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Abdulnasir Isah and Chang Phang*

Genocchi Wavelet-like Operational Matrix and its Application for Solving Non-linear Fractional Differential Equations

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Abstract: In this work, we propose a new operational method based on a Genocchi wavelet-like basis to obtain the numerical solutions of non-linear fractional order differential equations (NFDEs). To the best of our knowledge this is the first time a Genocchi wavelet-like basis is presented. The Genocchi wavelet-like operational matrix of a fractional derivative is derived through wavelet-polynomial transformation. These operational matrices are used together with the collocation method to turn the NFDEs into a system of non-linear algebraic equations. Error estimates are shown and some illustrative examples are given in order to demonstrate the accuracy and simplicity of the proposed technique.

Keywords: Genocchi wavelet-like basis; operational matrix of fractional derivatives; fractional order system; collocation points

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1 Introduction

Fractional differential equations (FDEs) are generalizations of ordinary differential equations to a non integer order equations. FDEs have attracted considerable interest consequential of their ability to formulate complex phenomena in the fields of physics, chemistry, engineering, aerodynamics, electrodynamics of complex medium, polymer rheology etc,[1–3]. Due to their extensive applications much work has been dedicated to developing nu-

merical schemes for their solution. Therefore this method is becoming a very attractive research area and as such, several methods are established. The Adomian decomposition method [4], variational iteration method [5], homotopy perturbation method [6] and predictor-corrector method [7] are some of the more famous methods. In particular, systems of FDEs are considered in [8] where the Adomian decomposition method was employed for the solutions of system of nonlinear fractional differential equations. In [9] the homotopy analysis method (HAM) was used for the solutions of a system of nonlinear fractional partial differential equations, while in [10], a modification of Adomian decomposition method was introduced and used for solving a system of nonlinear equations, which produce a series solution with accelerated convergence. Recently, the idea of approximating the solution of FDEs by a family of basis functions has been widely used. The most commonly used functions are sine-cosine functions and block pulse functions. Legendre polynomials, Chebyshev polynomials, Bernoulli polynomials and Laguerre polynomials are also regularly used. For instance, more recently, the operational matrix of Bernstein Polynomials (BPs) is used to solve the Bratu equation in [11]. Also in [12], operational matrices of integration, differentiation, dual and product based on Bernstein polynomials are introduced and utilized to solve fractional optimal control problems. On the other hand, wavelets are localized functions, which form the basis for $L^2(\mathbb{R})$, so that localized pulse problems can easily be approached and analysed [13]. They are successfully applied in system analysis, optimal control, signal analysis and many more areas see [14]. However, wavelets are just another basis set that offers considerable advantages over alternative basis sets and allow us to tackle problems not accessible with conventional numerical methods. These main advantages are discussed in [15]. Legendre wavelets[14, 16, 17], Chebyshev wavelets [18] and Bernoulli wavelets [19] are some of the wavelet methods applied for solving FDEs.

In this paper, we present a wavelet-like basis based on an important member of Appell polynomials called Genoc-

*Corresponding Author: **Chang Phang:** Department of Mathematics and Statistics, University Tun Hussein Onn Malaysia, 86400 Parit Raja, Batu Pahat Johor, Malaysia; Email: pchang@uthm.edu.my
Abdulnasir Isah: Department of Mathematics and Statistics, University Tun Hussein Onn Malaysia, 86400 Parit Raja, Batu Pahat Johor, Malaysia; Email: abdulnasir.isah@gmail.com

chi polynomial, though the polynomials are not based on orthogonal functions, they share some advantageous properties with the polynomials in the Appell family, such as Bernoulli polynomials, over other classical orthogonal polynomials when approximating an arbitrary function. These advantages are stated in [20].

Motivated by these advantages, in this paper we consider NFDEs

$$D^{\alpha_n} y_n(x) = f_n(x, y_1, y_2, \dots, y_n), \quad (1)$$

where, D^{α_n} is the derivative of order α_n in the sense of Caputo subject to initial conditions $y_i(0) = d_i$, $i = 1, 2, \dots, n$.

We introduce a new operational matrix of fractional order derivative based on Genocchi Wavelet-like basis, which we obtained through wavelet-polynomial transformation. This matrix is then utilized together with collocation method to reduce the NFDEs (1) to a system of algebraic equations. We compare our results with some existing standard results to clearly demonstrate the simplicity, applicability and accuracy of our method.

