

## Research Article

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# Stability and Analytic Solutions of an Optimal Control Problem on the Schrödinger Lie Group

DOI 10.1515/phys-2016-0060

Received August 27, 2016; accepted November 17, 2016

**Abstract:** The nonlinear stability and the existence of the periodic solutions for an optimal control problem on the Schrödinger Lie group are discussed. The analytic solutions via optimal homotopy asymptotic method of the dynamics and numerical simulations are presented, too.

**Keywords:** optimal control problem; nonlinear stability; periodic solutions; optimal homotopy asymptotic method

**PACS:** 02.20.Sv; 02.60.-x; 37J25; 65P40; 74H10; 37N20

## 1 Introduction

The role of the Schrödinger group in physics was always important and the literature on the subject is immense. The Schrödinger group and the Schrödinger algebra were first introduced by Niederer [13] and Hagen [14] as a non-relativistic limit of the vector-field realization of the conformal algebra. The Schrödinger group could play the space configuration role in some concrete mechanical problems, like other important Lie groups do:  $SO(3)$  for aircraft dynamics,  $SE(3)$  for underwater dynamics,  $SO(2) \times R^2$  for the rolling-penny dynamics,  $R^2 \times SO(3)$  for ball-plate problem, and so on. All these dynamics are modeled as invariant optimal control problems on the mentioned Lie groups. The study of the invariant optimal control problem on the Schrödinger Lie group naturally completes the short list from above. More details can be found in [5] or [9].

This article is concerned with some important comments on the behavior of a dynamical system arising from an optimal control problem on the Schrödinger Lie group presented in [10]. More specifically, we are interested in

improving the stability results, proving that some equilibrium states are nonlinearly stable and finding the periodic orbits around them, and finally, we propose an alternative method for finding analytic solution of the system.

The system, which can be put into the equivalent form:

$$\begin{cases} \dot{x}_1 = -x_2x_3 \\ \dot{x}_2 = -2x_1^2 + 2x_3^2 - x_4^2 \\ \dot{x}_3 = x_1x_2 - x_4x_5 \\ \dot{x}_4 = x_3x_5 \\ \dot{x}_5 = x_1x_4 + kx_4, \end{cases} \quad (1)$$

$k \in \mathbb{R}$ , has the Hamilton-Poisson realization (see [10]):

$$(\mathbb{R}^5, \Pi_-, H),$$

where

$$\Pi_- = \begin{bmatrix} 0 & 2x_1 & -x_2 & 0 & -x_4 \\ -2x_1 & 0 & 2x_3 & -x_4 & x_5 \\ x_2 & -2x_3 & 0 & -x_5 & 0 \\ 0 & x_4 & x_5 & 0 & -k \\ x_4 & -x_5 & 0 & k & 0 \end{bmatrix}$$

is the minus Lie-Poisson structure and

$$H(x_1, x_2, x_3, x_4, x_5) = \frac{1}{2}(x_1^2 + x_3^2 + x_4^2)$$

is the Hamiltonian. Also, there exists only one functionally independent Casimir of our Poisson configuration, given by  $C : \mathbb{R}^5 \rightarrow \mathbb{R}$ ,

$$C(x_1, x_2, x_3, x_4, x_5) = 2kx_1x_3 + \frac{k}{2}x_2^2 + x_1x_5^2 + x_2x_4x_5 - x_3x_4^2.$$

Following [10], the equilibrium states of the dynamics (1) are:

- $e_1^P = (P, 0, P, 0, 0)$  which are unstable for any  $P \in \mathbb{R}^*$ ;
- $e_2^P = (P, 0, -P, 0, 0)$  which are spectrally stable for  $\frac{P}{k} > 0$ ;
- $e_3^{PQ} = (0, P, 0, 0, Q)$  which are spectrally stable for any  $P, Q \in \mathbb{R}^*$ .

The paper is organized as follows: the second section is dedicated to the nonlinear stability results; also, the existence of the periodical orbits around the nonlinear stable

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states is proved. The existence of the Hamilton-Poisson realization offers the necessary geometric framework in order to obtain this results. Specific tools, such as the energy methods and the Moser Theorem, belong to this framework.

The Optimal Homotopy Asymptotic Method is presented in the third section. Using this method, the analytical approximate solution of the proposed dynamics is found in the fourth paragraph. A comparison with the Runge-Kutta 4th steps integrator obtained using MATHEMATICA 6.0 and numerical simulations are the subjects of the last part.

## 2 Nonlinear Stability Problems and Periodical Orbits

Let us present very briefly, the concept of nonlinear stable equilibrium point of a dynamical system. For more details, see [7].

**Definition 2.1.** *An equilibrium state  $x_e$  is said to be nonlinear stable if for each neighborhood  $U$  of  $x_e$  in  $D$  there is a neighborhood  $V$  of  $x_e$  in  $U$  such that trajectory  $x(t)$  initially in  $V$  never leaves  $U$ .*

*This definition supposes well-defined dynamics and a specified topology. In terms of a norm  $\| \cdot \|$ , nonlinear stability means that for each  $\varepsilon > 0$  there is  $\delta > 0$  such that if*

$$\|x(0) - x_e\| < \delta$$

then

$$\|x(t) - x_e\| < \varepsilon, (\forall)t > 0.$$

It is clear that nonlinear stability implies spectral stability; the converse is not always true.

