

Research Article

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On the fractional differential equations with not instantaneous impulses

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Abstract: Based on some previous works, an equivalent equations is obtained for the differential equations of fractional-order $q \in (1, 2)$ with non-instantaneous impulses, which shows that there exists the general solution for this impulsive fractional-order systems. Next, an example is used to illustrate the conclusion.

Keywords: fractional differential equations, impulses, non-instantaneous impulses, general solution

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1 Introduction

Fractional differential equations has gained much attention in literature because of its applications for description of hereditary properties in many fields, and some progresses were gotten in computation methods, controllability, existence etc. for fractional differential equations [1–6]. Moreover, impulsive fractional (partial) differential equations were widely studied [7–29] due to importance in description of some processes in which sudden, discontinuous jumps occur, and general solution has been

discovered for several impulsive fractional order systems in [30–35].

However, Hernandez and O'Regan in [36] pointed out that the instantaneous impulses (considered in almost all papers about impulsive differential equations) cannot characterize some processes such as evolution processes in pharmacotherapy, and presented a kind of impulsive differential equations with non-instantaneous impulses. Moreover, the existence of solution is considered for some fractional order systems with non-instantaneous impulses in [37, 38].

Based on the above-works, we will study the following fractional order system with non-instantaneous impulses.

$$\begin{cases} {}^C D_{0+}^q x(t) = f(t, x(t)), & (1.1a) \\ q \in (1, 2), t \in (s_k, t_{k+1}], k = 0, 1, \dots, N, \\ x(t) = g_k(t, x(t)), \quad t \in (t_k, s_k], k = 1, 2, \dots, N, & (1.1b) \\ x(0) = x_0, x'(0) = \bar{x}_0, \quad x_0, \bar{x}_0 \in \mathbb{R}. & (1.1c) \end{cases}$$

here ${}^C D_{0+}^q$ is the Caputo fractional derivative of order q . $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $g_k : (t_k, s_k] \times \mathbb{R} \rightarrow \mathbb{R}$ are some appropriate functions, and g_k denote non-instantaneous impulses, and $g'_k(s_k, x(s_k))$ exist (here $k = 1, 2, \dots, N$), and $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 \leq \dots \leq t_N \leq s_N \leq t_{N+1} = T$.

Next, let us introduce the concept of the fractional derivative and some conclusions in Section 2, and provide main result in section 3, and give an example to show the usefulness of the obtained result.

2 Preliminaries

Definition 2.1 [39]. The left-sided Riemann-Liouville fractional integral I_{a+}^p of order p ($p > 0$) for function x is defined as

$$I_{a+}^p x(t) = \frac{1}{\Gamma(p)} \int_a^t \frac{x(\tau)}{(t-\tau)^{1-p}} d\tau \quad (t > a, p > 0),$$

where $\Gamma(\cdot)$ is the Gamma function.

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Definition 2.2 [39]. The Caputo fractional derivative ${}^C D_{a+}^q x$ of order $q (q > 0)$ for a function x can be written as

$${}^C D_{a+}^q x(t) = \frac{1}{\Gamma(n-q)} \int_a^t \frac{x^{(n)}(\tau)}{(t-\tau)^{q+1-n}} d\tau$$

$$= (I_{a+}^{n-q} D^n x)(t), \quad t > a,$$

where $D = d/dt$ and $q \in (n-1, n)$.

Lemma 2.3 [39, 40]. If the function $g(t, x)$ is continuous, then the initial value problem

$$\begin{cases} {}^C D_{a+}^q x(t) = g(t, x(t)), & q \in (n-1, n), n \geq 1, \\ x^{(k)}(a) = x_a^k, & k = 0, 1, 2, \dots, n-1. \end{cases}$$

is equivalent to the following nonlinear Volterra integral equation of the second kind,

$$x(t) = \sum_{k=0}^{n-1} \frac{x_a^k}{k!} (t-a)^k + \frac{1}{\Gamma(q)} \int_a^t (t-\tau)^{q-1} g(\tau, x(\tau)) d\tau.$$

Lemma 2.4 [31]. Let ξ and ζ be two constants. The impulsive system

$$\begin{cases} {}^C D_{0+}^q x(t) = f(t, x(t)), \\ \quad q \in (1, 2), t \in J = [0, T], t \neq t_k (k = 1, \dots, m), \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\ \Delta x'|_{t=t_k} = \bar{I}_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\ x(0) = x_0, \quad x'(0) = \bar{x}_0. \end{cases}$$

is equivalent to the integral equation

$$x(t) = \begin{cases} x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f ds & \text{for } t \in J_0, \\ x_0 + \bar{x}_0 t + \sum_{i=1}^k I_i(x(t_i^-)) + \sum_{i=1}^k (t-t_i) \bar{I}_i(x(t_i^-)) \\ \quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f ds \\ \quad + \sum_{i=1}^k [\xi I_i(x(t_i^-)) + \zeta \bar{I}_i(x(t_i^-))] \\ \quad \left\{ \frac{1}{\Gamma(q)} \left[\int_0^{t_i} (t_i-s)^{q-1} f ds \right. \right. \\ \quad \left. \left. + \int_{t_i}^t (t-s)^{q-1} f ds - \int_0^{t_i} (t-s)^{q-1} f ds \right] \right. \\ \quad \left. + \frac{(t-t_i)}{\Gamma(q-1)} \int_0^{t_i} (t_i-s)^{q-2} f ds \right\} & \text{for } t \in J_k, 1 \leq k \leq m. \end{cases} \quad (2.2)$$

provided that the integral in (2.2) exists. Here $J_0 = [0, t_1]$ and $J_k = (t_k, t_{k+1}]$ ($k = 1, 2, \dots, m$).

