

Research Article

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Analytical solutions for the fractional diffusion-advection equation describing super-diffusion

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Abstract: This paper presents the alternative construction of the diffusion-advection equation in the range (1; 2). The fractional derivative of the Liouville-Caputo type is applied. Analytical solutions are obtained in terms of Mittag-Leffler functions. In the range (1; 2) the concentration exhibits the superdiffusion phenomena and when the order of the derivative is equal to 2 ballistic diffusion can be observed, these behaviors occur in many physical systems such as semiconductors, quantum optics, or turbulent diffusion. This mathematical representation can be applied in the description of anomalous complex processes.

Keywords: Fractional calculus; Non-local Transport processes; Caputo fractional derivative; Dissipative dynamics; Fractional advection-diffusion equation.

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1 Introduction

The diffusion-advection equation (DAE) describes the tendency of particles to be moved along by the fluid it is situated in (the convective terms arise when changing from

Lagrangian to Eulerian frames) and the diffusion refers to the dissipation/loss of a particles property (such as momentum) due to internal frictional forces [1]. The dynamical systems of fractional order are non-conservative and involve non-local operators [2–7]. Several approaches have been used for investigating anomalous diffusion, Langevin equations [8, 9], random walks [10, 11], or fractional derivatives, based on fractional calculus (FC) several works connected to anomalous diffusion processes may be found in [12–19]. Scher and Montroll [20] presented a stochastic model for the photocurrent transport in amorphous materials. Mainardi in [21] presented the interpretation of the corresponding Green function as a probability density, the fundamental equation was obtained from the conventional diffusion equation by replacing the second-order space derivative with a Riesz-Feller derivative and the first-order time derivative with a Liouville-Caputo derivative. Luchko in [22–24] presents the generalized time-fractional diffusion equation with variable coefficients. Jespersen in [25] presented a Riesz/Weyl form of the DAE considered Lévy flights subjected to external force fields, the corresponding Fokker-Planck equation contains a fractional spatial derivative. In the work [26], the fractional DE, DAE and the Fokker-Planck equation were presented, the equations were derived from basic random walk models. In the work [27] the authors proposed an alternative solution for the fractional DAE via derivatives of Liouville-Caputo type of order (0, 1). Based on the previous works developed by Gómez [27, 28], this paper explores the alternative construction of the DAE in the range (1; 2) for the space-time domain. The paper is organized as follows. In the next section, we present the fractional operators. In Section 3, the analytic solution of the fractional DAE is performed. Finally, some concluding remarks are drawn in Section 4.

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2 Basic Tools

The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \quad (1)$$

$$J^0 f(x) = f(x).$$

The Liouville-Caputo fractional derivative (C) of a function $f(x)$ is defined as [29]

$$\begin{aligned} D^\alpha f(x) &= J^{m-\alpha} \frac{d^m}{dx^m} [f(x)] \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dx, \end{aligned} \quad (2)$$

for $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, $x > 0$, $f \in \mathbb{C}^m$.

Also, the fractional derivative of $f(x)$ in the Liouville-Caputo sense satisfies the following relations

$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0, \quad (3)$$

$$D^\alpha J^\alpha f(x) = f(x).$$

Laplace transform to Liouville-Caputo fractional derivative is given by [29]

$$\mathcal{L}[{}_0^C D_t^\alpha f(t)] = S^\alpha F(S) - \sum_{k=0}^{m-1} S^{\alpha-k-1} f^{(k)}(0), \quad (4)$$

where

$$\mathcal{L}[{}_0^C D_t^\alpha f(t)] = s^\alpha F(s) - s^{\alpha-1} f(0) \quad 0 < \alpha \leq 1, \quad (5)$$

$$\begin{aligned} \mathcal{L}[{}_0^C D_t^\alpha f(t)] &= s^\alpha F(s) - s^{\alpha-1} f(0) - s^{\alpha-2} f'(0) \\ &1 < \alpha \leq 2. \end{aligned} \quad (6)$$

The inverse Laplace transform requires the introduction of the Mittag-Leffler function [30]

$$E_{\alpha,\beta}(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\alpha m + \beta)}, \quad (\alpha > 0), \quad (\beta > 0). \quad (7)$$