**Proposition 2.2.** *The equilibrium states  $e_2^P$  are nonlinearly stable if  $\frac{P}{k} > 0, P, k \in \mathbb{R}^*$ .*

*Proof.* We will use Arnold’s technique. Consider the function

$$\begin{aligned} F_\lambda(x_1, x_2, x_3, x_4, x_5) &= H(x_1, x_2, x_3, x_4, x_5) \\ &+ \lambda C(x_1, x_2, x_3, x_4, x_5) = \frac{1}{2}(x_1^2 + x_3^2 + x_4^2) + 2k\lambda x_1 x_3 \\ &+ \frac{k}{2}\lambda x_2^2 + \lambda x_1 x_5^2 + \lambda x_2 x_4 x_5 - \lambda x_3 x_4^2. \end{aligned}$$

The following conditions hold:

(i)  $\nabla F_\lambda(e_2^P) = 0$  if  $\lambda = \frac{1}{2k}$ ;

(ii) Taking now

$$W = \ker[dC(e_2^P)] = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

then, for all  $v \in W$ , i.e.  $v = (0, a, 0, b, c)$ ,  $a, b, c \in \mathbb{R}$ , we have

$$v \cdot \nabla^2 F_{\frac{1}{2k}}(e_2^P) \cdot v^t = \frac{1}{2}a^2 + \left(1 + \frac{P}{k}\right)b^2 + \frac{P}{k}c^2$$

which is positive definite under the restriction  $\frac{P}{k} > 0$ , and so

$$\nabla^2 F_{\frac{1}{2k}}(e_2^P)|_{W \times W}$$

is also positive definite.

Hence, via Arnold’s technique, the equilibrium states  $e_2^P$  with  $\frac{P}{k} > 0, P, k \in \mathbb{R}^*$  are nonlinear stable, as required.  $\square$

As a consequence, we can find the periodic orbits of the equilibrium points  $e_2^P$  for  $\frac{P}{k} > 0, P, k \in \mathbb{R}^*$ . More exactly, we can prove:

**Proposition 2.3.** *If  $\frac{P}{k} > 0, P, k \in \mathbb{R}^*$  the reduced dynamics to the coadjoint orbit*

$$2kx_1x_3 + \frac{k}{2}x_2^2 + x_1x_5^2 + x_2x_4x_5 - x_3x_4^2 = -2kP^2 \quad (2)$$

has near the equilibrium point  $e_2^P$  at least two periodic solution whose period is close to

$$\frac{\pi}{2\sqrt{2}|P|} \quad \text{and} \quad \frac{\pi}{\sqrt{kP + P^2}}.$$

*Proof.* We will use Moser-Weinstein theorem with zero eigenvalue, see [3] for details.

(i) The restriction of our dynamics (1) to the coadjoint orbit (2) gives rise to a classical Hamiltonian system.

(ii) The matrix of the linear part of our reduced dynamics to (1) has purely imaginary roots at the equilibrium of interest:

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm 2i\sqrt{2}P \quad \text{and} \quad \lambda_{4,5} = \pm i\sqrt{kP + P^2}.$$

(iii)

$$\text{Span}[\nabla C(e_2^P)] = \ker[A(e_2^P)] = \text{Span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

(iv) If  $\frac{P}{k} > 0, P, k \in \mathbb{R}^*$ , then the reduced Hamiltonian has a local minimum at the equilibrium state  $e_2^P$ .  $\square$

**Proposition 2.4.** *The equilibrium states  $e_3^{PQ} = (0, P, 0, 0, Q)$  are nonlinearly stable for any  $P, Q \in \mathbb{R}^*$ .*

*Proof.* We will use energy-Casimir method, see [2] for details. Let

$$F_\varphi(x_1, x_2, x_3, x_4, x_5) = H(x_1, x_2, x_3, x_4, x_5) + \varphi[C(x_1, x_2, x_3, x_4, x_5)] = \frac{1}{2}(x_1^2 + x_3^2 + x_4^2) + \varphi(2kx_1x_3 + \frac{k}{2}x_2^2 + x_1x_5^2 + x_2x_4x_5 - x_3x_4^2)$$

be the energy-Casimir function, where  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth real valued function.

Now, the first variation of  $F_\varphi$  is given by

$$\delta F_\varphi(x_1, x_2, x_3, x_4, x_5) = x_1\delta x_1 + x_3\delta x_3 + x_4\delta x_4 + \dot{\varphi}(2kx_1x_3 + \frac{k}{2}x_2^2 + x_1x_5^2 + x_2x_4x_5 - x_3x_4^2) \cdot [(2kx_3 + x_5^2)\delta x_1 + (kx_2 + x_4x_5)\delta x_2 + (2kx_1 - x_4^2)\delta x_3 + (x_2x_5 - 2x_3x_4)\delta x_4 + (2x_1x_5 + x_2x_4)\delta x_5]$$

so we obtain

$$\delta F_\varphi(e_3^{PQ}) = \dot{\varphi} \left( \frac{k}{2} P^2 \right) (Q^2 \delta x_1 + kP \delta x_2 + PQ \delta x_4)$$

that is equals zero for any  $P, Q \in \mathbb{R}^*$  if and only if

$$\dot{\varphi} \left( \frac{k}{2} P^2 \right) = 0. \tag{3}$$

The second variation of  $F_\varphi$  at the equilibrium of interest is given by

$$\delta^2 F_\varphi(e_3^{PQ}) = (\delta x_1)^2 + (\delta x_3)^2 + (\delta x_4)^2 + \ddot{\varphi} \left( \frac{k}{2} P^2 \right) (Q^2 \delta x_1 + kP \delta x_2 + PQ \delta x_4)^2.$$

If we choose now  $\varphi$  such that the relation (3) is valid and

$$\ddot{\varphi} \left( \frac{k}{2} P^2 \right) > 0,$$

then the second variation of  $F_\varphi$  at the equilibrium of interest is positive defined and so our equilibrium states  $e_3^{PQ}$  are nonlinearly stable.  $\square$

### 3 The Optimal Homotopy Asymptotic Method

Let us pass now to find the analytical approximate solutions for the nonlinear differential system given by the Eq. (1), with the boundary conditions

$$x_i(0) = A_i, \quad i = \overline{1, 5}, \tag{4}$$

using the Optimal Homotopy Asymptotic Method (OHAM) (see [11], [12] for details). In the beginning, we briefly present the description of this method.