3 Main result

For convenience, let $f = f(\tau, x(\tau))$ in this section. Consider condition (1.1a) in system (1.1) by using two different ap-

proaches:

$$(i) \begin{cases} {}^C D_{0+}^q x(t) = f(t, x(t)), \\ \quad \text{for } t \in (s_k, t_{k+1}]. \end{cases} = \begin{cases} {}^C D_{s_k+}^q x(t) = f(t, x(t)), \\ \quad \text{for } t \in (s_k, t_{k+1}]. \end{cases}$$

$$\Leftrightarrow \begin{cases} x(t) = x(s_k) + x'(s_k)(t-s_k) + \frac{1}{\Gamma(q)} \int_{s_k}^t (t-\tau)^{q-1} f d\tau, \\ \quad \text{for } t \in (s_k, t_{k+1}]. \end{cases} \quad (3.1)$$

$$(ii) \begin{cases} {}^C D_{0+}^q x(t) = f(t, x(t)), \\ \quad \text{for } t \in (s_k, t_{k+1}] \subset (0, T], \end{cases}$$

$$\Leftrightarrow \begin{cases} x(t) = C_k + B_k t + x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau, \\ \quad \text{for } t \in (s_k, t_{k+1}] \subset (0, T], \\ \quad \text{here } C_k \text{ and } B_k \text{ are constants.} \end{cases} \quad (3.2)$$

Next, substituting (i) into system (1.1), we get

$$\begin{cases} x(t) = x(s_k) + x'(s_k)(t-s_k) + \frac{1}{\Gamma(q)} \int_{s_k}^t (t-\tau)^{q-1} f d\tau, \\ \quad \text{for } t \in (s_k, t_{k+1}], k = 0, 1, \dots, N, \\ x(t) = g_k(t, x(t)), \quad \text{for } t \in (t_k, s_k], k = 1, 2, \dots, N, \\ x(0) = x_0, x'(0) = \bar{x}_0, \quad x_0, \bar{x}_0 \in \mathbb{R}. \end{cases}$$

That is,

$$\tilde{x}(t) = \begin{cases} x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau & \text{for } t \in (0, t_1], \\ g_k(t, x(t)) & \text{for } t \in (t_k, s_k], k = 1, 2, \dots, N, \\ g_k(s_k, x(s_k)) + g'_k(s_k, x(s_k))(t-s_k) \\ \quad + \frac{1}{\Gamma(q)} \int_{s_k}^t (t-\tau)^{q-1} f d\tau & \text{for } t \in (s_k, t_{k+1}], k = 1, \dots, N. \end{cases} \quad (3.3)$$

In fact, $\tilde{x}(t)$ satisfies conditions (1.1a)–(1.1c) in system (1.1). But, we will show that $\tilde{x}(t)$ isn't a solution of system (1.1). For system (1.1), we have

$$\{\text{system (1.1)}\} \begin{cases} g_k(t, x(t)) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau \\ \quad \text{for all } k \in \{1, 2, \dots, N\} \end{cases}$$

$$\rightarrow \begin{cases} {}^C D_{0+}^q x(t) = f(t, x(t)), \quad t \in (s_k, t_{k+1}], \\ \quad k = 0, 1, \dots, N, \\ x(t) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau, \\ \quad t \in (t_k, s_k], k = 1, 2, \dots, N, \\ x(0) = x_0, x'(0) = \bar{x}_0, \quad x_0, \bar{x}_0 \in \mathbb{R}. \end{cases} \quad (3.4)$$

$$\Leftrightarrow \begin{cases} {}^C D_{0+}^q x(t) = f(t, x(t)), \quad t \in (0, T], \\ x(0) = x_0, x'(0) = \bar{x}_0, \quad x_0, \bar{x}_0 \in \mathbb{R}. \end{cases}$$

And system (3.4) is equivalent to

$$x(t) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau \quad \text{for } t \in (0, T]. \quad (3.5)$$

Moreover, letting $g_k(t, x(t)) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau$ (for all $k \in \{1, 2, \dots, N\}$) in (3.3), we get

$$\tilde{x}(t) = \begin{cases} x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau & \text{for } t \in (0, t_1], \\ x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau & \text{for } t \in (t_k, s_k], \\ & k = 1, 2, \dots, N, \\ x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \left[\int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \right. \\ \left. + \int_{s_k}^t (t - \tau)^{q-1} f d\tau \right] + \frac{(t - s_k)}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau & \\ & \text{for } t \in (s_k, t_{k+1}], k = 1, 2, \dots, N. \end{cases} \quad (3.6)$$

Therefore, if $\tilde{x}(t)$ is a solution of system (1.1), then (3.6) is equivalent to (3.5). Thus,

$$\begin{aligned} & \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau \\ &= \frac{1}{\Gamma(q)} \left[\int_0^{s_k} (s_k - \tau)^{q-1} f d\tau + \int_{s_k}^t (t - \tau)^{q-1} f d\tau \right] \\ &+ \frac{(t - s_k)}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \\ &\text{for } t \in (s_k, t_{k+1}], k = 1, 2, \dots, N. \end{aligned} \quad (3.7)$$

Eq. (3.7) is an unfit equation, which means that $\tilde{x}(t)$ isn't a solution of system (1.1). Therefore, we will regard $\tilde{x}(t)$ as an **approximate solution** to seek the exact solution of system (1.1).

Substituting (ii) into system (1.1), we obtain

$$\begin{cases} x(t) = C_k + B_k t + x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau, \\ \quad \text{for } t \in (s_k, t_{k+1}], k = 0, 1, \dots, N, \\ x(t) = g_k(t, x(t)), \text{ for } t \in (t_k, s_k], k = 1, 2, \dots, N, \\ x(0) = x_0, x'(0) = \bar{x}_0, \quad x_0, \bar{x}_0 \in \mathbb{R}. \end{cases} \quad (3.8)$$

By initial conditions $x(s_k) = g_k(s_k, x(s_k))$ and $x'(s_k) = g'_k(s_k, x(s_k))$ (here $k = 1, 2, \dots, N$), we obtain $B_0 = 0$, $C_0 = 0$,

$$\begin{aligned} B_k &= g'_k(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau, \\ &k = 1, 2, \dots, N. \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} C_k &= g_k(s_k, x(s_k)) - x_0 - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \\ &- g'_k(s_k, x(s_k)) s_k + \frac{s_k}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau, \\ &k = 1, 2, \dots, N, \end{aligned} \quad (3.10)$$

