\mathbf{B}.\end{aligned} \tag{33}$$

If $\nabla_t \mathcal{B}$ along the curve, parallel to the Fermi–Walker terms, then

$$\begin{aligned}&-\frac{\partial}{\partial t}\left(\frac{\kappa^{\phi} m}{q}\right)\frac{\tau}{\beta} \\ &-\frac{\kappa^{\phi} m}{q}\left[\frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right]-\left(\beta_2\tau+\frac{\partial\beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]=0,\end{aligned} \tag{34}$$

$$\frac{\kappa^{\phi} m}{q}\left[\left(\beta_1\kappa-\beta_3\tau+\frac{\partial\beta_2}{\partial s}\right)\left[\frac{\tau}{\beta}\right]-\mathcal{C}\left[\frac{\kappa}{\beta}\right]\right]=0, \tag{35}$$

$$\frac{\kappa}{\beta}-\frac{\kappa^{\phi} m}{q}\left[\frac{\partial}{\partial t}\left[\frac{\kappa}{\beta}\right]+\left(\beta_2\tau+\frac{\partial\beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right]=0. \tag{36}$$

Case II. Only an electric induction $\mathcal{E}$ (no magnetic field $\mathcal{B}$), the equation of motion is

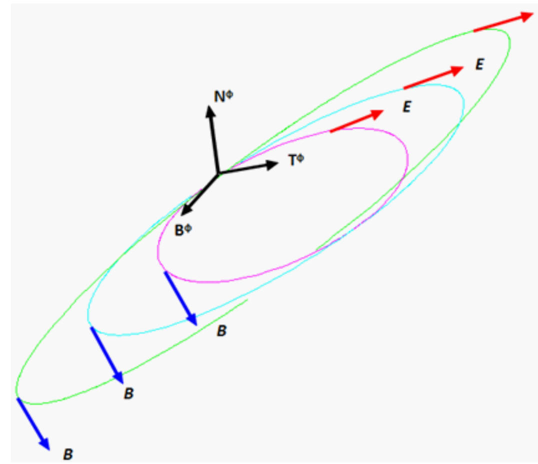
$$\mathcal{E}=\frac{m\kappa^{\phi}}{q}\mathbf{N}^{\phi}. \tag{37}$$

Then, we easily have

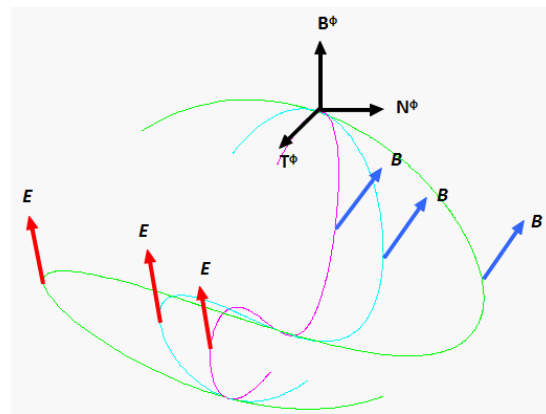
$$\begin{aligned}\nabla_s^{FW}\mathcal{E}=&\left[\frac{\partial}{\partial s}\left(\frac{m\kappa^{\phi}}{q}\right)\frac{\kappa}{\beta}+\frac{m\kappa^{\phi}\tau^{\phi}}{q}\frac{\tau}{\beta}\right]\mathbf{T} \\ &+\left[\frac{m\kappa^{\phi}\tau^{\phi}}{q}\frac{\kappa}{\beta}-\frac{\partial}{\partial s}\left(\frac{m\kappa^{\phi}}{q}\right)\frac{\tau}{\beta}\right]\mathbf{B},\end{aligned} \tag{38}$$