Some common Mittag-Leffler functions are [30, 31]

$$E_{1/2}(\pm z) = e^{z^2} [1 \pm \operatorname{erfc}(z)], \quad (8)$$

$$E_1(\pm z) = e^{\pm z},$$

$$E_2(-z^2) = \cos(z),$$

$$E_3(z) = \frac{1}{2} \left[e^{z^{1/3}} + 2e^{-(1/2)z^{1/3}} \cos\left(\frac{\sqrt{3}}{2}z^{1/3}\right) \right],$$

$$E_4(z) = \frac{1}{2} \left(\cos(z^{1/4}) + \cosh(z^{1/4}) \right),$$

$$\begin{aligned} E_{s/2,r}(z) &= \frac{z^{2\kappa(1-r)}}{s} \sum_{j=0}^{s-1} \beta_j^{1-(s/2+r)} (\exp(\beta_j z^{2\kappa})) (\beta_j^{s/2} \\ &+ \operatorname{erfc}(\beta_j^{1/2} z^\kappa)) - z^{-2n} \sum_{k=0}^{2n-1} \frac{z^k}{\Gamma(sk/2 + \mu)}, \end{aligned}$$

where $\kappa = 1/s$, $r = ns + \mu$, $n = 0, 1, 2, 3, \dots$, $\mu = 1, 2, 3, \dots$ [32, 33]. The $\operatorname{erfc}(z)$ denotes the error function [30]

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) dt. \quad (9)$$

3 Local Diffusion-Advection Equation

The equation (10) describes the processes of diffusion-advection

$$D \frac{\partial^2 C(x, t)}{\partial x^2} + \vartheta \frac{\partial C(x, t)}{\partial x} - \frac{\partial C(x, t)}{\partial t} = 0, \quad (10)$$

where C is the concentration, D is the diffusion coefficient and ϑ is the drift velocity, this equation predicts the concentration distribution onto one dimensional axis x .

3.1 Nonlocal Time Diffusion-Advection Equation

Based on the previous work developed by Gómez [27], we introduce an auxiliary parameter σ_t as follows

$$\frac{\partial}{\partial t} \rightarrow \frac{1}{\sigma_t^{1-\alpha}} \cdot \frac{\partial^\alpha}{\partial t^\alpha}, \quad n-1 < \alpha \leq n, \quad (11)$$

where n is integer, the parameter σ_t has dimensions of time (seconds). The authors of [34] used the Planck time, $t_p = 5.39106 \times 10^{-44}$ seconds, with the finality to preserve the dimensional compatibility, the σ_t parameter corresponds to the t_p in our calculations. Consider (11) in the eq. (10), the temporal fractional equation of order $\alpha \in (1, 2]$ becomes

$$\frac{\partial^\alpha C(x, t)}{\partial t^\alpha} - \vartheta t_p^{1-\alpha} \frac{\partial C(x, t)}{\partial x} - D t_p^{1-\alpha} \frac{\partial^2 C(x, t)}{\partial x^2} = 0. \quad (12)$$

Suppose the solution

$$C(x, t) = C_0 e^{ikx} u(t), \quad (13)$$

where k is the wave number in the x direction and C_0 is a constant. Substituting (13) into (12) we obtain

$$\frac{d^\alpha u(x)}{dt^\alpha} + (Dk^2 - i\vartheta k) t_p^{1-\alpha} u(t) = 0, \quad (14)$$

where

$$\omega^2 = (Dk^2 - i\vartheta k), \quad (15)$$

is the dispersion relation and

$$\tilde{\omega}^2 = (Dk^2 - i\vartheta k)t_p^{1-\alpha} = \omega^2 t_p^{1-\alpha}, \quad (16)$$

where $\tilde{\omega}^2$ is the fractional dispersion relation in the medium and ω^2 is the ordinary dispersion relation. From the fractional dispersion relation (15) we have

$$\omega = \delta - i\varphi, \quad (17)$$

substituting (17) into (16) we have

$$(\delta - i\varphi)^2 = \delta^2 - 2i\delta\varphi - \varphi^2, \quad (18)$$

where

$$\delta^2 - 2i\delta\varphi - \varphi^2 = (Dk^2 - i\vartheta k)t_p^{1-\alpha}, \quad (19)$$

solving for φ we obtain

$$\varphi = \frac{\vartheta k}{2\delta} t_p^{1-\alpha}, \quad (20)$$

and for δ

$$\delta = k\sqrt{Dt_p^{1-\alpha}} \left[\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{\vartheta^2}{k^2 D^2}} \right]^{\frac{1}{2}}, \quad (21)$$

substituting (21) into (20) we have

$$\varphi = \frac{\vartheta}{2} \cdot \frac{1}{\sqrt{D} \left[\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{\vartheta^2}{k^2 D^2}} \right]^{\frac{1}{2}}} \sqrt{t_p^{1-\alpha}}. \quad (22)$$

Now the fractional natural frequency is, $\tilde{\omega} = \delta - i\varphi$, where δ and φ are given by (21) and (22) respectively

$$\tilde{\omega} = \left(k\sqrt{D} \left[\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{\vartheta^2}{k^2 D^2}} \right]^{\frac{1}{2}} - i \frac{\vartheta}{2\sqrt{D} \left[\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{\vartheta^2}{k^2 D^2}} \right]^{\frac{1}{2}}} \right) \sqrt{t_p^{1-\alpha}}. \quad (23)$$

The equation (23) describes the real and the imaginary part of $\tilde{\omega}$ in terms of the wave number k , the viscous drag ϑ , the diffusion coefficient D and the fractional temporal components σ_t .