Substituting (3.9)-(3.10) into (3.8), we get

$$x(t) = \begin{cases} x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau & \text{for } t \in (0, t_1], \\ g_k(t, x(t)) & \text{for } t \in (t_k, s_k], k = 1, 2, \dots, N, \\ g_k(s_k, x(s_k)) - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \\ + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau \\ + (t - s_k) \left[g'_k(s_k, x(s_k)) - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] \\ & \text{for } t \in (s_k, t_{k+1}], k = 1, 2, \dots, N. \end{cases} \quad (3.11)$$

In fact, Eq. (3.11) satisfies conditions (1.1a)–(1.1c) and

$$\begin{aligned} & \{\text{Eq. (3.11)}\} \xrightarrow{g_k(t, x(t)) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau} \\ & \quad \text{for all } k \in \{1, 2, \dots, N\} \\ & \rightarrow x(t) = \begin{cases} x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau, \\ \quad \text{for } t \in (0, t_1], \\ x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau, \\ \quad \text{for } t \in (t_k, s_k], k = 1, 2, \dots, N, \\ x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau, \\ \quad \text{for } t \in (s_k, t_{k+1}], k = 1, 2, \dots, N. \end{cases} \\ & \Leftrightarrow \begin{cases} {}^c D_{0+}^q x(t) = f(t, x(t)), \quad t \in (0, T], \\ x(0) = x_0, x'(0) = \bar{x}_0, \quad x_0, \bar{x}_0 \in \mathbb{R} \end{cases} \\ & \quad \Leftrightarrow \lim_{\substack{g_k(t, x(t)) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau \\ \text{for } t \in (t_k, s_k], \text{ for all } k \in \{1, 2, \dots, N\}}} \{\text{system (1.1)}\}. \end{aligned}$$

Therefore, Eq. (3.11) satisfies all conditions of system (1.1), and it is a solution of system (1.1).

Remark 3.1. $\frac{1}{\Gamma(q)} \int_{s_k}^t (t - \tau)^{q-1} f d\tau$ ($k = 1, 2, \dots, N$) is a key part of the approximate solution $\tilde{x}(t)$, and it is not included in Eq. (3.11). Therefore, Eq. (3.11) is a **particular solution** of (1.1).

Theorem 3.1. Let ξ_k and ζ_k (here $k = 1, 2, \dots, N$) be some constants. System (1.1) is equivalent with the integral equation

$$x(t) = \begin{cases} x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau & \text{for } t \in (0, t_1], \\ g_k(t, x(t)) & \text{for } t \in (t_k, s_k], k = 1, 2, \dots, N, \\ g_k(s_k, x(s_k)) - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \\ + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau \\ + (t - s_k) \left[g'_k(s_k, x(s_k)) - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] \\ + \left\{ \xi_k \left[g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k \right. \right. \\ \left. \left. - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \right] \right. \\ \left. + \zeta_k \left[g'_k(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] \right\} \\ \times \left\{ \frac{1}{\Gamma(q)} \left[\int_0^{s_k} (s_k - \tau)^{q-1} f d\tau + \int_{s_k}^t (t - \tau)^{q-1} f d\tau \right. \right. \\ \left. \left. - \int_0^t (t - \tau)^{q-1} f d\tau \right] + \frac{(t - s_k)}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right\} \\ & \text{for } t \in (s_k, t_{k+1}], k = 1, 2, \dots, N. \end{cases} \quad (3.12)$$

provided that the integral in (3.12) exists.

Proof. ‘Sufficiency’; the solution of (1.1) for $t \in (0, t_1]$ satisfies

$$x(t) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau \quad \text{for } t \in (0, t_1], \quad (3.13)$$

and $x(t) = g_1(t, x(t))$ for $t \in (t_1, s_1]$.

For $t \in (s_1, t_2]$, the approximate solution of (1.1) is given by

$$\tilde{x}(t) = g_1(s_1, x(s_1)) + (t-s_1)g'_1(s_1, x(s_1)) + \frac{1}{\Gamma(q)} \int_{s_1}^t (t-\tau)^{q-1} f d\tau \quad \text{for } t \in (s_1, t_2]. \quad (3.14)$$

Let $e_1(t) = x(t) - \tilde{x}(t)$ for $t \in (s_1, t_2]$. Moreover, by the particular solution (3.11), the exact solution $x(t)$ of system (1.1) satisfies

$$\lim_{\substack{[g_1(s_1, x(s_1)) - x_0 - \bar{x}_0 s_1 - \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau] \rightarrow 0, \\ [g'_1(s_1, x(s_1)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau] \rightarrow 0}} x(t) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau \quad \text{for } t \in (s_1, t_2].$$

Thus,

$$\begin{aligned} \lim_{\substack{[g_1(s_1, x(s_1)) - x_0 - \bar{x}_0 s_1 - \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau] \rightarrow 0, \\ [g'_1(s_1, x(s_1)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau] \rightarrow 0}} e_1(t) &= \lim_{\substack{[g_1(s_1, x(s_1)) - x_0 - \bar{x}_0 s_1 - \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau] \rightarrow 0, \\ [g'_1(s_1, x(s_1)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau] \rightarrow 0}} \{x(t) - \tilde{x}(t)\} \\ &= \frac{1}{\Gamma(q)} \left[\int_0^t (t-\tau)^{q-1} f d\tau - \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau - \int_{s_1}^t (t-\tau)^{q-1} f d\tau \right] - \frac{t-s_1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau. \end{aligned} \quad (3.15)$$