which implies that

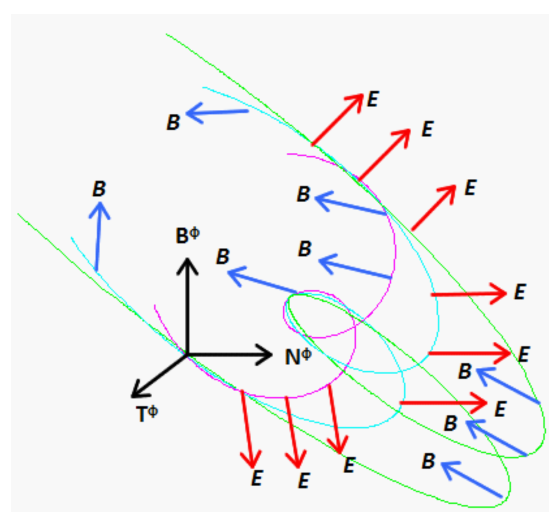
$$\frac{\partial}{\partial s}\left(\frac{m\kappa^{\phi}}{q}\right)\frac{\kappa}{\beta}+\frac{m\kappa^{\phi}\tau^{\phi}}{q}\frac{\tau}{\beta}=0, \tag{39}$$



(a) $A(t)=1$



(b) $A(t)=-t$



(c) $A(t)=t$

Figure 1: Binormal indicatrix of time helix is illustrated with Magenta, Cyan, Green color at the time $t=1.2$, $t=1.8$, $t=2.2$, respectively

$$\frac{m\kappa^{\phi}\tau^{\phi}}{q}\frac{\kappa}{\beta}-\frac{\partial}{\partial s}\left(\frac{m\kappa^{\phi}}{q}\right)\frac{\tau}{\beta}=0. \tag{40}$$

On the other hand, we obtain

$$\begin{aligned}\nabla_t\mathcal{E}=&\left[\frac{\partial}{\partial t}\left(\frac{m\kappa^{\phi}}{q}\right)\frac{\kappa}{\beta}+\frac{\partial}{\partial t}\frac{\kappa}{\beta}\right.\\&\left.+\left(\beta_2\tau+\frac{\partial\beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right]\mathbf{T}\\&+\left[\mathcal{C}\left[\frac{\tau}{\beta}\right]+\left(\beta_1\kappa-\beta_3\tau+\frac{\partial\beta_2}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\right]\mathbf{N}\\&+\left[-\frac{\partial}{\partial t}\left(\frac{m\kappa^{\phi}}{q}\right)\frac{\tau}{\beta}+\left(\beta_2\tau+\frac{\partial\beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]-\frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right]\right]\mathbf{B}.\end{aligned} \tag{41}$$

If $\nabla_t\mathcal{E}$ along the curve, parallel to the Fermi–Walker terms, then

$$\frac{\kappa}{\beta}-\frac{\kappa^{\phi}m}{q}\left[\frac{\partial}{\partial t}\left[\frac{\kappa}{\beta}\right]+\left(\beta_2\tau+\frac{\partial\beta_3}{\partial s}\right)\left[\frac{\tau}{\beta}\right]\right]=0, \tag{42}$$

$$\mathcal{C}\left[\frac{\tau}{\beta}\right]+\left(\beta_1\kappa-\beta_3\tau+\frac{\partial\beta_2}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]=0, \tag{43}$$

$$\begin{aligned}&-\frac{\partial}{\partial t}\left(\frac{m\kappa^{\phi}}{q}\right)\frac{\tau}{\beta}+\left(\beta_2\tau+\frac{\partial\beta_3}{\partial s}\right)\left[\frac{\kappa}{\beta}\right]\\&-\frac{\partial}{\partial t}\left[\frac{\tau}{\beta}\right]=0.\end{aligned} \tag{44}$$

5 Some pictures

In this section we draw some pictures corresponding to different cases by using a following example:

The time helix is parametrized by

$$\alpha_t(s)=(A(t)\cos(s),A(t)\sin(s),B(t)s), \tag{45}$$

where A, B are functions of time only.

6 Conclusions

Fermi-Walker transport and inextensible flows play an important role in geometric design and theorical physics.

In this paper, we have studied Fermi-Walker derivative and Fermi–Walker parallelism for binormal indicatrix. The aim of this work is to show inextensible flows of Fermi-Walker derivative by using curvatures of curves.

Furthermore, using the Frenet frame of the given curve, we present some partial differential equations. We have given some illustrations together with some examples, which we have used flows of Frenet frame and Fermi derivative in space. Finally, we construct the Fermi–Walker derivative in the motion of a charged particle.

In our future work under this theme, we propose to study the conditions on the Fermi-Walker derivative and Fermi-Walker parallelism for spherical indicatrix in Minkowski space.

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