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Energy and Wave function Analysis on Harmonic Oscillator Under Simultaneous Non-Hermitian Transformations of Co-ordinate and Momentum: Iso-spectral case

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Abstract: We present a complete energy and wavefunction analysis of a Harmonic oscillator with simultaneous non-hermitian transformations of co-ordinate ($x \rightarrow \frac{(x+i\lambda p)}{\sqrt{(1+\beta\lambda)}}$) and momentum ($p \rightarrow \frac{(p+i\beta x)}{\sqrt{(1+\beta\lambda)}}$) using perturbation theory under iso-spectral conditions. We observe that two different frequencies of oscillation (w_1, w_2)correspond to the same energy eigenvalue, - which can also be verified using a Lie algebraic approach.

Keywords: Non-Hermitian Harmonic oscillator; Perturbation theory; Wavefunction; iso-spectrum

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1 Introduction

In physics the generation of a new Hamiltonian typically relates to a transformation of co-ordinate

$$\bar{x} = \bar{x}(x, p) \tag{1}$$

or momentum

$$\bar{p} = \bar{p}(x, p) \tag{2}$$

or both. Here x, p are related to the original Hamiltonian:$H(x, p)$ and $\bar{x}, \bar{p}$ are related to the new Hamiltonian $H(\bar{x}, \bar{p}$. This type of transformation is well known in classical physics (canonical transformation) [1]. However in quantum physics one has to be careful about the commutation relation: as the commutation relation invariance results in iso-spectral behaviour of the Hamiltonian [2]. Mathematically

$$[x, p] = [\bar{x}, \bar{p}] = i\frac{h}{2\pi} = i\bar{h} \tag{3}$$

where h is Plank's constant. Of interest in this commutation relation is that one can have simultaneous transformations of co-ordinate and momentum. First the Hamiltonian must be written either in the momentum dimension or co-ordinate dimension. So momentum has to be defined in such a way that from the dimension point of view both co-ordinate and momentum are equally acceptable. In order to give an example of this we consider an exactly solvable model. The widely used exactly solvable model is the Harmonic Oscillator(HO) [1, 2], abd this plays a major role in understanding the limitations of various approximation methods such as the variational method, W.K.B. method, perturbation method, etc. Hence, for simplicity, we address the above commutation relation using HO as an example [2]. The Hamiltonian in old co-ordinate and momentum is written as

$$H = \frac{p^2}{2m} + \frac{w_0^2 m x^2}{2} \tag{4}$$

Here, m is mass and w_0 stands for the frequency of oscillation. The above expression can be written as a momentum base relation as

$$mH = \frac{p^2}{2} + \frac{w_0^2 m^2 x^2}{2} \tag{5}$$

This implies that the dimension of p remains the same as that of mw_0x. Here one can introduce new momentum as [3]

$$\bar{p} \rightarrow p + i\beta m w_0 x \tag{6}$$

where β is a simple numerical constant which can be varied arbitrarily in order to generate a large number of systems. Similarly one can write the Hamiltonian of the Harmonic Oscillator in co-ordinate base as

$$\frac{H}{mw_0^2} = \frac{p^2}{2m^2 w_0^2} + \frac{x^2}{2} \tag{7}$$

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In this case one can introduce a new co-ordinate $\bar{x}$ as

$$\bar{x} \to x + i\lambda \frac{p}{mw_0} \tag{8}$$

It is seen that the commutation relation is not invariant . However in the later part of the study we will consider its appropriate form [4–6]. Above, λ is a dimensionss parameter like β, which allows a large class of Hamiltonians to be generated using different numerical values. It is clear from the above two transformations that λ, β are dimensionless numbers. Further introducing" i " (complex factor) in the above transformation allows us to study the HO in complex space [4–6]. This is relevant due to the recent experiment in parity (P) - time (T) meta material [7]. Mathematically, the complex nature of transformation means that it must satisfy the following PT condition [8]