This means $e_1(t)$ is connected with $\lim_{\substack{[g_1(s_1, x(s_1)) - x_0 - \bar{x}_0 s_1 - \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau] \rightarrow 0, \\ [g'_1(s_1, x(s_1)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau] \rightarrow 0}} e_1(t)$, $g_1(s_1, x(s_1)) - x_0 - \bar{x}_0 s_1 - \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau$ and $g'_1(s_1, x(s_1)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau$. Therefore, suppose

$$\begin{aligned} e_1(t) &= \chi \left(g_1(s_1, x(s_1)) - x_0 - \bar{x}_0 s_1 - \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau, \right. \\ &\quad \left. g'_1(s_1, x(s_1)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau \right) \left\{ \frac{1}{\Gamma(q)} \left[\int_0^t (t-\tau)^{q-1} f d\tau \right. \right. \\ &\quad \left. \left. - \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau - \int_{s_1}^t (t-\tau)^{q-1} f d\tau \right] - \frac{t-s_1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau \right\} \quad \text{for } t \in (s_1, t_2]. \end{aligned} \quad (3.16)$$

where $\chi(\cdot, \cdot)$ is an undetermined function with $\chi(0, 0) = 1$. Thus,

$$\begin{aligned} x(t) &= \tilde{x}(t) + e_1(t) = g_1(s_1, x(s_1)) - \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau + (t-s_1) \left[g'_1(s_1, x(s_1)) - \frac{1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau \right] \\ &\quad + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau + \left[1 - \chi \left(g_1(s_1, x(s_1)) - x_0 - \bar{x}_0 s_1 - \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau, \right. \right. \\ &\quad \left. \left. g'_1(s_1, x(s_1)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau \right) \right] \left\{ \frac{1}{\Gamma(q)} \left[\int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau \right. \right. \\ &\quad \left. \left. + \int_{s_1}^t (t-\tau)^{q-1} f d\tau - \int_0^t (t-\tau)^{q-1} f d\tau \right] + \frac{t-s_1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau \right\} \quad \text{for } t \in (s_1, t_2]. \end{aligned} \quad (3.17)$$

On the other hand, letting $t_1 \rightarrow s_1$, we get

$$\begin{aligned} \lim_{t_1 \rightarrow s_1} &\begin{cases} {}^C D_{0+}^q x(t) = f(t, x(t)), & q \in (1, 2), t \in (s_k, t_{k+1}], k = 0, 1, \\ x(t) = g_1(t, x(t)), & t \in (t_1, s_1], \\ x(0) = x_0, x'(0) = \bar{x}_0, & x_0, \bar{x}_0 \in \mathbb{R}. \end{cases} \\ &= \begin{cases} {}^C D_{0+}^q x(t) = f(t, x(t)), & q \in (1, 2), t \in (s_k, t_{k+1}], k = 0, 1, \\ x(t_1) = x(s_1) = g_1(s_1, x(s_1)), \\ x'(t_1) = x'(s_1) = g'_1(s_1, x(s_1)), \\ x(0) = x_0, x'(0) = \bar{x}_0, & x_0, \bar{x}_0 \in \mathbb{R}. \end{cases} \\ &\Leftrightarrow \begin{cases} {}^C D_{0+}^q x(t) = f(t, x(t)), & q \in (1, 2), t \in (s_k, t_{k+1}], k = 0, 1, \\ x(s_1^+) - x(s_1^-) = g_1(s_1, x(s_1)) - x_0 - \bar{x}_0 s_1 - \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f(\tau, x(\tau)) d\tau, \\ x'(s_1^+) - x'(s_1^-) = g'_1(s_1, x(s_1)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f(\tau, x(\tau)) d\tau, \\ x(0) = x_0, x'(0) = \bar{x}_0, & x_0, \bar{x}_0 \in \mathbb{R}. \end{cases} \end{aligned} \quad (3.18)$$

Using **Lemma 2.4** for system (3.18), we get $1 - \chi(y, z) = \xi_1 y + \zeta_1 z$ for $\forall y, z \in \mathbb{R}$, here ξ_1 and ζ_1 are two constants. Thus,

$$\begin{aligned} x(t) = & g_1(s_1, x(s_1)) - \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau + (t - s_1) \left[g'_1(s_1, x(s_1)) - \frac{1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau \right] \\ & + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau + \left\{ \xi_1 \left[g_1(s_1, x(s_1)) - x_0 - \bar{x}_0 s_1 - \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau \right] \right. \\ & + \zeta_1 \left[g'_1(s_1, x(s_1)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau \right] \left. \right\} \left\{ \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau \right. \\ & + \int_{s_1}^t (t - \tau)^{q-1} f d\tau - \int_0^t (t - \tau)^{q-1} f d\tau \left. \right\} + \frac{(t-s_1)}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau \quad \text{for } t \in (s_1, t_2]. \end{aligned} \quad (3.19)$$

and $x(t) = g_2(t, x(t))$ for $t \in (t_2, s_2]$.

Next, for $t \in (s_k, t_{k+1}]$ (here $k \in \{1, 2, \dots, N\}$), the approximate solution of (1.1) is provided by

$$\tilde{x}(t) = g_k(s_k, x(s_k)) + (t - s_k) g'_k(s_k, x(s_k)) + \frac{1}{\Gamma(q)} \int_{s_k}^t (t - \tau)^{q-1} f d\tau \quad \text{for } t \in (s_k, t_{k+1}]. \quad (3.20)$$

Let $e_k(t) = x(t) - \tilde{x}(t)$ for $t \in (s_k, t_{k+1}]$. Moreover, by the particular solution (3.11), the exact solution $x(t)$ of system (1.1) satisfies

$$\begin{aligned} \lim_{t \rightarrow s_k^+} \left[g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \right] & \rightarrow 0, \\ \lim_{t \rightarrow s_k^+} \left[g'_k(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] & \rightarrow 0, \end{aligned} \quad x(t) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau \quad \text{for } t \in (s_k, t_{k+1}].$$

Thus,

$$\begin{aligned} & \lim_{t \rightarrow s_k^+} \left[g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \right] \rightarrow 0, \\ & \lim_{t \rightarrow s_k^+} \left[g'_k(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] \rightarrow 0, \\ & = \lim_{t \rightarrow s_k^+} \left\{ x(t) - \tilde{x}(t) \right\} \\ & = \lim_{t \rightarrow s_k^+} \left[g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \right] \rightarrow 0, \\ & \left[g'_k(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] \rightarrow 0, \\ & = \frac{-1}{\Gamma(q)} \left[\int_0^{s_k} (s_k - \tau)^{q-1} f d\tau + \int_{s_k}^t (t - \tau)^{q-1} f d\tau - \int_0^t (t - \tau)^{q-1} f d\tau \right] - \frac{t-s_k}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau, \end{aligned} \quad (3.21)$$

