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Alireza K. Golmankhaneh* and D. Baleanu

Non-local Integrals and Derivatives on Fractal Sets with Applications

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Abstract: In this paper, we discuss non-local derivatives on fractal Cantor sets. The scaling properties are given for both local and non-local fractal derivatives. The local and non-local fractal differential equations are solved and compared. Related physical models are also suggested.

Keywords: Fractal calculus, Non-local fractal derivatives, Scale change, Cantor set, Fractal dimension

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1 Introduction

Fractional calculus became an important tool which applied successfully in many branches of science, engineering etc [1–5]. The models based on fractional derivatives are crucial for describing processes with memory effects [6]. Local fractional has been defined on the real-line [7]. As it is well known the integer, fractional and complex order derivatives and integrals are defined on the real-line. Fractal analysis have been conducted by many researchers [8–10]. The fractal curves and the functions on fractal space are not differentiable in the sense of standard calculus. Using this as motivation recently a seminal paper has suggested F^α-calculus as a framework for the fractal sets and fractal curves [11–14]. F^α-calculus is generalized and applied in physics as a new and useful tool for modelling processes on fractals. Newtonian mechanics and Schrödinger equation on the fractal sets and curves are given [15–17]. The gauge integral is utilized to generalized F^α-calculus for unbound and singular functions [18]. The fractal grating is modeled by F^α-calculus and corresponding diffraction is presented [18]. One of the important aspects of fractional calculus was transferred recently to fractal derivatives. The concept of non-local fractal derivatives was introduced in [20]. In this manuscript our main aim is to define the fractal non-local derivatives and study their properties.

The outline of this work is as follows:

In Section 2 we summarize the basic definitions and properties of the the local fractional derivatives. In Section 3 the scaling properties of local and non-local derivatives are derived. We develop the theory of fractal local and non-local Laplace transformations in Section 4. In Section 5 the comparison of local and non-local linear fractal differential equations are presented. In Section 6 we indicate some illustrative applications. Section 7 contains our conclusion.

2 Preliminaries

In this section we recall some basic definitions and properties of the local fractal calculus (LFC) and non-local fractal calculus (NLFC) [11, 20].

2.1 Local fractal calculus

In the seminal paper local F^α-calculus is built on fractal Cantor set which is shown in Figure [1] [11]. The integral staircase function $S_F^\alpha(x)$ of order α for the triadic Cantor set

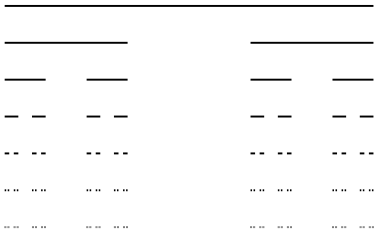


Figure 1: We present triadic Cantor set by iteration.

***Corresponding Author: Alireza K. Golmankhaneh:** Young Researchers and Elite Club, Urmia Branch, Islamic Azad university, Urmia, Iran, E-mail: alirezakhalili2002@yahoo.co.in
D. Baleanu: Department of Mathematics, Çankaya University, 06530 Ankara, Turkey and Institute of Space Sciences, P.O.BOX, MG-23, R 76900, Magurele-Bucharest, Romania, E-mail: dumitru@cankaya.edu.tr

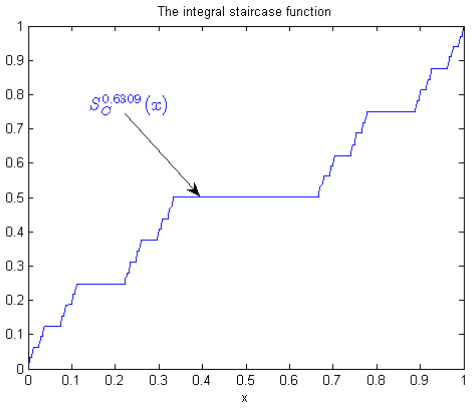


Figure 2: We indicate the integral staircase function for a triadic Cantor set F.