Substituting (16) into (14) we obtain

$$\frac{d^\alpha u(t)}{dt^\alpha} + \tilde{\omega}^2 u(t) = 0, \quad (24)$$

where

$$u(t) = E_\alpha(-\tilde{\omega}^2 t^\alpha). \quad (25)$$

The particular solution of equation (24) is

$$C(x, t) = C_0 \cdot e^{-ikx} \cdot E_\alpha(-\tilde{\omega}^2 t^\alpha), \quad (26)$$

now we will analyze the case when α takes different values.

When $\alpha = 3/2$, we have, $\tilde{\omega}^2 = \omega^2 t_p^{-1/2}$, substituting this expression in (26) we have

$$C(x, t) = C_0 \cdot e^{-ikx} \cdot E_{3/2}(-\tilde{\omega}^2 t^{3/2}), \quad (27)$$

where

$$\tilde{\omega} = \left(k\sqrt{D} \left[\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{\vartheta^2}{k^2 D^2}} \right]^{\frac{1}{2}} - i \frac{\vartheta}{2\sqrt{D} \left[\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{\vartheta^2}{k^2 D^2}} \right]^{\frac{1}{2}}} \right) \sqrt{t_p^{-1/2}}, \quad (28)$$

where, $E_{3/2}$ is given by (9), in this case $z = -\tilde{\omega}^2 t^{3/2}$. Substituting $E_{3/2}$ into (27) the solution is

$$C(x, t) = C_0 \cdot e^{ikx} \cdot \left[\frac{1}{3} \sum_{j=0}^2 \beta_j^{-3/2} \left(\exp(\beta_j z^{2/3}) \right) \left(\beta_j^{3/2} + \operatorname{erfc}(\beta_j^{1/2} z^{1/3}) \right) - z^{-2n} \sum_{k=0}^{2n-1} \frac{z^k}{\Gamma(3k/2 + \mu)} \right], \quad (29)$$

this equation represent the fractional concentration in the medium for $\alpha = 3/2$.

When $\alpha = 2$, we have, $\tilde{\omega}^2 = \omega^2 t_p^{-1}$, substituting this expression in (26) we have

$$C(x, t) = C_0 \cdot e^{ikx} \cdot E_2(-\tilde{\omega}^2 t^2), \quad (30)$$

where

$$\tilde{\omega} = \left(k\sqrt{D} \left[\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{\vartheta^2}{k^2 D^2}} \right]^{\frac{1}{2}} - i \frac{\vartheta}{2\sqrt{D} \left[\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{\vartheta^2}{k^2 D^2}} \right]^{\frac{1}{2}}} \right) \sqrt{t_p^{-1}}. \quad (31)$$

Substituting E_2 given by (8) into (30) the solution is

$$C(x, t) = \Re[C_0 \cdot e^{i(kx - \tilde{\omega}t)}], \quad (32)$$

$\Re$ indicates the real part and $\tilde{\omega} = (\delta - i\varphi)\sqrt{t_p^{-1}}$. The first exponential $e^{i(kx - \delta t\sqrt{t_p^{-1}})}$ gives the well-known plane-wave variation of the concentration. The second exponential $e^{-\varphi t\sqrt{t_p^{-1}}}$ gives and exponential decay in the amplitude of the wave.

In this case exists a physical relation given by

$$\alpha = \omega t_p = \frac{t_p}{T_0}, \quad 0 < t_p \leq T_0, \quad (33)$$

we can use this relation (33) in order to from equation (26) as

$$C(x, \tilde{t}) = C_0 \cdot e^{ikx} \cdot E_\alpha \left(-\alpha^{1-\alpha} \tilde{t}^\alpha \right). \quad (34)$$

Figure 1 and 2 show the simulation of the equation (34) for α values arbitrarily chosen between $[1.3, 2)$. For $\alpha \in [1.3, 2)$, we observe superdiffusion and for $\alpha = 2$ ballistic diffusion [26].

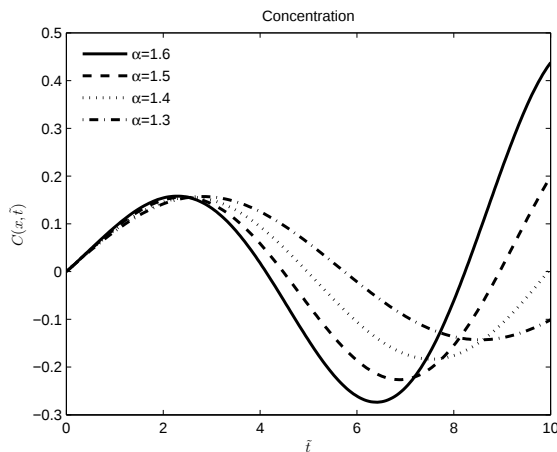


Figure 1: Concentration distribution for the temporal case. Simulation of equation (34) for $\alpha \in [1.3, 1.6]$.