$$PT\bar{x}(PT)^{-1} = -\bar{x} \tag{9}$$

$$PT\bar{p}(PT)^{-1} = -\bar{p} \tag{10}$$

So that the product $\bar{x}\bar{p}$ appearing in above commutation relation becomes PT invariant. Here P stands for parity transformation i.e under P ; $x \to -x; p \to -p$. Similarly under time reversal T ; $x \to x; p \to -p$; $i \to -i$. In other words one is likely to study the new Hamiltonian under complex transformation i.e under the PT transformations. We also note that when the co-ordinate and momentum simultaneously under go non-Hermitian transformations [4–6], its energy eigenvalue can be iso-spectral to the original Harmonic oscillator. Under iso-spectral conditions, the wavefunction of the transformed Hamiltonian differs drastically from that of the original Hamiltonian [4–6]. However, a complete picture on wavefunction is still in need of further study. Further, no such explicit calculations on wavefunctions are available at present [4–6]. Hence the aim of this paper is to present a complete picture of wavefunction and energy under iso-spectral conditions using perturbation theory under the PT transformation as discussed above.

2 Energy levels and Wavefunction of Simple Harmonic Oscillator (SHO)

We consider the case $m = w_0 = \bar{h} = 1$ and write the Hamiltonian of SHO [1, 2] as

$$H_{HO} = \frac{p^2}{2} + \frac{x^2}{2} \tag{11}$$

whose exact energy eigenvalues is [2]

$$E_n = \left(n + \frac{1}{2}\right) \tag{12}$$

and corresponding wavefunction is [2]

$$\psi_n = \sqrt{\frac{1}{\sqrt{\pi}2^n n!}} H_n(x) e^{-\frac{x^2}{2}} \tag{13}$$

where $H_n(x)$ is the Hermite polynomial.

3 SHO under non-Hermitian transformation of co-ordinate (x) and momentum (p)

Consider the non-Hermitian transformations of x and p as [4–6]

$$\bar{x} = x \to \frac{x + i\lambda p}{\sqrt{(1 + \beta\lambda)}} \tag{14}$$

and

$$\bar{p} = p \to \frac{p + i\beta x}{\sqrt{(1 + \beta\lambda)}} \tag{15}$$

In this transformation we note that the transformed co-ordinate and momentum preserve the commutation relation [4–6] i.e.

$$[x, p] = [\bar{x}, \bar{p}] = i \tag{16}$$

Now the new Hamiltonian with transformed x and p becomes non-Hermitian in nature and is

$$H = \frac{(p + i\beta x)^2}{2(1 + \lambda\beta)} + \frac{(x + i\lambda p)^2}{2(1 + \lambda\beta)} \tag{17}$$

4 Second Quantization and Hamiltonian

In order to solve the above Hamiltonian, we use the second quantization formalism as [2]

$$x = \frac{(a + a^+)}{\sqrt{2\omega}} \tag{18}$$

and

$$p = i\sqrt{\frac{\omega}{2}}(a^+ - a) \tag{19}$$

where the creation operator, a^+ and annhilation operator a satisfy the commutation relation

$$[a, a^+] = 1 \tag{20}$$

and ω is an unknown parameter. The Hamiltonian can be written as

$$H = H_D + H_N \tag{21}$$

where

$$H_D = \left[(1-\lambda^2)\omega + \frac{(1-\beta^2)}{\omega}\right]\frac{(2a^+a+1)}{4(1+\lambda\beta)} \tag{22}$$

and

$$H_N = U\frac{a^2}{4(1+\lambda\beta)} + V\frac{(a^+)^2}{4(1+\lambda\beta)} \tag{23}$$

$$V = \left[-\omega(1-\lambda^2) + \frac{(1-\beta^2)}{\omega} - 2(\lambda+\beta)\right] \tag{24}$$

$$U = \left[-\omega(1-\lambda^2) + \frac{(1-\beta^2)}{\omega} + 2(\lambda+\beta)\right] \tag{25}$$