Similarly to (3.16), suppose

$$\begin{aligned} e_k(t) = & \kappa \left(g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau, \right. \\ & g'_k(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \left. \right) \left\{ \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau \right. \\ & \left. - \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau - \int_{s_k}^t (t - \tau)^{q-1} f d\tau \right\} - \frac{t-s_k}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \quad \text{for } t \in (s_k, t_{k+1}]. \end{aligned} \quad (3.21)$$

where $\kappa(\cdot, \cdot)$ is an undetermined function with $\kappa(0, 0) = 1$. Thus,

$$\begin{aligned} x(t) = & \tilde{x}(t) + e_k(t) = g_k(s_k, x(s_k)) - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \\ & + (t - s_k) \left[g'_k(s_k, x(s_k)) - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau \\ & + \left[1 - \kappa \left(g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau, \right. \right. \\ & \left. \left. g'_k(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right) \right] \left\{ \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \right. \\ & \left. + \int_{s_k}^t (t - \tau)^{q-1} f d\tau - \int_0^t (t - \tau)^{q-1} f d\tau \right\} + \frac{t-s_k}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \quad \text{for } t \in (s_k, t_{k+1}]. \end{aligned} \quad (3.22)$$

Moreover, considering a special case $g_i(t, x(t)) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau$ for all $i \in \{1, 2, \dots, k-1\}$ and $t_k \rightarrow s_k$ in system (1.1), we have

$$\begin{aligned} \lim_{t_k \rightarrow s_k} & \begin{cases} {}^C D_{0+}^q x(t) = f(t, x(t)), & q \in (1, 2), t \in (s_i, t_{i+1}], i = 0, 1, \dots, k, \\ x(t) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau, x(\tau)) d\tau, & t \in (t_i, s_i], i = 1, 2, \dots, k-1, \\ x(t) = g_k(t, x(t)), & t \in (t_k, s_k], \\ x(0) = x_0, x'(0) = \bar{x}_0, & x_0, \bar{x}_0 \in \mathbb{R}. \end{cases} \\ = & \begin{cases} {}^C D_{0+}^q x(t) = f(t, x(t)), & q \in (1, 2), t \in (s_i, t_{i+1}], i = 0, 1, \dots, k, \\ x(t) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau, x(\tau)) d\tau, & t \in (t_i, s_i], i = 1, 2, \dots, k-1, \\ x(s_k) = x(t_k) = g_k(s_k, x(s_k)), \\ x'(s_k) = x'(t_k) = g'_k(s_k, x(s_k)), \\ x(0) = x_0, x'(0) = \bar{x}_0, & x_0, \bar{x}_0 \in \mathbb{R}. \end{cases} \\ \Leftrightarrow & \begin{cases} {}^C D_{0+}^q x(t) = f(t, x(t)), & q \in (1, 2), t \in (s_i, t_{i+1}], i = 0, 1, \dots, k, \\ x(t) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau, x(\tau)) d\tau, & t \in (t_i, s_i], i = 1, 2, \dots, k-1, \\ x(s_k^+) - x(s_k^-) = g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f(\tau, x(\tau)) d\tau, \\ x'(s_k^+) - x'(s_k^-) = g'_k(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f(\tau, x(\tau)) d\tau, \\ x(0) = x_0, x'(0) = \bar{x}_0, & x_0, \bar{x}_0 \in \mathbb{R}. \end{cases} \end{aligned} \quad (3.23)$$

Using **Lemma 2.4** and Eq. (3.23) for (3.24), we have $1 - \kappa(y, z) = \xi_k y + \zeta_k z$ for $\forall y, z \in \mathbb{R}$, here ξ_k and ζ_k are two constants. Thus,

$$\begin{aligned} x(t) = & g_k(s_k, x(s_k)) - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau + (t - s_k) \left[g'_k(s_k, x(s_k)) - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] \\ & + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau + \left\{ \xi_k \left[g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \right] \right. \\ & + \zeta_k \left[g'_k(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] \left\{ \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \right. \\ & \left. \left. + \int_{s_k}^t (t - \tau)^{q-1} f d\tau - \int_0^t (t - \tau)^{q-1} f d\tau \right\} + \frac{(t-s_k)}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right\}, \text{ for } t \in (s_k, t_{k+1}]. \end{aligned}$$

‘Necessity’; taking the fractional derivative to Eq. (3.12) for $t \in (s_k, t_{k+1}]$ (here $k = 1, 2, \dots, N$), we get

$$\begin{aligned} {}^C D_{0+}^q x(t) \Big|_{t \in (s_k, t_{k+1}]} &= {}^C D_{0+}^q \left\{ g_k(s_k, x(s_k)) - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau \right. \\ &\quad + (t - s_k) \left[g'_k(s_k, x(s_k)) - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] \\ &\quad + \left\{ \xi_k \left[g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \right] \right. \\ &\quad + \zeta_k \left[g'_k(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] \left\{ \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \right. \\ &\quad \left. \left. + \int_{s_k}^t (t - \tau)^{q-1} f d\tau - \int_0^t (t - \tau)^{q-1} f d\tau \right\} + \frac{(t-s_k)}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right\} \Big|_{t \in (s_k, t_{k+1}]} \\ &= f(t, x(t)) \Big|_{t \in (s_k, t_{k+1}]} + \frac{1}{\Gamma(q)} \left\{ \xi_k \left[g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \right] \right. \\ &\quad + \zeta_k \left[g'_k(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] \left. \right\} \\ &\quad \times \left[{}^C D_{s_k+}^q \left(\int_{s_k}^t (t - \tau)^{q-1} f(\tau, x(\tau)) d\tau \right) - {}^C D_{0+}^q \left(\int_0^t (t - \tau)^{q-1} f(\tau, x(\tau)) d\tau \right) \right] \Big|_{t \in (s_k, t_{k+1}]} \\ &= f(t, x(t)) \Big|_{t \in (s_k, t_{k+1}]} + \left\{ \xi_k \left[g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \right] \right. \\ &\quad + \zeta_k \left[g'_k(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] \left. \right\} \left[f(t, x(t)) \Big|_{t > s_k} - f(t, x(t)) \Big|_{t > 0} \right] \Big|_{t \in (s_k, t_{k+1}]} \\ &= f(t, x(t)) \Big|_{t \in (s_k, t_{k+1}]} . \end{aligned}$$