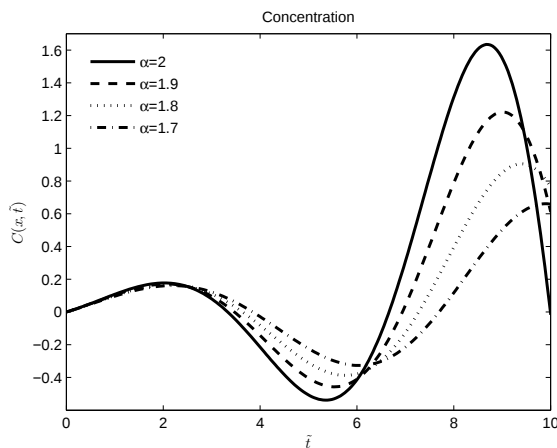


Figure 2: Concentration distribution for the temporal case. Simulation of equation (34) for $\alpha \in [1.3, 2]$. If $\alpha = 2$ we find the ballistic diffusion.

3.2 Nonlocal Space Diffusion-Advection Equation

Now, we consider

$$\frac{\partial}{\partial x} \rightarrow \frac{1}{\sigma_x^{1-\alpha}} \cdot \frac{\partial^\alpha}{\partial x^\alpha}, \quad n-1 < \alpha \leq n, \quad (35)$$

where n is integer, the parameter σ_x has dimensions of length (meters). In our calculations we used the Planck length, $l_p = 1.616199 \times 10^{-35}$ meters, with the finality to preserve the dimensional compatibility, the parameter $\sigma_x = l_p$. The spatial fractional equation of order $\alpha \in (1, 2]$ is

$$\frac{\partial^{2\alpha} C(x, t)}{\partial x^{2\alpha}} + \frac{\partial}{\partial t} l_p^{1-\alpha} \frac{\partial^\alpha C(x, t)}{\partial x^\alpha} - \frac{1}{D} l_p^{2(1-\alpha)} \frac{\partial C(x, t)}{\partial t} = 0, \quad (36)$$

A particular solution is given by

$$C(x, t) = C_0 e^{-\omega t} u(x), \quad (37)$$

where ω is the natural frequency and C_0 is a constant.

Substituting (37) into (36) we obtain

$$\frac{d^{2\alpha} u(x)}{dx^{2\alpha}} + \frac{\partial}{\partial t} l_p^{1-\alpha} \frac{d^\alpha u(x)}{dx^\alpha} + \frac{\omega}{D} l_p^{2(1-\alpha)} u(x) = 0. \quad (38)$$

The solution of (38) is given by

$$C(x, t) = C_0 e^{-\omega t} \cdot E_\alpha \left(-\frac{\partial}{\partial D} l_p^{1-\alpha} x^\alpha \right) \cdot E_{2\alpha} \left(-\left[\frac{\omega}{D} - \frac{\partial^2}{4D^2} \right] l_p^{2(1-\alpha)} x^{2\alpha} \right). \quad (39)$$

For the underdamped case, with $\left(\frac{\omega}{D} - \frac{\partial^2}{4D^2} \right) = 0$, $\partial = 2\sqrt{\omega D}$. Considering $\partial = 2\sqrt{\omega D}$ and $C(0) = C_0$ in equation (39), $\tilde{k}^2 = \frac{\omega}{D}$ is the wave vector and $\alpha^2 = \frac{\partial}{2D}$ is the damping factor. Now we will analyze the case when α takes different values.

When $\alpha = 3/2$, from equation (39) we have

$$C(x, t) = C_0 e^{-\omega t} \cdot E_{3/2} \left(-\frac{\partial}{2D} l_p^{-1/2} x^{3/2} \right) \cdot E_3 \left(-\left[\frac{\omega}{D} - \frac{\partial^2}{4D^2} \right] l_p^{-1} x^3 \right), \quad (40)$$

where $E_{3/2}$ is given by (9) and E_3 by (8), for the case of $E_{3/2}$, $z = -\frac{\partial}{2D} l_p^{-1/2} x^{3/2}$ and for E_3 , $z = -\left[\frac{\omega}{D} - \frac{\partial^2}{4D^2} \right] l_p^{-1} x^3$.