For equations of the general form:

$$L(F(\eta)) + N(F(\eta)) = 0, \tag{5}$$

subject to the boundary/initial conditions of the type:

$$B\left(F(\eta), \frac{dF(\eta)}{d\eta}\right) = 0. \tag{6}$$

we construct (see [11] and [12]) the homotopy given by:

$$\mathcal{H}\left[L(F(\eta, p)), H(\eta, C_i), N(F(\eta, p))\right], \tag{7}$$

where  $p \in [0, 1]$  is the embedding parameter,  $L$  is a linear operator and  $H(\eta, C_i)$ , ( $H \neq 0$ ) is an auxiliary convergence-control function, depending on the variable  $\eta$  and on the parameters  $C_1, C_2, \dots, C_s$ .

One can prove that the following properties hold:

1.

$$\mathcal{H}\left[L(F(\eta, 0)), H(\eta, C_i), N(F(\eta, 0))\right] = L(F(\eta, 0)) = L(F_0(\eta)) \tag{8}$$

2.

$$\mathcal{H}\left[L(F(\eta, 1)), H(\eta, C_i), N(F(\eta, 1))\right] = H(\eta, C_i)N(F(\eta, 1)). \tag{9}$$

If the function  $F$  is giving by:

$$F(\eta, p) = F_0(\eta) + pF_1(\eta, C_i), \tag{10}$$

then, by substituting (10) in (7), the following relation is obtained:

$$\mathcal{H}\left[L(F(\eta, p)), H(\eta, C_i), N(F(\eta, p))\right] = 0. \tag{11}$$

Consider the operator  $\mathcal{H}$  of the specific expression:

$$\mathcal{H}\left[L(F(\eta, p)), H(\eta, C_i), N(F(\eta, p))\right] = L(F_0(\eta)) + p\left[L(F_1(\eta, C_i)) - H(\eta, C_i)N(F_0(\eta))\right], \tag{12}$$

where the governing equations of  $F_0(\eta)$  and  $F_1(\eta, C_i)$  can be obtained by equating the coefficients of  $p^0$  and  $p^1$ , respectively:

$$L(F_0(\eta)) = 0, \quad B(F_0(\eta), \frac{dF_0(\eta)}{d\eta}) = 0 \tag{13}$$

$$\begin{aligned} L(F_1(\eta, C_i)) &= H(\eta, C_i)N(F_0(\eta)), \\ B(F_1(\eta, C_i), \frac{dF_1(\eta, C_i)}{d\eta}) &= 0, \quad i = \overline{1, s}. \end{aligned} \tag{14}$$

The expression of  $F_0(\eta)$  can be found solving the linear equation (13). Also, to compute  $F_1(\eta, C_i)$  we solve the equation (14), by taking into consideration that the nonlinear operator  $N$  presents the general form:

$$N(F_0(\eta)) = \sum_{i=1}^m h_i(\eta)g_i(\eta), \tag{15}$$

where  $m$  is a positive integer and  $h_i(\eta)$  and  $g_i(\eta)$  are known functions depending both on  $F_0(\eta)$  and  $N$ .

Although equation (14) is a nonhomogeneous linear equation, in most cases its solution can not be found.

In order to compute the function  $F_1(\eta, C_i)$  we will use the third modified version of OHAM (see [12] for details), consisting in the following steps:

– First we consider one of the following expressions for  $F_1(\eta, C_i)$ :

$$F_1(\eta, C_i) = \sum_{i=1}^m H_i(\eta, h_j(\eta), C_j)g_i(\eta), \quad j = \overline{1, s}, \tag{16}$$

or

$$\begin{aligned} F_1(\eta, C_i) &= \sum_{i=1}^m H_i(y, g_j(\eta), C_j)h_i(\eta), \quad j = \overline{1, s}, \\ B(F_1(\eta, C_i), \frac{dF_1(\eta, C_i)}{d\eta}) &= 0. \end{aligned} \tag{17}$$

These expressions of  $H_i(\eta, h_j(\eta), C_j)$  contain both linear combinations of the functions  $h_j$  and the parameters  $C_j, j = \overline{1, s}$ . The summation limit  $m$  is an arbitrary positive integer number.

– Next, by taking into account the equation (10), the first-order analytical approximate solution of the equations (5) - (6) is:

$$\bar{F}(\eta, C_i) = F(\eta, 1) = F_0(\eta) + F_1(\eta, C_i). \tag{18}$$

– Finally, the convergence-control parameters  $C_1, C_2, \dots, C_s$ , which determine the first-order approximate solution (18), can be optimally computed by means of various methods, such as: the least square method, the Galerkin method, the collocation method, the Kantorowich method or the weighted residual method and so on. The first option should be minimizing the square residual error:

$$J(C_1, C_2, \dots, C_s) = \int_{(D)} R^2(\eta, C_1, C_2, \dots, C_s) d\eta \tag{19}$$

where the residual  $R$  is given by

$$R(\eta, C_1, C_2, \dots, C_s) = N(\bar{f}(\eta, C_i)). \tag{20}$$

The unknown parameters  $C_1, C_2, \dots, C_s$  can be identified from the conditions:

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \dots = \frac{\partial J}{\partial C_s} = 0. \tag{21}$$

With these parameters known (called optimal convergence-control parameters), the first-order approximate solution given by Eq. (18) is well-determined.