4.1 Zero Energy Correction Method (Case Study for U=0)

Now we solve the the eigenvalue relation:

$$H\Psi_n(x) = \epsilon_n\Psi_n(x) \tag{26}$$

using perturbation theory as follows. Here we express

$$\epsilon_n = \epsilon_n^{(0)} + \sum_{m=1}^{k}\epsilon_n^{(m)} \tag{27}$$

The zeroth order energy $\epsilon_n^{(0)}$ satisfies the the following eigenvalue relation

$$H_D|\psi_n\rangle = H_D|n\rangle = \epsilon_n^{(0)}|n\rangle \tag{28}$$

where $\psi_n^{(0)}$ is the zeroth order wave function and $\epsilon_n^{(m)}$ is the mth order perturbation correction.

$$\epsilon_n^{(0)} = \frac{(2n+1)}{4(1+\lambda\beta)}\left[(1-\lambda^2)\omega + \frac{(1-\beta^2)}{\omega}\right] \tag{29}$$

and

$$\sum_{m=1}^{k}\epsilon_n^{(m)} = \epsilon_n^{(1)} + \epsilon_n^{(2)} + \epsilon_n^{(3)} + \dots \tag{30}$$

The energy correction terms will give zero contribution if the parameter is determined from non-diagonal terms of H_N [4]

Let the coefficient of a^2 be zero [4] i.e.

$$U = [-\omega(1-\lambda^2) + \frac{(1-\beta^2)}{\omega} + 2(\lambda+\beta)] = 0 \tag{31}$$

which leads to(considering positive sign)

$$\omega = \omega_1 = \frac{(1+\beta)}{(1-\lambda)} \tag{32}$$

In this case,

$$\epsilon_n^{(0)} = \left(n+\frac{1}{2}\right) \tag{33}$$

Now the perturbation correction term is

$$H_N = V\frac{(a^+)^2}{4(1+\lambda\beta)} = -\frac{(\lambda+\beta)}{1+\lambda\beta}(a^+)^2 \tag{34}$$

In this case one will notice that

$$\langle n|H_N|n-2\rangle = V\frac{\sqrt{n(n-1)}}{4(1+\lambda\beta)} \tag{35}$$

$$\langle n-2|H_N|n\rangle = 0 \tag{36}$$

Hence it follows that all orders of energy corrections will be zero. Let us consider explicitly corrections up to third order using a standard perturbation series given in literature [2, 4, 9–14], which can be written as

$$\epsilon_n^{(1)} = \langle\psi_n|H_N|\psi_n\rangle = 0 \tag{37}$$

$$\begin{aligned}\epsilon_n^{(2)} &= \sum_{k\neq n}\frac{\langle\psi_n|H_N|\psi_k\rangle\langle\psi_k|H_N|\psi_n\rangle}{(\epsilon_n^{(0)}-\epsilon_k^{(0)})} \\ &= \frac{\langle\psi_n|H_N|\psi_{n+2}\rangle\langle\psi_{n+2}|H_N|\psi_n\rangle}{(\epsilon_n^{(0)}-\epsilon_{n+2}^{(0)})} = 0\end{aligned} \tag{38}$$

$$\epsilon_n^{(3)} = \sum_{p,q}\frac{\langle\psi_n|H_N|\psi_p\rangle\langle\psi_p|H_N|\psi_q\rangle\langle\psi_q|H_N|\psi_n\rangle}{(\epsilon_n^{(0)}-\epsilon_p^{(0)})(\epsilon_n^{(0)}-\epsilon_q^{(0)})} = 0 \tag{39}$$

or

$$\epsilon_n^{(3)} = \frac{\langle\psi_n|H_N|\psi_{n+2}\rangle\langle\psi_{n+2}|H_N|\psi_{n+4}\rangle\langle\psi_{n+4}|H_N|\psi_n\rangle}{(\epsilon_n^{(0)}-\epsilon_{n+2}^{(0)})(\epsilon_n^{(0)}-\epsilon_{n+4}^{(0)})} = 0 \tag{40}$$