F is defined in [11] by

$$S_F^\alpha(x) = \begin{cases} \gamma^\alpha(F, a_0, x) & \text{if} \quad x \geq a_0 \\ -\gamma^\alpha(F, a_0, x) & \text{otherwise,} \end{cases} \tag{1}$$

where a_0 is an arbitrary real number. The plot of the integral staircase function is depicted in Figure [2]. F^α-derivative is defined for a function with this support as follows [11]

$$D_F^\alpha f(x) = \begin{cases} \text{F}-\lim_{y\to x} \frac{f(y)-f(x)}{S_F^\alpha(y)-S_F^\alpha(x)} & \text{if} \quad x \in F, \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

if the limit exists. For more details we refer the reader to [11].

2.2 Non-local fractal calculus

In this section, we review the non-local derivatives and basic definitions [20].

Definition 1. A function $f(S_F^\alpha(x))$, $x > 0$ is in the space $\mathcal{C}_{F,\rho}$, $\rho \in \Re$ if there exists a real number $p > \rho$, such $f(S_F^\alpha(x)) = S_F^\alpha(x)^p f_1(S_F^\alpha(x))$, where $f_1(S_F^\alpha(x)) \in \mathcal{C}_F^\alpha[a, b]$, and it is in the $\mathcal{C}_{F,\rho}^{n\alpha}[a, b]$ if and only if

$$(D_F^\alpha)^n f(S_F^\alpha(x)) \in \mathcal{C}_{F,\rho}, \quad n \in N. \tag{3}$$

Here and subsequently, we define the fractal left-sided Riemann-Liouville integral as follows

$${}_a\mathcal{I}_x^\beta f(x) := \frac{1}{\Gamma_F^\alpha(\beta)} \int_{S_F^\alpha(a)}^{S_F^\alpha(x)} \frac{f(t)}{(S_F^\alpha(x) - S_F^\alpha(t))^{\alpha-\beta}} d_F^\alpha t. \tag{4}$$

where $S_F^\alpha(x) > S_F^\alpha(a)$.

Definition 2. The fractal left-sided Riemann-Liouville derivative is defined as

$${}_a\mathcal{D}_x^\beta f(x) := \frac{1}{\Gamma_F^\alpha(n-\beta)} (D_F^\alpha)^n \int_{S_F^\alpha(a)}^{S_F^\alpha(x)} \frac{f(t)}{(S_F^\alpha(x) - S_F^\alpha(t))^{-n\alpha+\beta+\alpha}} d_F^\alpha t. \tag{5}$$

Definition 3. For A $f(x) \in C^{\alpha n}[a, b]$, $n\alpha - \alpha \leq \beta < \alpha n$ the fractal left-sided Caputo derivative is defined as

$${}_a^C\mathcal{D}_x^\beta f(x) := \frac{1}{\Gamma_F^\alpha(n-\beta)} \int_{S_F^\alpha(a)}^{S_F^\alpha(x)} (S_F^\alpha(x) - S_F^\alpha(t))^{n\alpha-\beta-\alpha} (D_F^\alpha)^n f(t) d_F^\alpha t. \tag{6}$$

Definition 4. The fractal Grünwald and Marchaud derivative of a function $f(x)$ with support of fractal sets is defined as

$${}^G\mathcal{D}^\beta f(x_0) = F - \lim_{n\to\infty} \frac{1}{\Gamma_F^\alpha(-\beta)} \left(\frac{S_F^\alpha(x_0)}{n}\right)^{-\beta} \sum_{k=0}^{n-1} \frac{\Gamma_F^\alpha(k-\beta)}{\Gamma_F^\alpha(k+1)} f\left(S_F^\alpha(x_0) - k\frac{S_F^\alpha(x_0)}{n}\right).$$

Definition 5. The generalized fractal standard Mittag-Leffler functions is defined as [20]

$$E_{F,\eta}^\alpha(x) = \sum_{k=0}^{\infty} \frac{S_F^\alpha(x)^k}{\Gamma_F^\alpha(\eta k + 1)}, \quad \eta > 0, \quad \nu \in \Re. \tag{7}$$

The fractal two parameter η, ν Mittag-Liffler function is defined as

$$E_{F,\eta,\nu}^\alpha(x) = \sum_{k=0}^{\infty} \frac{S_F^\alpha(x)^k}{\Gamma_F^\alpha(\eta k + \nu)}, \quad \eta > 0, \quad \nu \in \Re. \tag{8}$$