It should be emphasized that our procedure contains the auxiliary functions  $H_i(\eta, f_i, C_j), i = 1, \dots, m, j = 1, \dots, s$  which provides us with a simple way to adjust and control the convergence of the approximate solutions. It is very important to properly choose these functions  $H_i(\eta, f_i, C_j)$  which appear in the construction in the first-order approximation.

## 4 Application of Optimal Homotopy Asymptotic Method for Solving the Nonlinear Differential System (1)

Let us now apply the method described above to solve the nonlinear differential system given by the equations (1). We choose the linear operators:

$$\begin{aligned} L[x_1(t)] &= x_1'(t) + K_1x_1(t) \\ L[x_2(t)] &= x_2'(t) + K_1x_2(t) \\ L[x_3(t)] &= x_3'(t) + K_1x_3(t) \\ L[x_4(t)] &= x_4'(t) + K_1x_4(t) \\ L[x_5(t)] &= x_5'(t) + K_1x_5(t) - ax_4(t) \end{aligned} \tag{22}$$

where  $K_1 > 0$  is unknown parameter.

The corresponding linear equations for the initial approximations  $x_{i_0}, i = \overline{1, 5}$ , becomes

$$L[x_{i_0}(t)] = 0, \quad x_{i_0}(0) = A_i, \quad i = \overline{1, 5}, \tag{23}$$

whose solutions are

$$\begin{aligned} x_{1_0}(t) &= A_1 \cdot e^{-K_1t} \\ x_{2_0}(t) &= A_2 \cdot e^{-K_1t} \\ x_{3_0}(t) &= A_3 \cdot e^{-K_1t} \\ x_{4_0}(t) &= A_4 \cdot e^{-K_1t} \\ x_{5_0}(t) &= (A_5 + aA_4t) \cdot e^{-K_1t}. \end{aligned} \tag{24}$$

The nonlinear operators  $N[x_i(t)], i = \overline{1, 5}$ , are obtained from the equations (1) as follows:

$$\begin{aligned} N[x_1(t)] &= -K_1x_1(t) + x_2(t)x_3(t) \\ N[x_2(t)] &= -K_1x_2(t) + 2x_1^2(t) - 2x_3^2(t) + x_4^2(t) \\ N[x_3(t)] &= -K_1x_3(t) - x_1(t)x_2(t) + x_4(t)x_5(t) \\ N[x_4(t)] &= -K_1x_4(t) - x_3(t)x_5(t) \\ N[x_5(t)] &= -K_1x_5(t) - x_1(t)x_4(t). \end{aligned} \tag{25}$$

Substituting Eqs. (24) into Eqs. (25), we obtain

$$\begin{aligned} N[x_{1_0}(t)] &= -K_1x_{1_0}(t) + x_{2_0}(t)x_{3_0}(t) \\ N[x_{2_0}(t)] &= -K_1x_{2_0}(t) + 2x_{1_0}^2(t) - 2x_{3_0}^2(t) + x_{4_0}^2(t) \\ N[x_{3_0}(t)] &= -K_1x_{3_0}(t) - x_{1_0}(t)x_{2_0}(t) + x_{4_0}(t)x_{5_0}(t) \quad (26) \\ N[x_{4_0}(t)] &= -K_1x_{4_0}(t) - x_{3_0}(t)x_{5_0}(t) \\ N[x_{5_0}(t)] &= -K_1x_{5_0}(t) - x_{1_0}(t)x_{4_0}(t). \end{aligned}$$

**Remark 4.1.** *a) We have a large freedom to choose the linear operator  $L$ , such as:*

$$L[x_1(t)] = x_1'(t) + K_1x_1(t) + g_1(t)$$

*and therefore, the nonlinear operator  $N$  becomes*

$$N[x_1(t)] = -K_1x_1(t) + x_2(t)x_3(t) - g_1(t)$$

*where  $g_1(t)$  is an arbitrary continuous function that convergence-control the solution.*

*b) The function  $g_1(t)$  can be chosen as  $g_1(t) = B_1 \cos \omega t + C_1 \sin \omega t$  or  $g_1(t) = B_1 \cos \omega t + C_1 \sin \omega t + B_2 \cos \omega_2 t + C_2 \sin \omega_2 t$  or  $g_1(t) = B_1 \cos \omega t + C_1 \sin \omega t + B_2 \cos 3\omega t + C_2 \sin 3\omega t$  and so on.*

The auxiliary convergence-control functions  $H_i$  are chosen such that the product  $H_i \cdot N[x_{i_0}(t)]$  has the same form like  $N[x_{i_0}(t)]$ . Then, the first approximation is:

$$\begin{aligned} x_{i_1} &= [B_1 \cos \omega t + C_1 \sin \omega t + B_2 \cos 3\omega t + C_2 \sin 3\omega t \\ &+ B_3 \cos 5\omega_1 t + C_3 \sin 5\omega_1 t + B_4 \cos 7\omega_1 t + C_4 \sin 7\omega_1 t \\ &+ B_5 \cos \omega_2 t + C_5 \sin \omega_2 t + B_6 \cos 3\omega_2 t + C_6 \sin 3\omega_2 t \\ &+ B_7 \cos 5\omega_3 t + C_7 \sin 5\omega_3 t + B_8 \cos 7\omega_3 t + C_8 \sin 7\omega_3 t] \\ &\cdot e^{-K_1 t}. \quad (27) \end{aligned}$$

Using now the third-alternative of OHAM and the equations (18), the first-order approximate solution can be put in the form

$$\bar{x}_i(t, C_j) = x_{i_0}(t) + x_{i_1}(t, C_j), \quad i = \overline{1, 5} \quad (28)$$

where  $x_{i_0}(t)$  and  $x_{i_1}(t, C_j)$  are given by (24) and (27), respectively.