Therefore, Eq. (3.12) satisfies the condition (1.1a). Next, Eq. (3.12) also satisfies the conditions (1.1b) and (1.1c). Furthermore, by Eq. (3.12), we obtain

$$\begin{aligned}
 & \lim_{\substack{g_k(t, x(t)) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} d\tau \\ \text{for } t \in (t_k, s_k], \text{ for all } k \in \{1, 2, \dots, N\}}} x(t) \\
 &= \lim_{\substack{g_k(t, x(t)) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} d\tau \\ \text{for } t \in (t_k, s_k], \text{ for all } k \in \{1, 2, \dots, N\}}} \begin{cases} x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau & \text{for } t \in (0, t_1], \\ g_k(t, x(t)) & \text{for } t \in (t_k, s_k], k = 1, 2, \dots, N, \\ g_k(s_k, x(s_k)) - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau \\ \quad + (t - s_k) \left[g'_k(s_k, x(s_k)) - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] \\ \quad + \left\{ \xi_k \left[g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \right] \right. \\ \quad \left. + \zeta_k \left[g'_k(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] \right\} \\ \quad \times \left\{ \frac{1}{\Gamma(q)} \left[\int_0^{s_k} (s_k - \tau)^{q-1} f d\tau + \int_{s_k}^t (t - \tau)^{q-1} f d\tau - \int_0^t (t - \tau)^{q-1} f d\tau \right] \right. \\ \quad \left. + \frac{(t-s_k)}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right\} & \text{for } t \in (s_k, t_{k+1}], k = 1, 2, \dots, N. \end{cases} \\
 &= \begin{cases} x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau, & \text{for } t \in (0, t_1], \\ x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau, & \text{for } t \in (t_k, s_k], k = 1, 2, \dots, N, \\ x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f d\tau, & \text{for } t \in (s_k, t_{k+1}], k = 1, 2, \dots, N. \end{cases} \\
 &\Leftrightarrow \{\text{system (3.4)}\}
 \end{aligned}$$

Thus,

$$\lim_{\substack{g_k(t, x(t)) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} d\tau \\ \text{for } t \in (t_k, s_k], \text{ for all } k \in \{1, 2, \dots, N\}}} \{\text{Eq. (3.12)}\} \Leftrightarrow \lim_{\substack{g_k(t, x(t)) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} d\tau \\ \text{for } t \in (t_k, s_k], \text{ for all } k \in \{1, 2, \dots, N\}}} \{\text{system (1.1)}\}$$

So, Eq. (3.12) satisfies all conditions of system (1.1).

By “Sufficiency” and “Necessity”, system (1.1) is equivalent to Eq. (3.12). The proof is completed. $\square$

4 Example

Example 1. Let us consider the general solution of the impulsive fractional system

$$\begin{cases} {}^C D_{0+}^{\frac{5}{4}} x(t) = t, & t \in (0, \frac{\pi}{4}] \cup (\frac{\pi}{2}, \pi], \\ x(t) = \sin t, & t \in (\frac{\pi}{4}, \frac{\pi}{2}], \\ x(0) = 1, x'(0) = 1. \end{cases} \quad (4.1)$$

By **Theorem 3.1**, system (4.1) has a general solution

$$x(t) = \begin{cases} 1 + t + \frac{t^{\frac{9}{4}}}{\Gamma(\frac{13}{4})} & \text{for } t \in (0, \frac{\pi}{4}], \\ \sin t & \text{for } t \in (\frac{\pi}{4}, \frac{\pi}{2}], \\ 1 + \frac{t^{\frac{9}{4}} \Big|_{t=0} - (\frac{\pi}{2})^{\frac{9}{4}}}{\Gamma(\frac{13}{4})} - \frac{(\frac{\pi}{2})^{\frac{5}{4}}}{\Gamma(\frac{9}{4})} (t - \frac{\pi}{2}) \Big|_{t > \frac{\pi}{2}} - \left\{ \xi \left[\frac{\pi}{2} + \frac{(\frac{\pi}{2})^{\frac{9}{4}}}{\Gamma(\frac{13}{4})} \right] + \zeta \left[1 + \frac{(\frac{\pi}{2})^{\frac{5}{4}}}{\Gamma(\frac{9}{4})} \right] \right\} \\ \quad \times \left\{ \frac{1}{\Gamma(\frac{13}{4})} \left[\left(\frac{\pi}{2} \right)^{\frac{9}{4}} + \left(t - \frac{\pi}{2} \right)^{\frac{5}{4}} \left(t + \frac{5\pi}{8} \right) \Big|_{t > \frac{\pi}{2}} - t^{\frac{9}{4}} \Big|_{t > 0} \right] + \frac{(\frac{\pi}{2})^{\frac{5}{4}} (t - \frac{\pi}{2}) \Big|_{t > \frac{\pi}{2}}}{\Gamma(\frac{9}{4})} \right\} & \text{for } t \in (\frac{\pi}{2}, \pi]. \end{cases} \quad (4.2)$$

where ξ and ζ are two constants.

Eq. (4.2) for $t \in (0, \frac{\pi}{4}]$ satisfies fractional derivative condition in system (4.1) by **Lemma 2.3**, and for $t \in (\frac{\pi}{2}, \pi]$, we have