When $\alpha = 2$, from equation (39) we have

$$C(x, t) = C_0 e^{-\omega t} \cos \left(\sqrt{\frac{\partial}{2D}} l_p^{-1} x \right) \cdot E_4 \left(-\left[\frac{\omega}{D} - \frac{\partial^2}{4D^2} \right] l_p^{-2} x^4 \right), \quad (41)$$

where E_4 is given by (8), for $E_4, z = \left(-\left[\frac{\omega}{D} - \frac{g^2}{4D^2}\right] l_p^{-2} x^4\right)$.

In this case a physical relationship between α and l_p is given by

$$\alpha = \left(\frac{\omega}{D} - \frac{g^2}{4D^2}\right)^{\frac{1}{2}} l_p, \quad 0 < l_p \leq \frac{1}{\left(\frac{\omega}{D} - \frac{g^2}{4D^2}\right)^{\frac{1}{2}}}. \quad (42)$$

Then, the solution (39) for the underdamped case $\vartheta < 2\sqrt{\omega D}$ or $\eta < \tilde{K}_0$ takes the form

$$C(\tilde{x}, t) = C_0 e^{-\omega t} \cdot E_\alpha \left(-\frac{\vartheta}{2D\sqrt{\frac{\omega}{D} - \frac{g^2}{4D^2}}} \alpha^{1-\alpha} \tilde{x}^\alpha \right) \cdot E_{2\alpha} \left(-\alpha^{2(1-\alpha)} \tilde{x}^{2\alpha} \right), \quad (43)$$

where $\tilde{x} = \left(\frac{\omega}{D} - \frac{g^2}{4D^2}\right)^{\frac{1}{2}} x$.

Due to the condition $\vartheta < 2\sqrt{\omega D}$ we have

$$\frac{\vartheta}{2D\sqrt{\frac{\omega}{D} - \frac{g^2}{4D^2}}} = \frac{1}{3}, \quad 0 \leq \frac{\vartheta}{2D\sqrt{\frac{\omega}{D} - \frac{g^2}{4D^2}}} < \infty. \quad (44)$$

Thus, the solution (39) takes its final form

$$C(\tilde{x}, t) = C_0 e^{-\omega t} \cdot E_\alpha \left(-\frac{1}{3} \alpha^{1-\alpha} \tilde{x}^\alpha \right) \cdot E_{2\alpha} \left(-\alpha^{2(1-\alpha)} \tilde{x}^{2\alpha} \right). \quad (45)$$

Figures 3 and 4 show the simulation of equation (45) for $\alpha \in [1.3, 2]$.

In the overdamped case, $\eta > \tilde{K}_0$ or $\vartheta > 2\sqrt{\omega D}$, the solution of the equation (39) is given by

$$\tilde{C}(x, t) = \tilde{C}_0 e^{-\omega t} \cdot E_\alpha \left(-\frac{\vartheta}{2D} l_p^{1-\alpha} x^\alpha \right) \cdot E_2 \left(-\left[\frac{g^2}{4D^2} - \frac{\omega}{D}\right]^{\frac{1}{2}} l_p^{1-\alpha} x^\alpha \right), \quad (46)$$

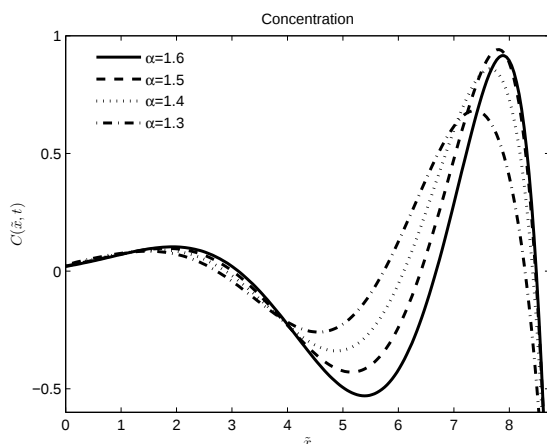


Figure 3: Concentration distribution for the underdamped spatial case. Simulation of the equation (45) for $\alpha \in [1.3, 1.6]$.

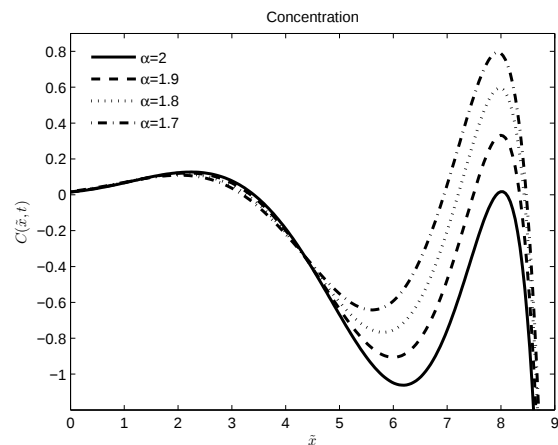


Figure 4: Concentration distribution for the underdamped spatial case. Simulation of the equation (45) for $\alpha \in [1.3, 2]$.