For the nonlinear differential system given by Eq. (1) the corresponding residual functions given by Eq. (20) become:

$$\begin{cases} R_1(t, B_j, C_j) = \dot{\bar{x}}_1 + \bar{x}_2\bar{x}_3 \\ R_2(t, B_j, C_j) = \dot{\bar{x}}_2 + 2\bar{x}_1^2 - 2\bar{x}_3^2 + \bar{x}_4^2 \\ R_3(t, B_j, C_j) = \dot{\bar{x}}_3 - \bar{x}_1\bar{x}_2 + \bar{x}_4\bar{x}_5 \\ R_4(t, B_j, C_j) = \dot{\bar{x}}_4 - \bar{x}_3\bar{x}_5 \\ R_5(t, B_j, C_j) = \dot{\bar{x}}_5 - \bar{x}_1\bar{x}_4 - k\bar{x}_4, \end{cases} \quad (29)$$

$k \in \mathbb{R}$ .

## 5 Numerical Examples and Discussions

In this section, the accuracy and validity of the OHAM technique is proved using a comparison of our approximate solutions with numerical results obtained via the fourth-order Runge-Kutta method in the following case: we consider the initial value problem given by Eq. (1) with initial conditions  $A_i = 0.1, i = \overline{1, 5}, k = 50$ .

The convergence-control parameters  $K_1, \omega, \omega_1, \omega_2, \omega_3, B_i, C_i, i = \overline{1, 8}$  are optimally determined by means of the least-square method.

For all unknown functions  $\bar{x}_i, i = \overline{1, 5}$ , we find  $K_1 = 2.278126552620239, \omega = 1.3762803403795802, \omega_1 = 0.47813539122122517, \omega_2 = 1.7711316031320425, \omega_3 = 0.7512679521244802$ .

The first-order approximate solutions given by the Eq. (28) are respectively:

– for  $\bar{x}_1$  :

$$\begin{aligned} \bar{x}_1(t) &= e^{-2.2781265526t} \\ &\cdot (0.1 - 1539.2036537865 \cos[1.3762803403t] \\ &+ 2146.5931060662 \cos[1.7711316031t] \\ &- 62.5789184515 \cos[2.3906769561t] \\ &- 2105.5202871126 \cos[3.3469477385t] \\ &+ 2145.1995165251 \cos[3.7563397606t] \\ &- 556.6739392471 \cos[4.1288410211t] \\ &- 228.1069705054 \cos[5.2588756648t] \\ &+ 200.2911465118 \cos[5.3133948093t] \\ &- 233.2354956045 \sin[1.3762803403t] \\ &+ 1734.4533467178 \sin[1.7711316031t] \\ &- 2010.4690691105 \sin[2.3906769561t] \\ &+ 498.9518111758 \sin[3.3469477385t] \\ &+ 801.0961373896 \sin[3.7563397606t] \\ &- 690.0244186296 \sin[4.1288410211t] \\ &+ 175.2647769735 \sin[5.2588756648t] \\ &- 131.0263973696 \sin[5.3133948093t]) \quad (30) \end{aligned}$$

– for  $\bar{x}_2$  :

$$\begin{aligned} \bar{x}_2(t) = e^{-2.2781265526t} & \\ \cdot (0.1 - 15260.7815681516 \cos[1.3762803403t] & \\ + 17108.5154597769 \cos[1.7711316031t] & \\ + 8461.8887319812 \cos[2.3906769561t] & \\ - 32725.7593591730 \cos[3.3469477385t] & \\ + 28565.1499752006 \cos[3.7563397606t] & \\ - 5471.1627975838 \cos[4.1288410211t] & \\ - 4896.0864779965 \cos[5.2588756648t] & \\ + 4218.2360359462 \cos[5.3133948093t] & \\ - 7070.7498432360 \sin[1.3762803403t] & \\ + 27060.1826811753 \sin[1.7711316031t] & \\ - 24769.1457444271 \sin[2.3906769561t] & \\ - 3740.4992886592 \sin[3.3469477385t] & \\ + 23255.2371244883 \sin[3.7563397606t] & \\ - 13866.5482555146 \sin[4.1288410211t] & \\ + 2281.1108262053 \sin[5.2588756648t] & \\ - 1610.8659095678 \sin[5.3133948093t]) & \quad (31) \end{aligned}$$