Definition 6. For a given function $f(S_F^\alpha(x))$ the fractal Laplace transform is denoted by $F(s)$ and defined as [20]

$$\mathcal{F}_F^\alpha(S_F^\alpha(s)) = \mathcal{L}_F^\alpha[f(x)] = \int_{S_F^\alpha(0)}^{S_F^\alpha(\infty)} f(x) e^{-S_F^\alpha(s) S_F^\alpha(x)} d_F^\alpha x, \tag{9}$$

where $S_F^\alpha(s)$ is limited by the values that the integral converges. The function $f(S_F^\alpha(x))$ is F-continuous and has following condition

$$\sup \frac{|f(S_F^\alpha(x))|}{e^{S_F^\alpha(c) S_F^\alpha(x)}} < \infty, \quad S_F^\alpha(c) \in \Re, \quad S_F^\alpha(x) > 0. \tag{10}$$

In view of the above conditions the fractal Laplace transform exists for all $S_F^\alpha(s) > S_F^\alpha(c)$. We follow the notation as

$\mathcal{L}_F^\alpha[f(x)] = \mathcal{F}_F^\alpha(S_F^\alpha(s))$ and $\mathcal{L}_F^\alpha[g(x)] = \mathcal{G}_F^\alpha(S_F^\alpha(s))$.

Remark 1. We denote that if we choose $\beta = \alpha$ then we have

$$ {}_a\mathcal{D}_x^\alpha f(x) = D_{F,x}^\alpha f(x)|_{x=S_F^\alpha(a)}. \tag{11}$$

3 Scale properties of fractal local and non-local fractal calculus

In this section we study the scale properties of the LFC and NLFC.

3.1 Scale change on the local fractal derivatives

A function $f(S_F^\alpha(x))$ is called fractal homogenous of degree-$m\alpha$ or invariant under fractal rescalings if we have

$$ f(S_F^\alpha(\lambda x)) = \lambda^{m\alpha} f(S_F^\alpha(x)), \tag{12}$$

where for some m and for all λ. The fractals have self-similar properties, namely for the case of function with the fractal Cantor set support we choose $m = 1$ and $\lambda = 1/3^n$, $n = 1, 2, \ldots$ then

$$ f(S_F^\alpha(\frac{1}{3^n}x)) = (\frac{1}{3^n})^\alpha f(S_F^\alpha(x)), \tag{13}$$

where $\alpha = 0.6$ is the dimension of triadic Cantor set. The fractal derivative of the fractal homogenous function $f(S_F^\alpha(x))$ rescaling as follows

$$ D_F^\alpha f(S_F^\alpha(\lambda x)) = \lambda^{m\alpha-\alpha} f(S_F^\alpha(x)). \tag{14}$$

3.2 Scale change on the non-local fractal derivatives

By a scale change of the fractal function $f(S_F^\alpha(x))$, we imply

$$ x \to \lambda x \Rightarrow S_F^\alpha(\lambda x) = \lambda^\alpha S_F^\alpha(x), \tag{15}$$

and using Eq. (5) and choosing $a = 0$ we derive

$$ {}_0\mathcal{D}_x^\beta(f(S_F^\alpha(\lambda x))) = \lambda^{\beta\alpha}\, {}_0\mathcal{D}_{\lambda x}^\beta(f(S_F^\alpha(\lambda x))), \tag{16}$$

which is called scale change on the non-local fractal derivatives.

4 Laplace transformation on fractals

We provide some important lemmas that are useful for finding the fractal Laplace transforms of function $f(S_F^\alpha(x))$.

Lemma 1. The fractal Laplace transform of the non-local fractal Caputo derivative of order $m\alpha - \alpha < \beta \le m\alpha$, $m \in N$ is

$$ \mathcal{L}_F^\alpha\{{}_0^C\mathcal{D}_x^\beta f(x)\} = \frac{(S_F^\alpha(s))^{m\alpha}\mathcal{F}_F^\alpha(s) - (S_F^\alpha(s))^{m\alpha-\alpha} f(S_F^\alpha(0))}{S_F^\alpha(s)^{m\alpha-\beta}} \times \frac{-(S_F^\alpha(s))^{m\alpha-2\alpha} D_x^\alpha f(x)|_{x=S_F^\alpha(0)} - \ldots - D_x^{m\alpha-\alpha} f(x)|_{x=S_F^\alpha(0)}}{1}. \tag{17}$$

Proof: We first compute the Laplace fractal transform of the fractal Caputo fractional derivative of order β as follows

$$ \begin{aligned} &\mathcal{L}_F^\alpha\{{}_0^C\mathcal{D}_x^\beta f(x)\} \\ &= \mathcal{L}_F^\alpha\{{}_0\mathcal{J}_x^{m\alpha-\beta}(D_x^\alpha)^m f(x)\} \\ &= \frac{\mathcal{L}_F^\alpha[(D_x^\alpha)^m f(x)]}{s^{m\alpha-\beta}} \end{aligned} \tag{18}$$

In view of Eq. (28) which completes the proof.