Here second order correction is zero due to $\langle\psi_n|H_N|\psi_{n+2}\rangle = \delta_{n,n+4}$ and third order correction is zero due to $\langle\psi_{n+4}|H_N|\psi_n\rangle = \delta_{n+4,n+2}$. Similarly we note that all correction terms $\epsilon_n^{(m)}$ will be zero. Hence

$$\epsilon_n = \epsilon_n^{(0)} = E_n^{(0)} = \left(n+\frac{1}{2}\right) \tag{41}$$

$$|\psi_n\rangle = \left(\frac{\sqrt{\omega_1}}{\sqrt{\pi}2^n n!}\right)^{\frac{1}{2}} H_n(\sqrt{\omega_1}x)e^{-\omega_1\frac{x^2}{2}} \tag{42}$$

with

$$\langle\psi_n|\psi_n\rangle = 1 \tag{43}$$

Form the above, analysis it is seen that the total energy of the Harmonic oscillator and the Harmonic oscillator with non-hermitian transformation remains the same.

4.2 Corresponding Wavefunction using Perturbation Theory

Here we find the wavefunction as

$$\begin{aligned}\Psi_n^{(k)} =&|\psi_n\rangle + f_{\lambda,\beta}\frac{\sqrt{(n+2)!}}{2\sqrt{n!}}|\psi_{n+2}\rangle \\ &+ (f_{\lambda,\beta})^2\frac{\sqrt{(n+4)!}}{8\sqrt{n!}}|\psi_{n+4}\rangle \\ &+ (f_{\lambda,\beta})^3\frac{\sqrt{(n+6)!}}{48\sqrt{n!}}|\psi_{n+6}\rangle + \ldots \end{aligned} \tag{44}$$

where $f_{\lambda,\beta} = \frac{(\lambda+\beta)}{(1+\lambda\beta)}$. In its compact form one can write,

$$\Psi_n^{(k)} = \sum_{k=0}\left[\frac{(\lambda+\beta)}{(1+\lambda\beta)}\right]^k\sqrt{\frac{(n+2k)!}{n!}}|\psi_{n+2k}\rangle_{\omega_1} \tag{45}$$

The normalization condition here can be written as [9–12]

$$\langle\psi_n|\Psi_n^{(k)}\rangle = 1 \tag{46}$$

and so also the eigenvalue relation

$$\langle\psi_n|H|\Psi_n^{(k)}\rangle = E_n = \left(n+\frac{1}{2}\right) \tag{47}$$

4.3 Zero Energy Correction Method (Case Study for V=0)

Let the coefficient of $(a^+)^2$ be zero [4] i.e.

$$V = [-\omega(1-\lambda^2) + \frac{(1-\beta^2)}{\omega} - 2(\lambda+\beta)] = 0 \tag{48}$$

which leads to

$$\omega = \omega_2 = \frac{(1-\beta)}{(1+\lambda)} \tag{49}$$

In this case, ω is calculated using similarity transformation [6] and remains the same as ω_2. Now the perturbation term becomes

$$H_N = U\frac{a^2}{4(1+\lambda\beta)} = \frac{(\lambda+\beta)}{1+\lambda\beta}a^2 \tag{50}$$