Now we will analyze the case when α takes different values.

If $\alpha = 3/2$, from equation (46) we have

$$\tilde{C}(x, t) = \tilde{C}_0 e^{-\omega t} \cdot E_{3/2} \left(-\frac{\vartheta}{2D} l_p^{-1/2} x^{3/2} \right) \cdot E_{3/2} \left(-\left[\frac{g^2}{4D^2} - \frac{\omega}{D}\right]^{\frac{1}{2}} l_p^{-1/2} x^{3/2} \right), \quad (47)$$

where $E_{3/2}$ is given by (9), for the case of $E_{3/2}$, $z_1 = -\frac{\vartheta}{2D} l_p^{-1/2} x^{3/2}$ and $z_2 = \left(-\left[\frac{g^2}{4D^2} - \frac{\omega}{D}\right]^{\frac{1}{2}} l_p^{-1/2} x^{3/2}\right)$.

Now, from $\alpha = 2$ we have

$$\tilde{C}(x, t) = \tilde{C}_0 e^{-\omega t} \cdot E_2 \left(-\frac{\vartheta}{2D} l_p^{-1} x^2 \right) \cdot E_2 \left(-\left[\frac{g^2}{4D^2} - \frac{\omega}{D}\right]^{\frac{1}{2}} l_p^{-1} x^2 \right), \quad (48)$$

substituting E_2 given by (8) into (48) we obtain the solution

$$\tilde{C}(x, t) = \tilde{C}_0 e^{-\omega t} \cdot \cos \left(\sqrt{\frac{\vartheta}{2D}} l_p^{-1} x \right) \cdot \cos \left(\sqrt{\left[\frac{g^2}{4D^2} - \frac{\omega}{D}\right]^{\frac{1}{2}} l_p^{-1} x} \right). \quad (49)$$

In this case a physical relation is given by

$$\alpha = \left(\frac{g^2}{4D^2} - \frac{\omega}{D}\right)^{\frac{1}{2}} l_p, \quad 0 < l_p \leq \frac{1}{\left(\frac{g^2}{4D^2} - \frac{\omega}{D}\right)^{\frac{1}{2}}}. \quad (50)$$

substituting the relation (50), the solution (46) takes the form

$$\tilde{C}(\tilde{x}, t) = \tilde{C}_0 e^{-\omega t} \cdot E_\alpha \left(-\frac{\vartheta}{2D\sqrt{\frac{g^2}{4D^2} - \frac{\omega}{D}}} \alpha^{1-\alpha} \tilde{x}^\alpha \right). \quad (51)$$

$$\cdot E_{\alpha} \left(-\alpha^{1-\alpha} \tilde{x}^{\alpha} \right),$$

where $\tilde{x} = \left(\frac{g^2}{4D^2} - \frac{\omega}{D} \right)^{\frac{1}{2}} x$.

Due the condition $g > 2\sqrt{\omega D}$, we have

$$\frac{g}{2D\sqrt{\frac{g^2}{4D^2} - \frac{\omega}{D}}} = 2, \quad 1 < \frac{g}{2D\sqrt{\frac{g^2}{4D^2} - \frac{\omega}{D}}} < \infty. \quad (52)$$

Then, the solution (46) is given by

$$\begin{aligned} \tilde{C}(\tilde{x}, t) = \tilde{C}_0 e^{-\omega t} \cdot E_{\alpha} \left(-2\alpha^{1-\alpha} \tilde{x}^{\alpha} \right) \cdot \\ \cdot E_{\alpha} \left(-\alpha^{1-\alpha} \tilde{x}^{\alpha} \right). \end{aligned} \quad (53)$$

Figures 5 and 6 show the simulation of the equation (53) for $\alpha \in [1.3, 2)$, where the values of α are arbitrarily chosen.

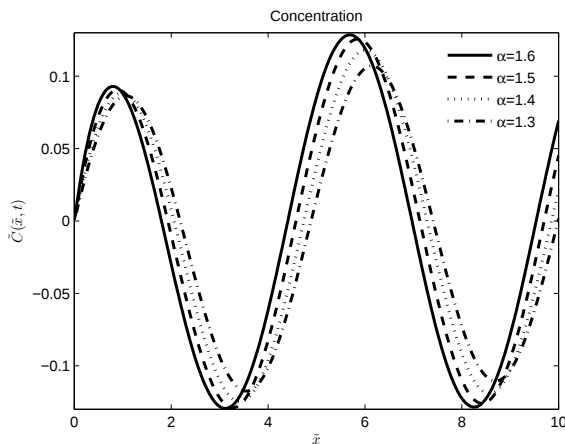


Figure 5: Concentration distribution for the underdamped spatial case. Simulation of the equation (45) for $\alpha \in [1.3, 1.6]$.