Lemma 2. For a given ζ, $\mu > 0$, $S_F^\alpha(a) \in \Re$ and $S_F^\alpha(s)^\zeta > |S_F^\alpha(a)|$ the fractal Laplace transform is

$$ \begin{aligned} &\mathcal{L}_F^{\alpha,-1}\left[\frac{S_F^\alpha(s)^{\zeta-\mu}}{S_F^\alpha(s)^\zeta + S_F^\alpha(a)}\right] \\ &= S_F^\alpha(x)^{\mu-1} E_{F,\zeta,\mu}^\alpha(-S_F^\alpha(a) S_F^\alpha(x)^\zeta). \end{aligned} \tag{19}$$

Proof: Using the series expansion we have

$$ \begin{aligned} \frac{S_F^\alpha(s)^{\zeta-\mu}}{S_F^\alpha(s)^\zeta + S_F^\alpha(a)} &= \frac{1}{S_F^\alpha(s)^\mu}\frac{1}{1+\frac{S_F^\alpha(a)}{S_F^\alpha(s)^\zeta}} && (20) \\ &= \frac{1}{S_F^\alpha(s)^\mu}\sum_{n=0}^\infty\left(\frac{-S_F^\alpha(a)}{S_F^\alpha(s)^\zeta}\right)^n \\ &= \sum_{n=0}^\infty \frac{(-S_F^\alpha(a))^n}{S_F^\alpha(s)^{n\zeta+\mu}} && (21) \end{aligned}$$

The inverse fractal Laplace transform of Eq. (20) leads to

$$ \begin{aligned} &\sum_{n=0}^\infty \frac{(-S_F^\alpha(a))^n\, S_F^\alpha(x)^{n\zeta+\mu-1}}{\Gamma_F^\alpha(n\zeta+\mu)} \\ &= S_F^\alpha(x)^{\mu-1}\sum_{n=0}^\infty \frac{(-S_F^\alpha(a)\, S_F^\alpha(x)^\zeta)^n}{\Gamma_F^\alpha(n\zeta+\mu)} \\ &= S_F^\alpha(x)^{\mu-1} E_{F,\zeta,\mu}^\alpha(-S_F^\alpha(a) S_F^\alpha(x)^\zeta). \end{aligned} \tag{22}$$

Lemma 3. Suppose $\zeta \geq \mu > 0$, $S_F^\alpha(a) \in \Re$ and $S_F^\alpha(s)^{\zeta-\mu} > |S_F^\alpha(a)|$ then we have

$$\mathcal{L}_F^{\alpha,-1}\left[\frac{1}{(S_F^\alpha(s)^\zeta + S_F^\alpha(a)S_F^\alpha(s)^\mu)^{n+1}}\right] = S_F^\alpha(x)^{\zeta(n+1)-1}\sum_{k=0}^{\infty}\frac{-(S_F^\alpha(a))^k}{\Gamma_F^\alpha(k(\zeta-\mu)+(n+1)\zeta)}\binom{n+k}{k}S_F^\alpha(x)^{k(\zeta-\mu)}. \quad (23)$$

Proof: Let us use following expression

$$\frac{1}{(1+S_F^\alpha(x))^{n+1}} = \sum_{k=0}^{\infty}\binom{k+n}{k}(-S_F^\alpha(x))^k. \quad (24)$$

Therefore we can write

$$\frac{1}{(S_F^\alpha(s)^\zeta + S_F^\alpha(a)S_F^\alpha(s)^\mu)^{n+1}} = \frac{1}{(S_F^\alpha(s)^\zeta)^{n+1}}\frac{1}{(1+\frac{S_F^\alpha(a)}{S_F^\alpha(s)^{\zeta-\mu}})^{n+1}} = \frac{1}{(S_F^\alpha(s))^{n+1}}\sum_{k=0}^{\infty}\binom{n+k}{k}\left(\frac{-S_F^\alpha(a)}{S_F^\alpha(s)^{\zeta-\mu}}\right)^k.$$

The proof is complete.