In this case we note that

$$\langle\phi_n|H_N|\phi_{n+2}\rangle = U\frac{\sqrt{[(n+1)(n+2)]}}{4(1+\lambda\beta)} \tag{51}$$

$$\langle\phi_{n+2}|H_N|\phi_n\rangle = 0 \tag{52}$$

Hence [9–14]

$$\epsilon_n^{(1)} = \langle\phi_n|H_N|\phi_n\rangle = 0 \tag{53}$$

$$\begin{aligned}\epsilon_n^{(2)} &= \sum_{k\neq n}\frac{\langle\phi_n|H_N|\phi_k\rangle\langle\phi_k|H_N|\phi_n\rangle}{(\epsilon_n^{(0)}-\epsilon_k^{(0)})} \\ &= \frac{\langle\phi_n|H_N|\phi_{n-2}\rangle\langle\phi_{n-2}|H_N|\phi_n\rangle}{(\epsilon_n^{(0)}-\epsilon_{n-2}^{(0)})} = 0\end{aligned} \tag{54}$$

$$\epsilon_n^{(3)} = \sum_{p,q}\frac{\langle\phi_n|H_N|\phi_p\rangle\langle\phi_p|H_N|\phi_q\rangle\langle\phi_q|H_N|\phi_n\rangle}{(\epsilon_n^{(0)}-\epsilon_p^{(0)})(\epsilon_n^{(0)}-\epsilon_q^{(0)})} = 0 \tag{55}$$

or

$$\epsilon_n^{(3)} = \frac{\langle\phi_n|H_N|\phi_{n-2}\rangle\langle\phi_{n-2}|H_N|\phi_{n-4}\rangle\langle\phi_{n-4}|H_N|\phi_n\rangle}{(\epsilon_n^{(0)}-\epsilon_{n-2}^{(0)})(\epsilon_n^{(0)}-\epsilon_{n-4}^{(0)})} = 0 \tag{56}$$

Here second order correction is zero due to $\langle\phi_n|H_N|\phi_{n-2}\rangle = \delta_{n,n-2}$ and third order correction is zero due to $\langle\phi_{n-4}|H_N|\phi_n\rangle = \delta_{n-2,n-4}$. Similarly one can notice all correction terms $\epsilon_n^{(m)}$ will be zero. Hence

$$\epsilon_n = \epsilon_n^{(0)} = E_n^{(0)} = \left(n+\frac{1}{2}\right) \tag{57}$$

which is the same as the energy level of harmonic oscillator as given in Eq. (2) and

$$|\phi_n\rangle = (\frac{\sqrt{\omega_2}}{\sqrt{\pi}2^n n!})^{\frac{1}{2}}H_n(\sqrt{\omega_2}x)e^{-\omega_2\frac{x^2}{2}} \tag{58}$$

4.4 Corresponding Wavefunction using Perturbation Theory

Here we consider the wavefunction as

$$\begin{aligned}\Phi_n^{(k)} =&|\phi_n\rangle + f_{\lambda,\beta}\frac{\sqrt{n!}}{2\sqrt{(n-2)!}}|\phi_{n-2}\rangle \\ &+ (f_{\lambda,\beta})^2\frac{\sqrt{n!}}{8\sqrt{(n-4)!}}|\phi_{n-4}\rangle \\ &+ (f_{\lambda,\beta})^3\frac{\sqrt{n!}}{48\sqrt{(n-6)!}}|\phi_{n-6}\rangle + \ldots \end{aligned} \tag{59}$$

In its compact form

$$\Phi_n^{(k)} = \sum_{k=0}(\frac{(\lambda+\beta)}{(1+\lambda\beta)})^k\frac{\sqrt{n!}}{\sqrt{(n-2k)!}2^k k!}|\phi_{n-2k}\rangle_{\omega_2} \tag{60}$$

Here we note that for $x\to\infty$ i.e.

$$\phi_n(x\to\infty)\to 0 \tag{61}$$

and

$$\Phi_n^{(k)}(x\to\infty)\to 0 \tag{62}$$

In this case, the normalization condition can be written as [9–12]

$$\langle\phi_n|\Phi_n^{(k)}\rangle = 1 \tag{63}$$

and so also the eigenvalue relation

$$\langle\phi_n|H|\Phi_n^{(k)}\rangle = E_n = \left(n+\frac{1}{2}\right) \tag{64}$$

