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Regarding on the exact solutions for the nonlinear fractional differential equations

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Abstract: In this work, we have considered the modified simple equation (MSE) method for obtaining exact solutions of nonlinear fractional-order differential equations. The space-time fractional equal width (EW) and the modified equal width (mEW) equation are considered for illustrating the effectiveness of the algorithm. It has been observed that all exact solutions obtained in this paper verify the nonlinear ordinary differential equations which was obtained from nonlinear fractional-order differential equations under the terms of wave transformation relationship. The obtained results are shown graphically.

Keywords: Nonlinear fractional differential equation; the modified Riemann-Liouville derivative; the modified simple equation method; exact solution

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1 Introduction

Fractional calculus, was introduced by Leibniz in 1695, as a generalization of ordinary calculus [1]. Physicists, engineers and mathematicians have always taken intense interest in nonlinear problems related to various scientific applications, such as fluid dynamics, plasma physics, high-energy physics, geochemistry, chemical physics, viscoelastic materials, elastic media, optical fibers, signal processing, polymers, chemical kinematics and biomathematics [2–4]. The most considerable advantage of utilizing nonlinear fractional differential equations (NFDEs) is their non-local property, which means that the next status of a system depends both upon its current status and all of its historical status. For instance, by using the fractional differential equations, the fluid-dynamic traffic model can eliminate the deficiency arising from the assumptions of continuum traffic flow [5]. Considering their importance, the exploration of exact solutions of NFDEs became a crucial case and matter of interest for researchers in recent years. As a result, numerous influential methods have been proposed. Some of these include as the tanh-sech method [6], the (G'/G)-expansion method [7], the $(G'/G, 1/G)$-expansion method [8], the first integral method [9], the exponential function method [10, 11], the sub equation method [12], the trial equation method [13], ansatz method [14], the modified Kudryashov method [15], variational iteration method [16] and others [17–19].

There are different approaches to the generalization of the notion of differentiation to fractional orders such as Caputo, Grünwald-Letnikow, Riemann-Liouville and the Generalized Functions approach [20, 21]. Among these approaches, the modified Riemann-Liouville derivative is mostly used by mathematicians.

2 The Modified Simple Equation Method

We consider a NFDE with independent variables x and t given by:

$$F(u, D_t^\alpha u, D_x^\alpha u, D_t^\alpha D_t^\alpha u, D_t^\alpha D_x^\alpha u, D_x^\alpha D_x^\alpha u, \dots) = 0, \quad 0 < \alpha < 1, \tag{1}$$

where $u(x, t)$ is an unknown function, F is a polynomial in u and various partial fractional derivatives in which the highest order derivatives and nonlinear terms are involved.

To find the exact solution of Eq. (1), we introduce a fractional complex transformationn as follows:

$$u(x, t) = u(\xi), \xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}. \tag{2}$$

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Applying Eq. (2), Eq. (1) can be converted to a nonlinear ordinary differential equation (ODE). When integrated as many times as possible, the integration constant can be selected as zero.

The exact solution of the reduced equation can be represented by a polynomial in $\frac{\zeta'(\xi)}{\zeta(\xi)}$ as follows;

$$u(\xi) = \sum_{n=0}^{m} a_n \left[\frac{\zeta'(\xi)}{\zeta(\xi)}\right]^n, \tag{3}$$

where $a_n, (n = 0, 1, 2, \dots, m), (a_m \neq 0)$ are unknown constants and $\zeta(\xi)$ is an unknown function that needs to be found.

Here the positive integer m (the balancing number) can be calculated by linking the homogeneous balance principle with the highest order derivative term and the highest order nonlinear term which appears in the reduced equation.

By substituting Eq. (3) into the reduced equation, we obtain a polynomial of $\zeta^{-j}(\xi)$ which also includes the derivatives of $\zeta(\xi)$. By equating all the coefficients of $\zeta^{-j}(\xi)$ to zero ($j \geq 0$), we obtain an algebraic equation system which can be solved for a_n $(n = 0, 1, 2, \dots, m)$, c and $\zeta(\xi)$. Finally substitution of a_n, c and $\zeta(\xi)$ into Eq. (3) provides the specification of the exact solution of Eq. (1)

In the other expansion methods, such as the tanh-function method, the exp-function method, the $(\frac{G'}{G})$-expansion method, the auxiliary function method, the exact solutions are expressed in terms of some pre-defined functions. However some fresh solutions can be found with the usage of the MSE method, since φ is not a pre-defined function nor is it a solution of any pre-defined equation. This is the main advantage of the proposed method [22, 23].