Lemma 4. For $\zeta \geq \mu$, $\zeta > \xi$, $S_F^\alpha(a) \in \Re$, $S_F^\alpha(s)^{\zeta-\mu} > |S_F^\alpha(a)|$ and $|S_F^\alpha(s)^\zeta + S_F^\alpha(a)S_F^\alpha(s)^\mu|$ we have

$$\mathcal{L}_F^{\alpha,-1}\left[\frac{S_F^\alpha(s)^\xi}{S_F^\alpha(s)^\zeta + S_F^\alpha(a)S_F^\alpha(s)^\mu + S_F^\alpha(b)}\right] = S_F^\alpha(x)^{\zeta-\xi-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-S_F^\alpha(b))^n(-S_F^\alpha(a))^k}{\Gamma_F^\alpha(k(\zeta-\mu)+(n+1)\zeta-\xi)}\binom{n+k}{k}S_F^\alpha(x)^{k(\zeta-\mu)+n\zeta}. \quad (25)$$

Proof: Since we can write

$$\frac{S_F^\alpha(s)^\xi}{S_F^\alpha(s)^\zeta + S_F^\alpha(a)S_F^\alpha(s)^\mu + S_F^\alpha(b)} = \frac{S_F^\alpha(s)^\xi}{S_F^\alpha(s)^\zeta + S_F^\alpha(a)S_F^\alpha(s)^\mu}\frac{1}{1+\frac{S_F^\alpha(b)}{S_F^\alpha(s)^\zeta + S_F^\alpha(a)S_F^\alpha(s)^\mu}} = \sum_{n=0}^{\infty}\frac{S_F^\alpha(s)^\xi(-S_F^\alpha(b))^n}{S_F^\alpha(s)^\zeta + S_F^\alpha(a)S_F^\alpha(s)^\mu}, \quad (26)$$

according to the Lemma 3. the proof is complete.

Some important formulas of the local fractal calculus are given below : [11, 20]:

$$\mathcal{L}_F^\alpha[S_F^\alpha(x)^n] = \frac{\Gamma_F^\alpha(n+1)}{S_F^\alpha(s)^{n+1}},$$

$$\mathcal{L}_F^\alpha\left[\int_{S_F^\alpha(0)}^{S_F^\alpha(x)} f(S_F^\alpha(t))d_F^\alpha t\right] = \mathcal{L}_F^\alpha\left[{}_0I_x^\alpha f(S_F^\alpha(t))\right] = \frac{\mathcal{F}_F^\alpha(s)}{s},$$

$$\mathcal{L}_F^\alpha[S_F^\alpha(x)^n f(S_F^\alpha(x))] = (-1)^n(D_F^\alpha)^n\mathcal{F}_F^\alpha(s),$$

$$\mathcal{L}_F^\alpha\left[\int_{S_F^\alpha(0)}^{S_F^\alpha(x)} f(S_F^\alpha(x)-S_F^\alpha(t))g(S_F^\alpha(t))d_F^\alpha t\right] = \mathcal{F}_F^\alpha(S_F^\alpha(s))\mathcal{G}_F^\alpha(S_F^\alpha(s)), \quad (27)$$

and

$$\mathcal{L}_F^\alpha[(D_F^\alpha)^n f(S_F^\alpha(x))] = (S_F^\alpha(s))^{n\alpha}\mathcal{F}_F^\alpha(s) - (S_F^\alpha(s))^{n\alpha-1}f(S_F^\alpha(0)) - (S_F^\alpha(s))^{n\alpha-2}D_F^\alpha f(x)|_{x=S_F^\alpha(0)} - \ldots - (D_F^\alpha)^{n-1}f(x)|_{x=S_F^\alpha(0)}. \quad (28)$$

Remark 2. If we choose $\alpha = 1$ we obtain the standard result.