5 Comparision with Similarity Transformation using Lie-algebra [5]

In the above we note that two different frequencies(w_1, w_2)corresponds to the same energy eigenvalue. Now we compare our results with that of Zhang et al. [5] using Lie-algebra as follows.Previous authors consider the Hamiltonian

$$H = s_0\left(a^+a + \frac{1}{2}\right) + s_1(a^+)^2 + s_2a^2 + s_3a^+ + s_4a \quad (65)$$

having energy eigenvalue

$$\epsilon_n = \sqrt{s_0^2 - 4s_1s_2}\left(n + \frac{1}{2}\right) + \frac{s_2s_3^2 + s_1s_4^2 - s_0s_3s_4}{s_0^2 - 4s_0s_1s_2} \quad (66)$$

Case-I $w = w_1 = \frac{1+\beta}{1-\lambda}$.

In this case we have the Hamiltonian

$$H = \left(a^+a + \frac{1}{2}\right) - \frac{(\lambda+\beta)}{2(1+\lambda\beta)}(a^+)^2 \quad (67)$$

Now comparing we get

$$s_0 = 1 \quad (68)$$

$$s_1 = -\frac{(\lambda+\beta)}{2(1+\lambda\beta)} \quad (69)$$

$$s_2 = 0 \quad (70)$$

$$s_3 = 0 \quad (71)$$

$$s_4 = 0 \quad (72)$$

Hence the

$$\epsilon_n = \left(n + \frac{1}{2}\right) \quad (73)$$

which is the same result as given earlier using perturbation theory.

Case-II $w = w_2 = \frac{1-\beta}{1+\lambda}$.

In this case we have the Hamiltonian

$$H = (a^+a + \frac{1}{2}) + \frac{(\lambda+\beta)}{2(1+\lambda\beta)}a^2 \quad (74)$$

Now comparing we get

$$s_0 = 1 \quad (75)$$

$$s_2 = \frac{(\lambda+\beta)}{2(1+\lambda\beta)} \quad (76)$$

$$s_1 = 0 \quad (77)$$

$$s_3 = 0 \quad (78)$$

$$s_4 = 0 \quad (79)$$

Hence the

$$\epsilon_n = \left(n + \frac{1}{2}\right) \quad (80)$$

which, once again the same as given earlier using perturbation theory. We see that the results of Lie-algebra match those of perturbation theory for energy level calculation.

6 Comparision with Similarity Transformation [6]

It is worth mentioning that Fernandez [6] has calculated a groundstate wave function of this oscillator using similarity transformation.The explict expression for the wave function [6] is

$$|\phi_0\rangle = \left(\frac{\omega_2}{\pi}\right)^{\frac{1}{4}} e^{-\omega_2\frac{x^2}{2}} \quad (81)$$

It is easy to check that our result for $n = 0$ remains the same as that of Fernandez [6]. However, we do not have literature for further comparison.

7 Conclusion

In this paper, we suggest a simpler procedure for calculating energy levels and wave function of the non-Hermitian harmonic oscillator under simultaneous transfromation of co-ordinate and momentum using perturbation theory. Further more the energy eigenvalue calculated using perturbation theory matches that of the Lie algebric method [5]. At this point it is necessary to mention that the ground state wave function calculated by Fernandez [6] refers to the zeroth order wave function as reflected in VI(b).However we present a complete picture on wave function. In all cases, we show that the energy levels remain the same as for the simple Harmonic oscillator. If the parameter ω is determined using variational principle [13] i.e. $\frac{d\epsilon_n^{(0)}}{d\omega} = 0$, then one has to calculate all orders of perturbation corrections i.e. $\sum_{m=2}^{k}\epsilon_n^{(m)}$ because $\langle n|H_N|n+2\rangle \neq 0$ so also $\langle n+2|H_N|n\rangle \neq 0$ and it will be a cumbersome process. However, if the parameter is determined either using