3 Applications

In the current section, the exact solutions of the space-time fractional EW equation and the space-time fractional modified EW equation NPDEs are obtained by using the MSE method.

3.1 The space-time fractional EW equation

We first consider the space-time fractional EW equation in the form:

$$D_t^{\alpha}u(x,t) + \varepsilon D_x^{\alpha}u^2(x,t) - \delta D_{xxt}^{3\alpha}u(x,t) = 0, \tag{4}$$

where ε and δ are positive parameters. This equation is used to model nonlinear dispersive waves [24]. Employing the transformation (2), Eq. (4) reduced to an ODE as follows:

$$-cu' + \varepsilon k(u^2)' + \delta ck^2 u''' = 0. \tag{5}$$

Integrating (5) once and taking the constant of the integration as zero, it becomes:

$$-cu + \varepsilon ku^2 + \delta ck^2 u'' = 0 \tag{6}$$

Balancing the highest order derivative term u'' with the highest order nonlinear term u^2, the balancing number is determined to be $m = 2$. Then assume the exact solution of the Eq. (6) as:

$$u(\xi) = a_0 + a_1\left(\frac{\zeta'(\xi)}{\zeta(\xi)}\right) + a_2\left(\frac{\zeta'(\xi)}{\zeta(\xi)}\right)^2. \tag{7}$$

We obtain an algebraic equation system which follows by substituting Eq. (7) into Eq. (6) and equating all the coefficients of $\zeta^{-j}(\xi), (j = 0, 1, \dots, 4)$ to zero:

$$\begin{aligned}
\zeta^0(\xi) &: \varepsilon ka_0^2 - ca_0 = 0, \\
\zeta^1(\xi) &: ck^2a_1\delta\zeta''' + 2\varepsilon ka_0a_1\zeta' - ca_1\zeta' = 0, \\
\zeta^2(\xi) &: 2ck^2\delta a_2\zeta'\zeta''' + 2\varepsilon ka_0a_2(\zeta')^2 + \varepsilon ka_1^2(\zeta')^2 \\
&\quad + 2ck^2\delta a_2(\zeta'')^2 - 3ck^2\delta a_1\zeta''\zeta' \\
&\quad - ca_2(\zeta')^2 = 0, \\
\zeta^3(\xi) &: 2\varepsilon ka_1(\zeta')^3a_2 - 10ck^2\delta a_2(\zeta')^2\zeta'' \\
&\quad + 2ck^2\delta a_1(\zeta')^3 = 0, \\
\zeta^4(\xi) &: \varepsilon ka_2^2(\zeta')^4 + 6ck^2\delta a_2(\zeta')^4 = 0.
\end{aligned} \tag{8}$$

From the first and last equations of the system above we obtain for a_0 and a_2 :

$$a_0 = \frac{c}{\varepsilon k}, \ a_2 = \frac{6ck\delta}{\varepsilon}. \tag{9}$$

Thereafter by substituting Eq. (9) into the remaining equations of the Eq. system (8), we find

$$\begin{aligned}
\zeta^1(\xi) &: ck^2\delta a_1\zeta''' + ca_1\zeta' = 0, \\
\zeta^2(\xi) &: -\frac{12c^2k^3\delta^2\zeta'\zeta'''}{\varepsilon} - \frac{6kc^2\delta(\zeta')^2}{\varepsilon} + \varepsilon ka_1^2(\zeta')^2 \\
&\quad - \frac{12c^2k^3\delta^2(\zeta'')^2}{\varepsilon} - 3ck^2\delta a_1\zeta''\zeta' = 0, \\
\zeta^3(\xi) &: -10ck^2\delta a_1(\zeta')^3 + \frac{60c^2k^3\delta^2(\zeta')^2\zeta''}{\varepsilon} = 0.
\end{aligned} \tag{10}$$

It follows from the last equation of the system above that,

$$a_1 = \frac{6ck\delta\zeta''}{\varepsilon\zeta'}. \tag{11}$$

By substituting Eq. (11) into the remaing equations of the Eq. system (10), an ODE system results. Next, the solution of this system for $\zeta(\xi)$ to show that:

$$\zeta(\xi) = C_1 + C_2e^{\pm\frac{\xi}{\sqrt{-\delta k}}}, \tag{12}$$

where C_1 and C_2 are arbitrary constants. Finally, we substitute Eq. (12) into Eq. (7) and the following exact solutions of the space-time fractional EW equation resulted:

$$u(\xi) = \frac{c}{\varepsilon k}\left(1 - \frac{6\left(C_2 \cosh\left(\frac{\xi}{\sqrt{-\delta k}}\right) \pm \sinh\left(\frac{\xi}{\sqrt{-\delta k}}\right)\right)}{\left(C_1 + C_2\left(\cosh\left(\frac{\xi}{\sqrt{-\delta k}}\right) \pm \sinh\left(\frac{\xi}{\sqrt{-\delta k}}\right)\right)\right)} + \frac{6\left(C_2\left(\cosh\left(\frac{\xi}{\sqrt{-\delta k}}\right) \pm \sinh\left(\frac{\xi}{\sqrt{-\delta k}}\right)\right)^2\right)}{\left(C_1 + C_2\left(\cosh\left(\frac{\xi}{\sqrt{-\delta k}}\right) \pm \sinh\left(\frac{\xi}{\sqrt{-\delta k}}\right)\right)\right)^2}\right).$$

Here $\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}$. The figures of the obtained solutions are generated by setting special values for the parameters as follows:

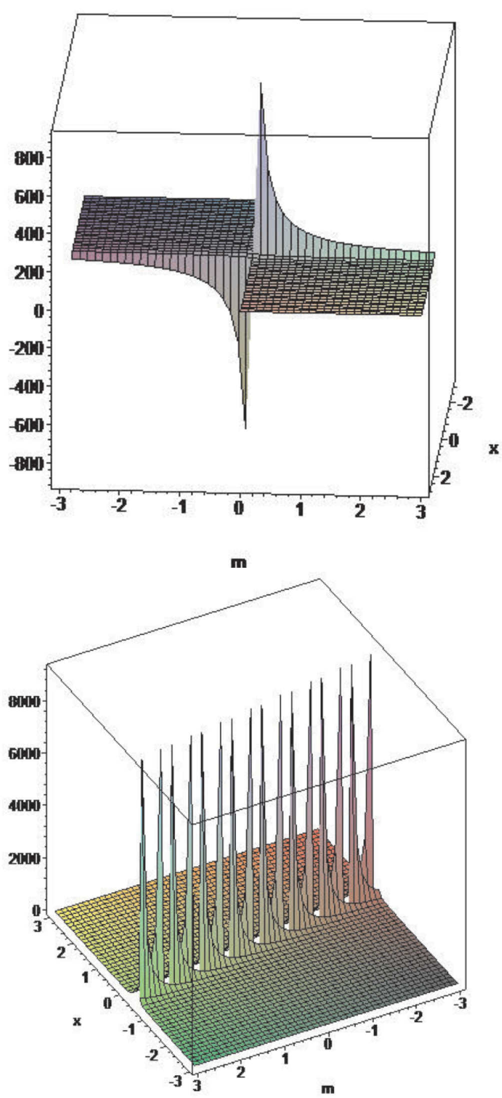

Figure 1: The exact solution for the EW equation with $\alpha = 0.5$ and $\alpha = 1$ respectively when $\delta = 2$, $c = 0.5$, $\varepsilon = 0.5$, $C_1 = 1$, $C_2 = -1$ and $k = -3$.

3.2 The space-time fractional modified EW equation

The space-time fractional modified equal width equation (mEW) can be written in the following form [24]:

$$D_t^\alpha u(x,t) + \varepsilon D_x^\alpha u^3(x,t) - \delta D_{xxt}^{3\alpha} u(x,t) = 0. \quad (13)$$

Here ε and δ are positive parameters. We perform the transformation (2) and reduce Eq. (13) to an ODE:

$$-cu' + \varepsilon k(u^3)' + \delta ck^2 u''' = 0. \quad (14)$$

Negleting the constant of integration, we integrate Eq. (14) once with respect to ξ and thus obtain:

$$-cu + \varepsilon ku^3 + \delta ck^2 u'' = 0. \quad (15)$$

Balancing the highest order derivative terms and nonlinear terms in Eq.(15), the balancing number as $m = 1$. According to MSE method, the exact solution turns into

$$u(\xi) = a_0 + a_1 \frac{\zeta'(\xi)}{\zeta(\xi)}. \quad (16)$$

Here after the following substitutions are made:

$$\begin{aligned}
\zeta^0(\xi) &: \quad \varepsilon k a_0^3 - c a_0 = 0, && (17)\\
\zeta^1(\xi) &: \quad \delta ck^2 a_1 \zeta''' - c a_1 \zeta' + 3\varepsilon k a_0^2 a_1 \zeta' = 0, &&\\
\zeta^2(\xi) &: \quad -3\delta ck^2 a_1 \zeta''\zeta' + 3\varepsilon k a_0 a_1^2 (\zeta')^2 = 0, &&\\
\zeta^3(\xi) &: \quad 2\delta ck^2 a_1 + \varepsilon k a_1^3 (\zeta')^3 = 0. &&
\end{aligned}$$