The rest of the paper is organized as follows: In Section 2, we introduce some mathematical preliminaries of fractional calculus and properties of Genocchi polynomials. Section 3 is devoted to the basic formulation of the Genocchi wavelet-like basis and function approximation. In Section 4, we derive the Genocchi wavelet-like operational matrix of fractional derivative. Section 5 is devoted to the numerical method for solving NFDEs. While in section 6 the proposed method is applied to some numerical examples. Finally, a conclusion is given in section 7.

2 Preliminaries

In this section, we recall some basic definitions and properties of fractional calculus which will be used in this work.

Definition 2.1. [1, 2] The Riemann-Liouville integral I of fractional order α of a function $f(t)$ is given by

$$I^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, & \alpha > 0 \\ f(t), & \alpha = 0 \end{cases} \quad (2)$$

Some of the properties of I^α are

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t), \quad \alpha > 0, \beta > 0 \quad (3)$$

$$I^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\beta+\alpha}. \quad (4)$$

The Riemann-Liouville fractional derivative of order $\alpha > 0$ is also defined by

$$(D_t^\alpha f)(t) = \left(\frac{d}{dt} \right)^m (I^{m-\alpha} f)(t), \\ (\alpha > 0, m-1 < \alpha \leq m)$$

Some properties of D_t^α are as follows [2]:

$$D_t^\alpha (D_t^{-\beta} f(t)) = D_t^{\alpha-\beta} f(t), \quad \alpha, \beta \in \mathbb{R}^+ \text{ and } \alpha > \beta. \quad (5)$$

$$D_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha} \quad (6)$$

Definition 2.2. [1, 2] The Caputo fractional derivative D^α of a function $f(t)$ is defined as:

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (7) \\ n-1 < \alpha \leq n, n \in \mathbb{N}.$$

It has the following properties

$$D^\alpha C = 0, \quad (C \text{ is constant}) \quad (8)$$

$$D^\alpha t^\beta = \begin{cases} 0, \beta \in \mathbb{N} \cup \{0\} & \text{and } \beta < [\alpha] \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, \beta \in \mathbb{N} \cup \{0\} & \text{and } \beta \geq [\alpha] \\ \text{or } \beta \notin \mathbb{N} & \text{and } \beta > [\alpha], \end{cases} \quad (9)$$

where $[\alpha]$ denote the smallest integer greater than or equal to α and $\lfloor \alpha \rfloor$ denotes the largest integer less than or equal to α .

Similar to the integer order differentiation, the Caputo fractional differential operator is a linear operator, since,

$$D^\alpha (\lambda f(t) + \mu g(t)) = \lambda D^\alpha f(t) + \mu D^\alpha g(t) \quad (10)$$

where λ and μ are constants.

2.1 Genocchi polynomials

Genocchi numbers and polynomials have been extensively studied in many different contexts in branches of mathematics. Both Genocchi number G_n and polynomials $G_n(x)$ are respectively defined by means of the exponential generating functions [21–23].

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad (|t| < \pi) \quad (11)$$

$$\frac{2te^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad (|t| < \pi) \quad (12)$$

where $G_n(x)$ is the Genocchi polynomial of degree n and is given by

$$G_n(x) = \sum_{k=0}^n \binom{n}{k} G_{n-k} x^k \quad (13)$$

G_{n-k} here is the Genocchi number. The first few Genocchi polynomials are;

$$G_0(x) = 0$$

$$G_1(x) = 1$$

$$G_2(x) = 2x - 1$$

$$G_3(x) = 3x^2 - 3x$$

$$G_4(x) = 4x^3 - 6x^2 + 1$$

$$G_5(x) = 5x^4 - 10x^3 + 5x.$$

We list some of the properties of Genocchi polynomials below

$$\int_0^1 G_n(x) G_m(x) dx = \frac{2(-1)^n n! m!}{(m+n)!} G_{m+n} \quad n, m \geq 1 \quad (14)$$

$$\frac{dG_n(x)}{dx} = nG_{n-1}(x), \quad n \geq 1 \quad (15)$$

$$G_n(1) + G_n(0) = 0, \quad n > 1 \quad (16)$$

3 Genocchi Wavelet-like Basis

Genocchi wavelet-like basis $\psi_{n,m}(t) = \psi(k, n, m, t)$ have four arguments: k can assume any positive integer, m is the order for Genocchi polynomials and t is the normalized time. They are defined on the interval $[0, 1]$ by