– for  $\bar{x}_3$  :

$$\begin{aligned} \bar{x}_3(t) = e^{-2.2781265526t} & \\ \cdot (0.1 + 139431.2629341998 \cos[1.3762803403t] & \\ - 210104.0681072855 \cos[1.7711316031t] & \\ + 29008.5479815552 \cos[2.3906769561t] & \\ + 218012.8029626905 \cos[3.3469477385t] & \\ - 273969.5008187874 \cos[3.7563397606t] & \\ + 98280.7262884159 \cos[4.1288410211t] & \\ + 7067.2891846270 \cos[5.2588756648t] & \\ - 7727.0604254157 \cos[5.3133948093t] & \\ + 23048.1402575979 \sin[1.3762803403t] & \\ - 162846.7585419209 \sin[1.7711316031t] & \\ + 209729.6427073796 \sin[2.3906769561t] & \\ - 126581.7269534985 \sin[3.3469477385t] & \\ + 7138.6204759150 \sin[3.7563397606t] & \\ + 44499.4492217200 \sin[4.1288410211t] & \\ - 34929.8074534369 \sin[5.2588756648t] & \\ + 28628.4924527697 \sin[5.3133948093t]) & \quad (32) \end{aligned}$$

– for  $\bar{x}_4$  :

$$\begin{aligned} \bar{x}_4(t) = e^{-2.2781265526t} & \\ \cdot (0.1 + 54326.5181638183 \cos[1.3762803403t] & \\ - 81607.7493425815 \cos[1.7711316031t] & \\ + 12684.9219591446 \cos[2.3906769561t] & \\ + 72363.6896622957 \cos[3.3469477385t] & \\ - 87183.8189010923 \cos[3.7563397606t] & \\ + 29386.8516596795 \cos[4.1288410211t] & \\ + 2874.7475267167 \cos[5.2588756648t] & \\ - 2845.1607279812 \cos[5.3133948093t] & \\ + 8009.4374135866 \sin[1.3762803403t] & \\ - 60498.9564060210 \sin[1.7711316031t] & \\ + 76948.1979148196 \sin[2.3906769561t] & \\ - 44204.7821282308 \sin[3.3469477385t] & \\ + 3333.9079540144 \sin[3.7563397606t] & \\ + 13491.0951467330 \sin[4.1288410211t] & \\ - 7813.5442603879 \sin[5.2588756648t] & \\ + 6208.0968314566 \sin[5.3133948093t]) & \quad (33) \end{aligned}$$

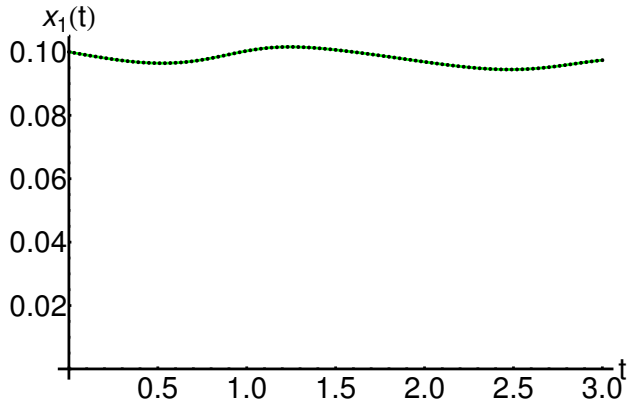
– for  $\bar{x}_5$  :

$$\begin{aligned} \bar{x}_5(t) = e^{-2.2781265526t} & \\ \cdot (0.1 + 5t + 666138.0110648051 \cos[1.3762803403t] & \\ - 934387.5696542863 \cos[1.7711316031t] & \\ + 20014.4603906560 \cos[2.3906769561t] & \\ + 1.0440 \cdot 10^6 \cos[3.3469477385t] & \\ - 1.1650 \cdot 10^6 \cos[3.7563397606t] & \\ + 366120.7836513140 \cos[4.1288410211t] & \\ + 52146.2063124158 \cos[5.2588756648t] & \\ - 49094.7503855498 \cos[5.3133948093t] & \\ + 154782.8235333784 \sin[1.3762803403t] & \\ - 860435.8794097108 \sin[1.7711316031t] & \\ + 987341.6977387308 \sin[2.3906769561t] & \\ - 397550.2105036524 \sin[3.3469477385t] & \\ - 178937.7585760594 \sin[3.7563397606t] & \\ + 259533.5676211552 \sin[4.1288410211t] & \\ - 116116.0139910701 \sin[5.2588756648t] & \\ + 92653.0757278590 \sin[5.3133948093t]) & \quad (34) \end{aligned}$$

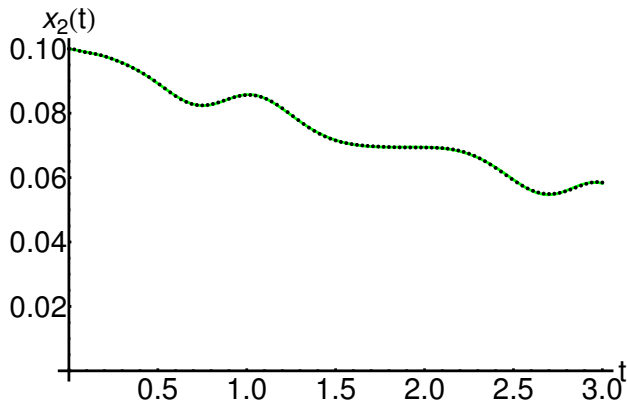
Finally, Tables 1 – 5 emphasizes the accuracy of the OHAM technique by comparing the approximate analytic solutions  $\bar{x}_i$ ,  $i = \overline{1, 5}$  presented above with the corresponding numerical integration values.

The Figures 1 – 5 present a comparison between the analytical approximate solutions obtained by OHAM

method and numerical results obtained by Runge-Kutta 4th steps integrator. We can see that the analytical approximate solutions offer us the same numerical results as Runge-Kutta 4th integrator.