$$\begin{aligned} & {}^C D_{0+}^{\frac{5}{4}} x(t) \Big|_{t \in (\frac{\pi}{2}, \pi]} \\ &= {}^C D_{0+}^{\frac{5}{4}} \left\{ 1 + \frac{t^{\frac{9}{4}} \Big|_{t>0} - (\frac{\pi}{2})^{\frac{9}{4}}}{\Gamma(\frac{13}{4})} - \frac{(\frac{\pi}{2})^{\frac{5}{4}}}{\Gamma(\frac{9}{4})} (t - \frac{\pi}{2}) \Big|_{t>\frac{\pi}{2}} \right. \\ &\quad - \left\{ \xi \left[\frac{\pi}{2} + \frac{(\frac{\pi}{2})^{\frac{9}{4}}}{\Gamma(\frac{13}{4})} \right] + \zeta \left[1 + \frac{(\frac{\pi}{2})^{\frac{5}{4}}}{\Gamma(\frac{9}{4})} \right] \right\} \\ &\quad \times \left\{ \frac{1}{\Gamma(\frac{13}{4})} \left[(\frac{\pi}{2})^{\frac{9}{4}} + (t - \frac{\pi}{2})^{\frac{5}{4}} (t + \frac{5\pi}{8}) \Big|_{t>\frac{\pi}{2}} - t^{\frac{9}{4}} \Big|_{t>0} \right] \right. \\ &\quad \left. \left. + \frac{(\frac{\pi}{2})^{\frac{5}{4}} (t - \frac{\pi}{2}) \Big|_{t>\frac{\pi}{2}}}{\Gamma(\frac{9}{4})} \right\} \right\}_{t \in (\frac{\pi}{2}, \pi]} \\ &= {}^C D_{0+}^{\frac{5}{4}} \left\{ \frac{t^{\frac{9}{4}} \Big|_{t>0}}{\Gamma(\frac{13}{4})} - \frac{1}{\Gamma(\frac{13}{4})} \left\{ \xi \left[\frac{\pi}{2} + \frac{(\frac{\pi}{2})^{\frac{9}{4}}}{\Gamma(\frac{13}{4})} \right] \right. \right. \\ &\quad \left. \left. + \zeta \left[1 + \frac{(\frac{\pi}{2})^{\frac{5}{4}}}{\Gamma(\frac{9}{4})} \right] \right\} \left[(t - \frac{\pi}{2})^{\frac{5}{4}} (t + \frac{5\pi}{8}) \Big|_{t>\frac{\pi}{2}} - t^{\frac{9}{4}} \Big|_{t>0} \right] \right\}_{t \in (\frac{\pi}{2}, \pi]} \\ &= \left\{ t \Big|_{t>0} - \frac{1}{\Gamma(\frac{13}{4})} \left\{ \xi \left[\frac{\pi}{2} + \frac{(\frac{\pi}{2})^{\frac{9}{4}}}{\Gamma(\frac{13}{4})} \right] + \zeta \left[1 + \frac{(\frac{\pi}{2})^{\frac{5}{4}}}{\Gamma(\frac{9}{4})} \right] \right\} \right. \\ &\quad \left. \left[{}^C D_{\frac{\pi}{2}+}^{\frac{5}{4}} \left((t - \frac{\pi}{2})^{\frac{5}{4}} (t + \frac{5\pi}{8}) \Big|_{t>\frac{\pi}{2}} \right) - {}^C D_{0+}^{\frac{5}{4}} \left(t^{\frac{9}{4}} \Big|_{t>0} \right) \right] \right\}_{t \in (\frac{\pi}{2}, \pi]} \\ &= \left\{ t \Big|_{t>0} - \left\{ \xi \left[\frac{\pi}{2} + \frac{(\frac{\pi}{2})^{\frac{9}{4}}}{\Gamma(\frac{13}{4})} \right] + \zeta \left[1 + \frac{(\frac{\pi}{2})^{\frac{5}{4}}}{\Gamma(\frac{9}{4})} \right] \right\} \right. \\ &\quad \left. \times \left[t \Big|_{t>\frac{\pi}{2}} - t \Big|_{t>0} \right] \right\}_{t \in (\frac{\pi}{2}, \pi]} = t \Big|_{t \in (\frac{\pi}{2}, \pi]}. \end{aligned}$$

Therefore, Eq. (4.2) (for $t \in (0, \frac{\pi}{4}] \cup (\frac{\pi}{2}, \pi]$) satisfies fractional derivative condition in system (4.1). Meanwhile, Eq. (4.2) satisfies the non-instantaneous impulses condition in system (4.1). Thus, Eq. (4.2) is the general solution of (4.1).

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References

[1] Yang X.J., Machado J.A.T., Baleanu D., Cattani C., On exact traveling-wave solutions for local fractional Korteweg-de Vries equation, *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 2016, 26(8), 110-118.

[2] Yang X.J., Machado J.A.T., Hristov J., Nonlinear dynamics for local fractional Burgers' equation arising in fractal flow, *Nonlinear Dynamics*, 2015, 84(1), 3-7.

[3] Yang X.J., Machado J.A.T., Srivastava H.M., A new numerical technique for solving the local fractional diffusion equation, *Appl. Math. Comput.*, 2016, 274, 143-151.

[4] Kailasavalli S., Baleanu D., Suganya S., Arjunan M. M., Exact controllability of fractional neutral integro-differential systems with state-dependent delay in Banach spaces, *Analele Stiintifice ale Universitatii Ovidius Constanta-Seria Matematica*, 2016, 24(1), 29-55.

[5] Suganya S., Baleanu D., Arjunan M.M., A note on fractional neutral integro-differential inclusions with state-dependent delay in Banach spaces, *Journal of Computational Analysis and Applications*, 2016, 20(7), 1302-1317.

[6] Suganya S., Baleanu D., Selvarasu S., Arjunan M.M., About the Existence Results of Fractional Neutral Integrodifferential Inclusions with State-Dependent Delay in Fréchet Spaces, *Journal of Function Spaces*, vol. 2016, Article ID 6165804, 9 pages, 2016.

[7] Yukunthorn W., Ntouyas S.K., Tariboon J., Impulsive Multiorders Riemann-Liouville Fractional Differential Equations, *Discrete Dynamics in Nature and Society*, vol. 2015, Article ID 603893, 9 pages, 2015.

[8] Thaiprayoon C., Tariboon J., Ntouyas S.K., Impulsive fractional boundary-value problems with fractional integral jump conditions, *Boundary Value Problems*, vol. 2014, article 17, 16 pages, 2014.

[9] Zhang X., Zhang X., Liu Z., Ding W., Cao H., Shu T., On the general solution of impulsive systems with Hadamard fractional derivatives, *Math. Prob. Eng.*, vol. 2016, Article ID 2814310, 12 pages, 2016.