The important formulas of non-local fractal calculus are as follows [20]:

$${}_0\mathcal{J}_x^\beta(S_F^\alpha(x))^\eta = \frac{\Gamma_F^\alpha(\eta+1)}{\Gamma_F^\alpha(\eta+\beta+1)}(S_F^\alpha(x))^{\eta+\beta},$$

$${}_0\mathcal{D}_x^\beta(S_F^\alpha(x))^\eta = \frac{\Gamma_F^\alpha(\eta+1)}{\Gamma_F^\alpha(\eta-\beta+1)}(S_F^\alpha(x))^{\eta-\beta}.$$

$${}_0\mathcal{D}_x^\beta(c\,\chi_F^\alpha) = \frac{c}{\Gamma_F^\alpha(1-\beta)}(S_F^\alpha(x))^{-\beta},$$

$$\mathcal{L}_F^\alpha[{}_0\mathcal{J}_x^\beta f(x)] = \frac{\mathcal{F}_F^\alpha(S_F^\alpha(s))}{S_F^\alpha(s)^\beta}. \quad (29)$$

where c is constant.

Remark 3. If we choose $\beta = \alpha$ then we arrive at to the local fractal derivative whose order is equal the dimension of the fractal.

5 Comparison between the local fractal differential and non-local fractal differential

In this section, we compare the local and non-local fractal differential equations.

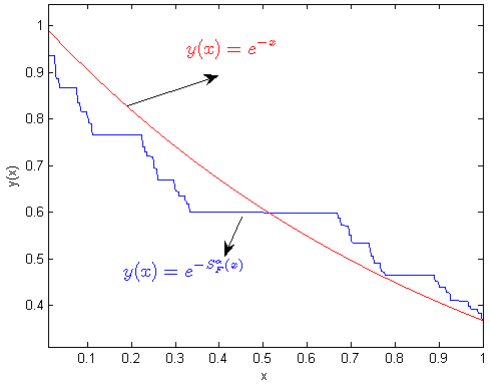


Figure 3: We plot the solution of Eq. (30).

Example 1. Consider the linear local fractal differential equation

$$D_F^\alpha y(x) + y(x) = 0, \tag{30}$$

with initial-value

$$y(x)|_{x=S_F^\alpha(0)} = 1, \tag{31}$$

Hence the solution to Eq. (30) is

$$y(x) = e^{-S_F^\alpha(x)}, \tag{32}$$

where $\alpha = 0.6309$ is the γ-dimension of the triadic Cantor set [11, 20].

In Figure 3 we give the graph of Eq. (32).

Example 2. Consider linear non-local fractal differential equation as

$${}_0^C\mathcal{D}_x^\beta y(x) + y(x) = 0, \tag{33}$$

with the initial condition

$$y(x)|_{x=S_F^\alpha(0)} = 1, \quad D_F^\alpha y(x)|_{x=S_F^\alpha(0)} = 0. \tag{34}$$

In view of Eq. (17) we have

$$\mathcal{L}_F^\alpha\{{}_0^C\mathcal{D}_x^\beta f(x)\} = \frac{(S_F^\alpha(s))^\alpha \mathcal{F}_F^\alpha(s) - 1}{S_F^\alpha(s)^{\alpha-\beta}}. \tag{35}$$

Applying the fractal Laplace transformation on both sides of Eq. (33) and using Eq. (17) we obtain

$$\frac{(S_F^\alpha(s))^\alpha \mathcal{F}_F^\alpha(s) - 1}{S_F^\alpha(s)^{\alpha-\beta}} + \mathcal{F}_F^\alpha(s) = 0. \tag{36}$$

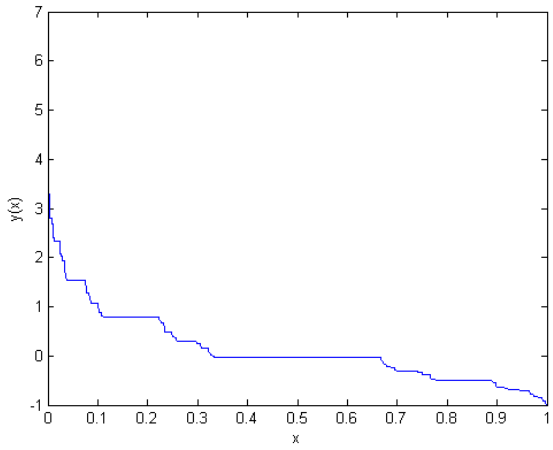
It follows that

$$\mathcal{F}_F^\alpha(s) = \frac{S_F^\alpha(s)^{\beta-\alpha}}{1 + S_F^\alpha(s)^\beta}, \tag{37}$$