From the solution of the first and the last equation of the above system, we find the values of a_0 and a_1 as follows

$$a_0 = \pm\sqrt{\frac{c}{\varepsilon k}}, \quad a_1 = \pm\sqrt{-\frac{2\delta ck}{e}}$$

By substituting these valuse into the second equation of Eq. system (17), an ordinary differential equation system is obtained. From the solutions of this system

$$\zeta(\xi) = C_1 + C_2 e^{\pm\sqrt{-\frac{2}{\delta}\frac{\xi}{k}}},$$

where C_1 and C_2 are arbitrary constants. Subsequently, we have the following exact solutions of the space-time fractional modified equal width equation

$$u(\xi) = \mp \frac{\sqrt{c}\left(-C_1 + C_2\left(\cosh\left(\sqrt{-\frac{2}{\delta}\frac{\xi}{k}}\right) + \sinh\left(\sqrt{-\frac{2}{\delta}\frac{\xi}{k}}\right)\right)\right)}{\sqrt{\varepsilon k}\left(C_1 + C_2\left(\cosh\left(\sqrt{-\frac{2}{\delta}\frac{\xi}{k}}\right) + \sinh\left(\sqrt{-\frac{2}{\delta}\frac{\xi}{k}}\right)\right)\right)},$$

where $\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}$. We represent graphically the obtained exact solutions of the modified EW equation by setting special values for the parameters as follows:

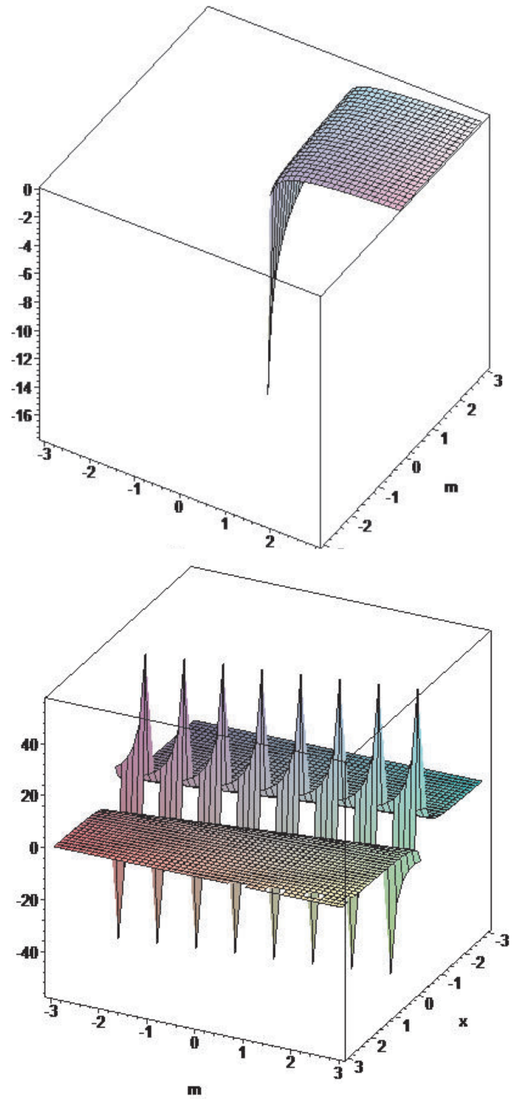

Figure 2: The exact solution for the modified EW equation with $\alpha = 0.5$ and $\alpha = 1$ respectively when $\delta = 2$, $c = 0.5$, $\varepsilon = 0.5$, $C_1 = 1$, $C_2 = -1$ and $k = -3$.

4 Conclusions

In the current paper, the MSE method has been successfully employed to obtain exact solutions of the space-time fractional EW equation and the modified EW equation. The generated solutions serve to illustrate several new features of waves and can be more useful in theoretical and numerical studies of the considered equation. Moreover, the solutions which are obtained in this work are different from the ones which are given in [24]. To the best of our knowledge, symbolic computation systems (such as MATLAB, MAPLE and MATHEMATICA) played a critical role in the computations. The modified simple equation method is a reliable and effective method. Also it can be applied to many other NFDEs and NFDE systems appearing in mathematical physics and nonlinear sciences.

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