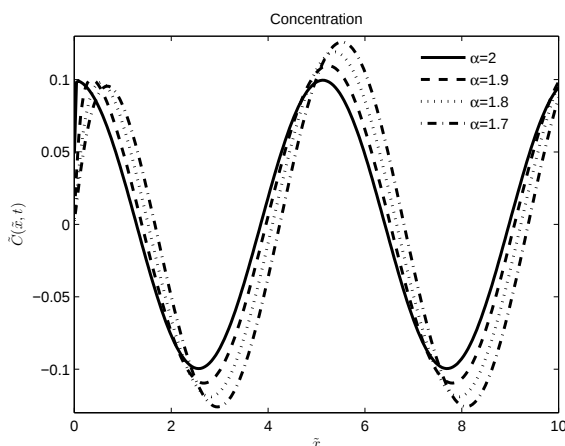


Figure 6: Concentration in the overdamped spatial case. Simulation of the equation (45) for $\alpha \in [1.3, 2]$.

3.3 Nonlocal Time-Space Diffusion-Advection Equation

Now we consider the fractional DAE, when $t = 0$, $x \geq 0$ and $x = L$, $C(0, t) = 0$ and initial conditions $0 < x < L$, $t = 0$: $T(t, 0) = T_0 > 0$ and $0 < x < L$, $t = 0$: $\frac{\partial C}{\partial x}|_{x \rightarrow \infty} = 0$. Applying the Fourier method we have

$$C(t, x) = X(x)T(t), \quad (54)$$

$$X(x)\dot{T}(t) = DX''(x)T(t),$$

$$\frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{DT(t)} = C$$

$$x(0) = 0; \quad T(t) = \beta \exp(CDt).$$

The full solution of the equation (10) is

$$C(x, t) = \sum_{m=1}^{\infty} \beta_m \cdot E_{\alpha} \left(-D\sigma_t^{1-\alpha} \lambda_m^2 \tilde{t}^{\alpha} \right) \quad (55)$$

$$\begin{aligned} \cdot \Im \left[E_{i\alpha} \left(\lambda_m \sigma_x^{1-\alpha} \tilde{x}^{\alpha} \right) \right] + \sum_{m=1}^{\infty} \left[\int_0^t f_m(\tau) d\tau \right] \\ \cdot E_{\alpha} \left(-D\sigma_t^{1-\alpha} \lambda_m^2 \tilde{t}^{\alpha} \right) \cdot \Im \left[E_{i\alpha} \left(\lambda_m \sigma_x^{1-\alpha} \tilde{x}^{\alpha} \right) \right]. \end{aligned}$$

where $\Im$ indicates the imaginary part, when $\alpha = 1$, we have the classical solution

$$\begin{aligned} C(x, t) = \sum_{m=1}^{\infty} \beta_m \cdot \exp(-D\lambda_m^2 t) \cdot \sin(\lambda_m x) \\ + \sum_{m=1}^{\infty} \left[\int_0^t f_m(\tau) d\tau \right] \cdot \exp(-D\lambda_m^2 t) \cdot \sin(\lambda_m x). \end{aligned} \quad (56)$$

where, $\tilde{t} = \omega t$, $\tilde{x} = \left(\frac{\omega}{D} - \frac{g^2}{4D^2} \right)^{\frac{1}{2}} x$ are a dimensionless parameters and β is a constant. Figures 7, 8, 9, 10 and 11 show

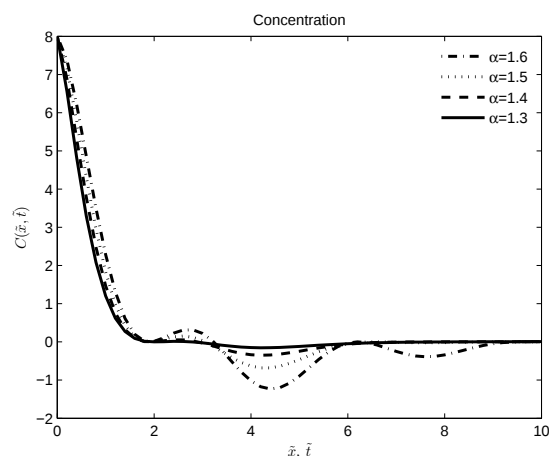


Figure 7: Concentration in space-time. Simulation of the equation (55) for $\alpha \in [1.3, 1.6]$.