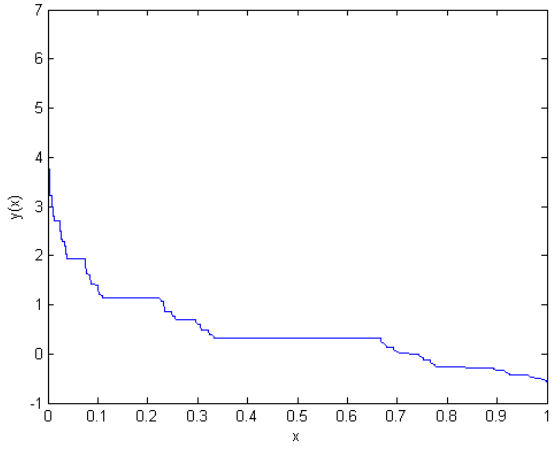
using fractal inverse Laplace transform Eq. (19) we arrive at the solution of Eq. (33) as follows

$$y(x) = S_F^\alpha(x)^{\alpha-1} E_{F,\beta,\alpha}^\alpha\left(-S_F^\alpha(x)^\beta\right). \tag{38}$$

In Figure 4 we present the graph of Eq.(38).



(a) If we choose $\beta = 0.33$ in Eq. (38)



(b) If we choose $\beta = 0.25$ in Eq. (38)

Figure 4: We draw the graph of Eq. (38).

6 Application of non-local fractal differential equations

In this section we provide the applications and new models to non-local fractal derivatives [20].

Fractal Abel's tautochrone: As a first example we generalized Abel's problem which is the curve of quick descent on the fractal time-space. Using the conservation of energy in the fractal space the differential equation of the motion a particle is

$$D_{F,t}^\alpha \mathfrak{s}_F^\alpha = \frac{d_F^\alpha \mathfrak{s}_F^\alpha}{d_F^\alpha t} = -\sqrt{2g_F^\alpha(S_F^\alpha(y) - S_F^\alpha(y_0))}, \tag{39}$$

where $\mathfrak{s}_F^\alpha$ is fractal arc length, and g_F^α fractal space gravitational constant, and y is the high particle from the refer-

ence of potential. As a result we have

$$S_F^\alpha(T) = -\frac{1}{\sqrt{2g_F^\alpha}} \int_{S_F^\alpha(A)}^{S_F^\alpha(B)} \frac{1}{\sqrt{(S_F^\alpha(y) - S_F^\alpha(\eta))}} d_F^\alpha \mathfrak{s}_F^\alpha. \quad (40)$$

Let us consider

$$\mathfrak{s}_F^\alpha = h_F^\alpha(S_F^\alpha(\eta)), \quad (41)$$

so that we have

$$S_F^\alpha(T) = -\frac{1}{\sqrt{2g_F^\alpha}} \int_{S_F^\alpha(y)}^{S_F^\alpha(0)} (S_F^\alpha(y) - S_F^\alpha(\eta))^{-1/2} D_{F,\eta}^\alpha h_F^\alpha(\eta) d_F^\alpha \eta. \quad (42)$$

Utilizing $D_{F,\eta}^\alpha h_F^\alpha(S_F^\alpha(y)) = f(S_F^\alpha(y))$ we arrive at

$$S_F^\alpha(T) = -\frac{1}{\sqrt{2g_F^\alpha}} \int_{S_F^\alpha(y)}^{S_F^\alpha(0)} (S_F^\alpha(y) - S_F^\alpha(\eta))^{-1/2} f(S_F^\alpha(y)) d_F^\alpha \eta. \quad (43)$$

It follows

$$\frac{\sqrt{2g_F^\alpha}}{\Gamma(\frac{1}{2})} S_F^\alpha(T) = {}_0\mathcal{D}_y^{1/2} f(y). \quad (44)$$

The solution of Eq.(44) is called the fractal cycloid.