**Figure 1:** Comparison between the approximate solutions  $\bar{x}_1$  given by Eq. (30) and the corresponding numerical solutions: — numerical solution, ..... OHAM solution.

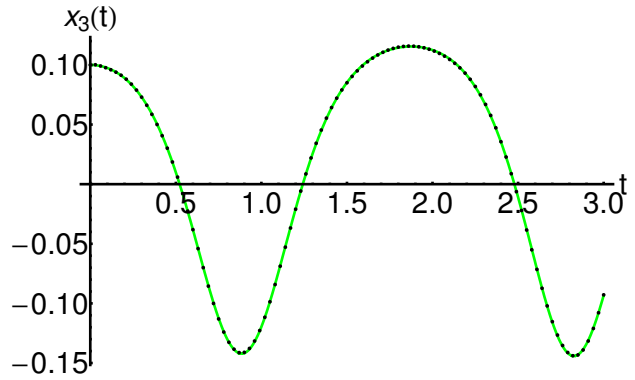


**Figure 2:** Comparison between the approximate solutions  $\bar{x}_2$  given by Eq. (31) and the corresponding numerical solutions: — numerical solution, ..... OHAM solution.

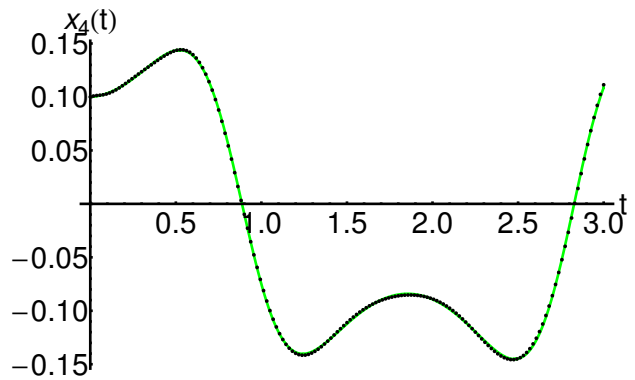
For the analytic approximate solutions given by Eqs. (30) - (34), the corresponding numerical value of the square residual are presented in Table 6.

## 6 Conclusion

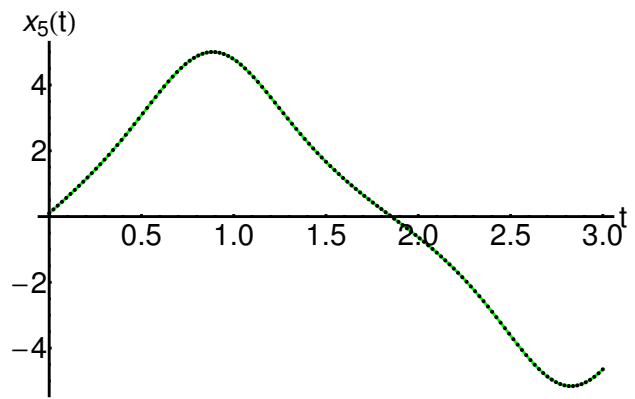
The main goal of this paper is to present some comments about the nonlinear stability and the periodic orbits



**Figure 3:** Comparison between the approximate solutions  $\bar{x}_3$  given by Eq. (32) and the corresponding numerical solutions: — numerical solution, ..... OHAM solution.



**Figure 4:** Comparison between the approximate solutions  $\bar{x}_4$  given by Eq. (33) and the corresponding numerical solutions: — numerical solution, ..... OHAM solution.



**Figure 5:** Comparison between the approximate solutions  $\bar{x}_5$  given by Eq. (34) and the corresponding numerical solutions: — numerical solution, ..... OHAM solution.

for some equilibrium points of a dynamical system. Its solutions are the optimal controls that minimize a cost function which steer a left-invariant drift-free control system on

**Table 1:** Comparison between OHAM results given by Eq. (30) and numerical results for  $a = 50$  (relative error:  $\varepsilon_{x_1} = \left| \bar{x}_{1\text{OHAM}} - x_{1\text{numeric}} \right|$ )

$t$	$x_{1\text{numeric}}$	$\bar{x}_{1\text{OHAM}}$ given by Eq. (30)	$\varepsilon_{x_1}$
0	0.1	0.1	0
3/10	0.0973010122	0.0972949303	$6.0819 \cdot 10^{-6}$
3/5	0.0965692103	0.0965689390	$2.7124 \cdot 10^{-7}$
9/10	0.0991855835	0.0991976718	0.0000120883
6/5	0.1015502519	0.1015493062	$9.4564 \cdot 10^{-7}$
3/2	0.1006520393	0.1006486235	$3.4157 \cdot 10^{-6}$
9/5	0.0984666978	0.0984700813	$3.3834 \cdot 10^{-6}$
21/10	0.0961160804	0.0961129077	$3.1726 \cdot 10^{-6}$
12/5	0.0945455752	0.0945487305	$3.1552 \cdot 10^{-6}$
27/10	0.0951860425	0.0951870172	$9.7470 \cdot 10^{-7}$
3	0.0973969241	0.0973980345	$1.1103 \cdot 10^{-6}$

**Table 2:** Comparison between OHAM results given by Eq. (31) and numerical results for  $a = 50$  (relative error:  $\varepsilon_{x_2} = \left| \bar{x}_{2\text{OHAM}} - x_{2\text{numeric}} \right|$ )