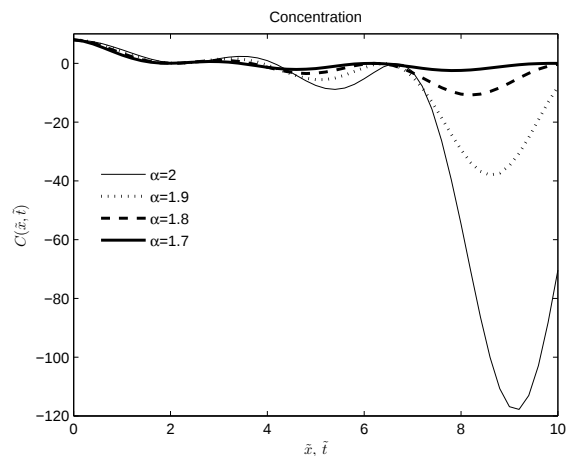


Figure 8: Concentration in space-time. Simulation of the equation (55) for $\alpha \in [1.7, 2]$.

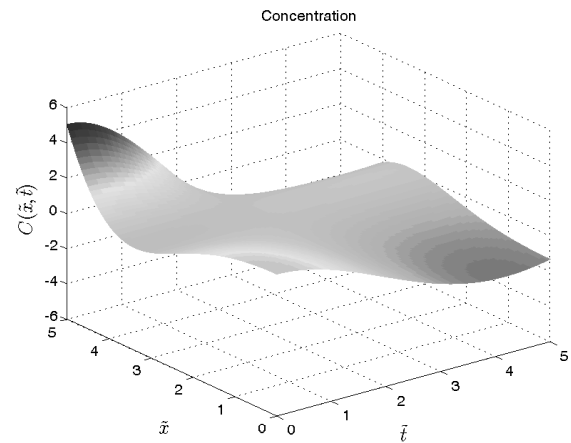


Figure 10: Concentration in space-time. Simulation of the equation (55) for $\alpha = 1.9$.

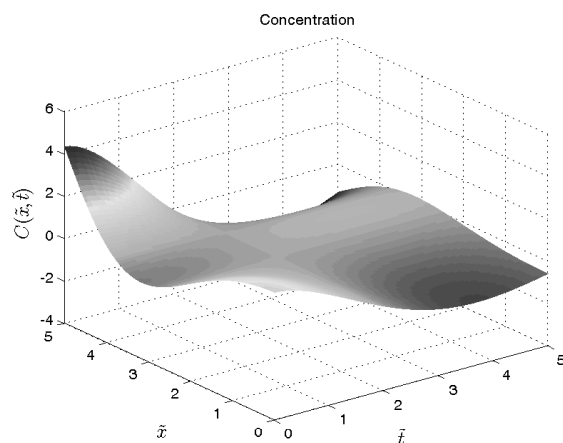


Figure 9: Concentration in space-time. Simulation of the equation (55) for $\alpha = 1.7$.

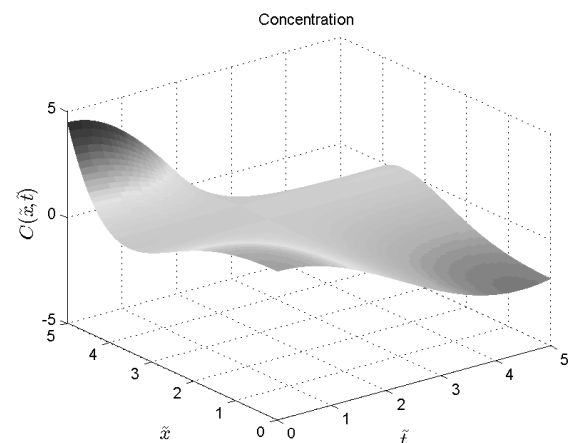


Figure 11: Concentration in space-time. Simulation of the equation (55) for $\alpha = 2.0$.

the simulation of equation (55), where the values of α are arbitrarily chosen.

4 Conclusions

In this paper we introduced an alternative representation of the fractional DAE in the range $(1, 2)$, the nonlocal equations were examined separately; with fractional spatial derivative and with fractional temporal derivative. In particular, a one dimensional model was considered in detail. Our results indicate that the fractional order α has an important influence on the concentration. For the temporal case, in the range $\alpha \in (1, 2)$ the diffusion is fast (superdiffusion phenomena and mixed diffusion-wave behavior) and when $\alpha = 2$ we see ballistic diffusion. In the

spatial case, in the range $\alpha \in (1, 2)$, the diffusion exhibits an increment of the amplitude and the behavior becomes anomalous dispersive (the diffusion increases with increasing order of α), we observe the Markovian Lévy flights [26].

The methodology proposed in this work can be potentially useful to study rotating flow, Richardson turbulent diffusion, diffusion of ultracold atoms in an optical lattice and turbulent systems.

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