Fractal models for the viscoelasticity: We generalize the viscoelasticity models to the fractal mediums in the case of ideal solids and ideal liquids. Namely, the fractal ideal solids described by

$$\sigma_F^\alpha(t) = E_F^\alpha \epsilon_F^\alpha(t), \quad (45)$$

which is called Hooke's Law of fractal elasticity. Where σ_F^α is fractal stress, ϵ_F^α is fractal strain which occurs under the applied stress and E_F^α is the elastic modulus of the fractal material.

The fractal ideal fluid can be modeled and described by Newton's Law of fractal viscosity as follows

$$\sigma_F^\alpha(t) = \lambda_F^\alpha D_F^\alpha \epsilon_F^\alpha(t), \quad (46)$$

where λ_F^α is the viscosity of the fractal material. But in nature we have real martials which have properties between the ideal solids and ideal liquids. It is clear that in the Hooke's Law of fractal elasticity Eq. (45) fractal stress is proportional to the 0-order derivative of the fractal strain and in Newton's Law of fractal viscosity the stress is proportional to the α-order derivative of the fractal strain. Therefore, more general model is

$$\sigma_F^\alpha(t) = E_F^\alpha (\chi_F^\alpha)^\beta {}_0\mathcal{D}_x^\beta \epsilon_F^\alpha(t), \qquad \chi_F^\alpha = \frac{\lambda_F^\alpha}{E_F^\alpha}, \quad (47)$$

which is called fractal Blair's model. Here, we suggest the fractional non-local order fractal derivative β as an index

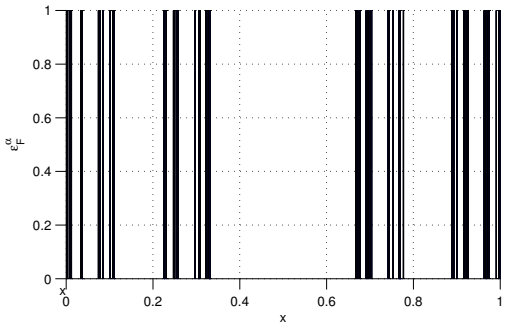


Figure 5: We sketch $\epsilon_F^\alpha(t) = \chi_F^\alpha$ which is characteristic function of the triadic Cantor set.

of memory. Namely, if we choose $\beta = 0$ the process is equivalent to "nothing forgotten" and the case of $\beta = \alpha$ the process is memoryless. Hence if we choose $0 < \beta < \alpha$ it shows the processes with memory on fractals.

If we choose

$$\epsilon_F^\alpha(t) = \chi_F^\alpha, \quad (48)$$

where χ_F^α is characteristic function of the triadic Cantor set. In Figure 5 we plot the $\epsilon_F^\alpha(t)$. Utilizing Eq. (47) we obtain the fractal stress as follows

$$\sigma_F^\alpha(t) = E_F^\alpha (\chi_F^\alpha)^\beta \frac{1}{\Gamma_F^\alpha(1-\beta)} (S_F^\alpha(t))^{-\beta}. \quad (49)$$

In Figure 6 we show the graph of $\sigma_F^\alpha(t)$ fractal stress.

Remark 4. If we choose $\beta = 0$ and $\beta = \alpha$ in Eq. (47) we will have the fractal stress and the fractal strain relations for the cases of fractal ideal solids and the fractal ideal fluids, respectively.

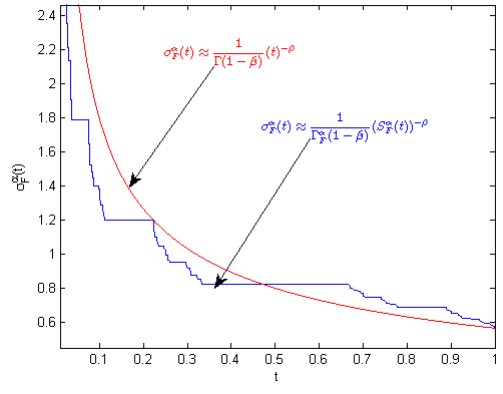


Figure 6: We sketch $\sigma_F^\alpha(t)$ for the fractal stress substituting $\beta = 0.5$

7 Conclusion

In this paper we generalized fractal calculus involving the non-local derivatives. The scaling properties of local and non-local derivatives are studied because they are important in physical applications. Using an illustrative example we compared the local and non-local linear fractal differential equations. We also suggested some applications for the new non-local fractal differential equations.

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