[10] Yukunthorn W., Suantai S., Ntouyas S.K., Tariboon J., Boundary value problems for impulsive multi-order Hadamard fractional differential equations, *Boundary Value Problems*, vol. 2015, article 148, 13 pages, 2015.

[11] Fu X., Liu X., Lu B., On a new class of impulsive fractional evolution equations, *Adv. Differ. Equ.*, vol. 2015, article 227, 16 pages, 2015.

[12] Yukunthorn W., Ahmad B., Ntouyas S.K., Tariboon J., On Caputo-Hadamard type fractional impulsive hybrid systems with nonlinear fractional integral conditions, *Nonlinear Anal.: HS*, 2016, 19, 77-92.

[13] Ahmad B., Sivasundaram S., Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations, *Nonlinear Anal.: HS*, 2009, 3, 251-258.

[14] Ahmad B., Sivasundaram S., Existence of solutions for impulsive integral boundary value problems of fractional order, *Nonlinear Anal.: HS*, 2010, 4, 134-141.

[15] Zhang X., Shu T., Liu Z., Ding W., Peng H., He J., On the concept of general solution for impulsive differential equations of fractional-order $q \in (2, 3)$, *Open math.*, 2016, 14, 452-473.

[16] Ahmad B., Wang G., Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order, *Comput. Math. Appl.*, 2010, 59, 1341-1349.

[17] Tian Y., Bai Z., Existence results for the three-point impulsive boundary value problem involving fractional differential equations, *Comput. Math. Appl.*, 2010, 59, 2601-2609.

[18] Cao J., Chen H., Some results on impulsive boundary value problem for fractional differential inclusions, *Electron. J. Qual. Theory Differ. Equ.*, 2010, 11, 1-24.

- [19] Wang G., Ahmad B., Zhang L., Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order, *Nonlinear Anal. Theory Methods Appl.*, 2011, 74, 792-804.
- [20] Wang G., Ahmad B., Zhang L., Some existence results for impulsive nonlinear fractional differential equations with mixed boundary conditions, *Comput. Math. Appl.*, 2010, 59, 1389-1397.
- [21] Feckan M., Zhou Y., Wang J.R., On the concept and existence of solution for impulsive fractional differential equations, *Commun. Nonlinear Sci. Numer. Simulat.*, 2012, 17, 3050-3060.
- [22] Stamova I., Stamov G., Stability analysis of impulsive functional systems of fractional order, *Commun. Nonlinear Sci. Numer. Simulat.*, 2014, 19, 702-709.
- [23] Zhang X., On impulsive partial differential equations with Caputo-Hadamard fractional derivatives, *Adv. Differ. Equ.*, vol. 2016, article 281, 21 pages, 2016.
- [24] Abbas S., Benchohra M., Upper and lower solutions method for impulsive partial hyperbolic differential equations with fractional order, *Nonlinear Anal. HS*, 2010, 4, 406-413.
- [25] Abbas S., Benchohra M., Impulsive partial hyperbolic functional differential equations of fractional order with state-dependent delay, *Fract. Calc. Appl. Anal.*, 2010, 13, 225-242.
- [26] Abbas S., Agarwal R.P., Benchohra M., Darboux problem for impulsive partial hyperbolic differential equations of fractional order with variable times and infinite delay, *Nonlinear Anal. HS*, 2010, 4, 818-829.
- [27] Abbas S., Benchohra M., Gorniewicz L., Existence theory for impulsive partial hyperbolic functional differential equations involving the Caputo fractional derivative, *Scientiae Mathematicae Japonicae*, 2010, 72 (1), 49-60.
- [28] Benchohra M., Seba D., Impulsive partial hyperbolic fractional order differential equations in Banach spaces, *J. Fract. Calc. Appl.*, 2011, 1 (4), 1-12.
- [29] Guo T., Zhang K., Impulsive fractional partial differential equations, *Appl. Math. Comput.*, 2015, 257, 581-590.
- [30] Zhang X., Zhang X., Zhang M., On the concept of general solution for impulsive differential equations of fractional order $q \in (0,1)$, *Appl. Math. Comput.*, 2014, 247, 72-89.
- [31] Zhang X., On the concept of general solutions for impulsive differential equations of fractional order $q \in (1, 2)$, *Appl. Math. Comput.*, 2015, 268, 103-120.
- [32] Zhang X., The general solution of differential equations with Caputo-Hadamard fractional derivatives and impulsive effect, *Adv. Differ. Equ.*, vol. 2015, article 215, 16 pages, 2015.
- [33] Zhang X., Agarwal P., Liu Z., Peng H., The general solution for impulsive differential equations with Riemann-Liouville fractional order $q \in (1, 2)$, *Open Math.*, 2015, 13, 908-930.
- [34] Zhang X., Shu T., Cao H., Liu Z., Ding W., The general solution for impulsive differential equations with Hadamard fractional derivative of order $q \in (1, 2)$, *Adv. Differ. Equ.*, vol. 2016, article 14, 36 pages, 2016.
- [35] Zhang X., Zhang X., Liu Z., Peng H., Shu T., Yang, S., The General Solution of Impulsive Systems with Caputo-Hadamard Fractional Derivative of Order $q \in C(\mathbb{R}(q) \in (1, 2))$, *Math. Prob. Eng.*, vol. 2016, Article ID 8101802, 20 pages, 2016.
- [36] Hernandez E., O'Regan D., On a new class of abstract impulsive differential equations, *Proc. Amer. Math. Soc.*, 2013, 141, 1641-1649.
- [37] Li P.L., Xu C.J., Mild solution of fractional order differential equations with not instantaneous impulses, *Open Math.*, 2015, 13, 436-443.
- [38] Suganya S., Baleanu D., Kalamani P., Arjunan M.M., On fractional neutral integro-differential systems with state-dependent delay and non-instantaneous impulses, *Adv. Differ. Equ.*, vol. 2015, article 372, 39 pages, 2015.
- [39] Kilbas A.A., Srivastava H.H., Trujillo J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam (2006).
- [40] Diethelm K., Ford N.J., *Analysis of fractional differential equations*, *J. Math. Anal. Appl.*, 2002, 265, 229-248.