$\langle n|H_N|n+2\rangle$ or $\langle n+2|H_N|n\rangle$ as per the above procedure then the energy levels and wave function are obtained more easily. Further we note that

$$\begin{aligned}\langle\phi_n|H_D|\phi_n\rangle =&\langle\psi_n|H_D|\psi_n\rangle = \langle\phi_n|H|\Phi_n^{(k)}\rangle\\ =&\langle\psi_n|H|\Psi_n^{(k)}\rangle = \left(n+\frac{1}{2}\right)\end{aligned} \tag{82}$$

This is directly due to non-commuting operators i.e. $[H, H_D] \neq 0$ corresponding to different wave functions. Further,one can use nonlinear perturbation series [12] and conclude that if the parameter ω is determined using the condition $\langle n|H_N|n+2\rangle$ or $\langle n+2|H_N|n\rangle$, the calculations radily give the desired result. The final question is to consider is the possibility of realising this complex space **PT** invariant analysis on HO which generates a large class of isospectral system. We judge the answer to this question to be yes; provided an experiment is carried out using "unidirectional parity-time metamaterial " at optical frequencies [7] and observing spectra like the one in "Raman effect".

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References

[1] Biswas S.N., Classical Mechanics, Ist ed,Books and Allied (P) Ltd Calcutta, India, 1998

[2] Schiff L.I., Quantum Mechanics, 3rd ed, McGraw-Hill, Singapore, 1985

[3] Ahmed Z., Pseudo-Hermiticity of Hamiltonians under gauge-like transformation :real spectrum of non-Hermitian Hamiltonians, Phys. Lett., 2002, A 294, 287-291

[4] Rath B. and Mallick P., Zero energy correction method for non-Hermitian Harmonic oscillator with simultaneous transformation of co-ordinate and momentum. arxiv:1501.06161(quant-ph)

[5] Zhang H.B., Jiang G.Y. and Wang G.C., Unified algebric method to non-Hermitian systems with Lie algebric linear structure, J. Math. Phys., 2015, 56, 072103 (This paper also reflects iso-spectral condition as reported in [4])

[6] Fernandez F.M., Non-Hermitian Hamiltonians and Similarity transfromation, arxiv:1502.02694[quant-ph];Int. J. Theo. Phys., 2015,55,843-850 (This paper uses similarity transformation to reflect iso-spectral condition reported in [4])

[7] Feng L., Xu X.L., Fegadolli W.S., Lu M.H/, Oliveira J.E.B., Almeida V.R., Chen Y.F. and Scherer A., Experimental demonstration of a unidirectional reflectionless parity time metamaterial at optical frequencies. Nature Materials, 2013, 12, 108-113

[8] Bender C.M. and Boettcher S., Real Spectra in Non-Hermitian Hamiltonians Having $\mathcal{PT}$ Symmetry. Phys. Rev. Lett, 1998, 80(24), 5243-5246

[9] Zettili N., Quantum Mechanics:Concepts and applications, 2nd ed, John Wiley, New York, 2001

[10] Landau L.D. and Lifshtiz E.M., Quantum Mechanics, 3rd ed, Elsevier, Amsterdam, 2011.

[11] Rath B., A new approach on wave function and energy level calculation through perturbation theory, J. Phys. Soc. Jpn, 1998, 67(9), 3044-3049

[12] Rath B., Case study of the convergency of nonlinear perturbation series: Morse-Feshbach nonlinear series. Int. J. Mod. Phys, 1999, A14(13), 2103-2115

[13] Rath B., Second quantization, variational principle and perturbation theory for the anharmonic oscillator. Eur. J. Phys, 1990, 11, 184-185

[14] Rath B., Energy level calculation through perturbation theory. Phys. Rev 1990, A 42(5), 2520-2523