$t$	$x_{2\text{numeric}}$	$\bar{x}_{2\text{OHAM}}$ given by Eq. (31)	$\varepsilon_{x_2}$
0	0.1	0.1	0
3/10	0.0956818544	0.0955888648	0.0000929895
3/5	0.0854263073	0.0853278319	0.0000984753
9/10	0.0844198193	0.0844564095	0.0000365902
6/5	0.0813803193	0.0815382784	0.0001579590
3/2	0.0716411788	0.0715346587	0.0001065201
9/5	0.0694602859	0.0694944333	0.0000341473
21/10	0.0690947046	0.0691593470	0.0000646424
12/5	0.0631772363	0.0630046297	0.0001726066
27/10	0.0546946007	0.0549241257	0.0002295250
3	0.0582053904	0.0584816646	0.0002762741

**Table 3:** Comparison between OHAM results given by Eq. (32) and numerical results for  $a = 50$  (relative error:  $\varepsilon_{x_3} = \left| \bar{x}_{3\text{OHAM}} - x_{3\text{numeric}} \right|$ )

$t$	$x_{3\text{numeric}}$	$\bar{x}_{3\text{OHAM}}$ given by Eq. (32)	$\varepsilon_{x_3}$
0	0.1	0.1	0
3/10	0.0730628883	0.0729788225	0.0000840657
3/5	-0.0383195601	-0.0381175983	0.0002019617
9/10	-0.1416782781	-0.1410317691	0.0006465089
6/5	-0.0207374035	-0.0209948054	0.0002574018
3/2	0.0852662107	0.0850579261	0.0002082845
9/5	0.1144957877	0.1149014176	0.0004056299
21/10	0.1050522552	0.1045957626	0.0004564926
12/5	0.0353041031	0.0356142019	0.0003100987
27/10	-0.1157519839	-0.1153154206	0.0004365633
3	-0.0932578835	-0.0928098350	0.0004480484

the Schrödinger Lie group from an initial state to a final state (see [10]).

A lot of similar left-invariant drift-free control system on matrix Lie groups have been studied for instance in [6], [4], [1], [8].

**Table 4:** Comparison between OHAM results given by Eq. (33) and numerical results for  $a = 50$  (relative error:  $\varepsilon_{x_4} = \left| \bar{x}_{4\text{OHAM}} - x_{4\text{numeric}} \right|$ )

$t$	$x_{4\text{numeric}}$	$\bar{x}_{4\text{OHAM}}$ given by Eq. (33)	$\varepsilon_{x_4}$
0	0.1	0.1	0
3/10	0.1232652896	0.1236011513	0.0003358616
3/5	0.1385857661	0.1395600389	0.0009742728
9/10	-0.0094601354	-0.0098563956	0.0003962601
6/5	-0.1387713846	-0.1397827486	0.0010113640
3/2	-0.1122445225	-0.1110756747	0.0011688478
9/5	-0.0848234365	-0.0860525897	0.0012291532
21/10	-0.0986190756	-0.0975498076	0.0010692680
12/5	-0.1407648758	-0.1422493287	0.0014844528
27/10	-0.0868394736	-0.0858697722	0.0009697013
3	0.1087050948	0.1113663977	0.0026613028

**Table 5:** Comparison between OHAM results given by Eq. (34) and numerical results for  $a = 50$  (relative error:  $\varepsilon_{x_5} = \left| \bar{x}_{5\text{OHAM}} - x_{5\text{numeric}} \right|$ )

$t$	$x_{5\text{numeric}}$	$\bar{x}_{5\text{OHAM}}$ given by Eq. (34)	$\varepsilon_{x_5}$
0	0.1	0.1	0
3/10	1.7320338592	1.7342008834	0.0021670242
3/5	3.7978866255	3.7953347600	0.0025518655
9/10	5.0024384519	4.9983900602	0.0040483916
6/5	3.6279293668	3.6330997784	0.0051704116
3/2	1.6646925061	1.6610151733	0.0036773328
9/5	0.2287079437	0.2314404046	0.0027324609
21/10	-1.0961112512	-1.0968882365	0.0007769853
12/5	-2.8949929928	-2.8954177486	0.0004247557
27/10	-4.8662294199	-4.8621480187	0.0040814011
3	-4.6466195413	-4.6458791721	0.0007403691

**Table 6:** Numerical value of the square residual  $\int_0^{t_{\max}} R_i^2(t, B_j, C_j) dt, i = \overline{1, 5}$  given by Eq. (29) for  $a = 50$

$\int_0^{t_{\max}} R_1^2(t, B_j, C_j) dt$	$6.442716947059657 \cdot 10^{-8}$
$\int_0^{t_{\max}} R_2^2(t, B_j, C_j) dt$	0.00001343436774938075
$\int_0^{t_{\max}} R_3^2(t, B_j, C_j) dt$	0.00005505233000649044
$\int_0^{t_{\max}} R_4^2(t, B_j, C_j) dt$	0.0005461686827488749
$\int_0^{t_{\max}} R_5^2(t, B_j, C_j) dt$	0.013216857418541084

In the last section the approximate analytic solutions of the considered controlled system (1) are established using the optimal homotopy asymptotic method (OHAM). Both numerical simulations via Mathematica 6.0 software and approximations deviations were presented. The accuracy of our results is pointed out by means of the approximate residual of the solutions. We can see that the proposed method, even if it does not use a Poisson integrator, offers better results than the Lie-Trotter integrator used in [10].

**Acknowledgement:** We would like to thank the referees very much for their valuable comments and suggestions.

**Conflict of Interests:** The authors declare that there is no conflict of interests regarding the publication of this paper.

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