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k-1}{2}} \sqrt{\frac{1}{R(m)}} G_m(2^{k-1}t - n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}} \\ 0, & \text{otherwise.} \end{cases} \quad (17)$$

where,

$$R(m) = \frac{2(-1)^m (m!)^2}{(2m)!} G_{2m} \quad (18)$$

$m = 1, \dots, M, n = 1, \dots, 2^{k-1}$, G_{2m} is the Genocchi Number and $G_m(t)$ is the Genocchi polynomial. The coefficient $\sqrt{\frac{1}{R(m)}}$ is for the normality.

Since we assume that k is a positive integer only, hence, this Genocchi wavelet-like basis is not wavelets in the framework of wavelet analysis. However, this family of Genocchi polynomials in the interval of $[0, 1]$ (what we

called Genocchi wavelet-like basis) still inherit the advantages of using wavelets in solving fractional differential equations numerically.

Most of the advantages that motivated us to use this Genocchi wavelet-like basis are inherited from Genocchi polynomials. For example, this Genocchi wavelet-like basis has fewer of terms than Legendre wavelets and thus when approximating arbitrary functions, we need less CPU time. There are less computational errors when using this Genocchi wavelet-like basis, since the coefficients of individual terms of this basis are smaller when compared to that of Legendre wavelets, which we know are related to the computational errors in the product. The operational matrix of derivative based on these functions has fewer of non zero elements. Apart from that, using a collocation method together with other wavelets basis, for instance, Legendre and Chebyshev wavelets, requires the use of the zeros of Chebyshev polynomials as the suitable collocation points [24, 25]. Thus, another important advantage of using this Genocchi wavelet-like basis is that one can use suitably equally spaced collocation points.

3.1 Function approximations

Suppose that $\{\psi_{1,1}, \dots, \psi_{1,M}, \psi_{2,1}, \dots, \psi_{2,M}, \dots, \psi_{(2^{k-1}),1}, \dots, \psi_{(2^{k-1}),M}\} \subset L^2[0, 1]$ is the set of Genocchi wavelet-like basis. We pause here and claim that this is the set of linearly independent elements of $L^2[0, 1]$. To prove this claim it is enough to show that

$$\text{Gram}(\psi_{1,1}(t), \dots, \psi_{2^{k-1},M}(t)) \neq 0$$

where, $\text{Gram}(\psi_{1,1}(t), \dots, \psi_{2^{k-1},M}(t))$ is the Gram determinant defined in [26] as

$$\text{Gram}(\psi_{1,1}(t), \dots, \psi_{2^{k-1},M}(t)) = \begin{vmatrix} \langle \psi_{1,1}(t), \psi_{1,1}(t) \rangle & \dots & \langle \psi_{1,1}(t), \psi_{1,M}(t) \rangle & \dots \\ \langle \psi_{1,2}(t), \psi_{1,1}(t) \rangle & \dots & \langle \psi_{1,2}(t), \psi_{1,M}(t) \rangle & \dots \\ \vdots & \dots & \vdots & \dots \\ \langle \psi_{2^{k-1},M}(t), \psi_{1,1}(t) \rangle & \dots & \langle \psi_{2^{k-1},M}(t), \psi_{1,M}(t) \rangle & \dots \\ \langle \psi_{1,1}(t), \psi_{2^{k-1},M}(t) \rangle & & & \\ \langle \psi_{1,2}(t), \psi_{2^{k-1},M}(t) \rangle & & & \\ \vdots & & & \\ \langle \psi_{2^{k-1},M}(t), \psi_{2^{k-1},M}(t) \rangle \end{vmatrix} \quad (19)$$

To show this determinant is not equal to zero, we first reduce the matrix to an upper triangular by Gaussian elimination and it is not difficult to see that the elements of the diagonal of the reduced matrix are given by

$$\text{Diag}(k, M) = \frac{1}{R(m)} a(m-1), \quad m = 1, 2, \dots, M$$

where,

$$a(n) = \frac{(n!(n+1)!)^2}{(2n!)(2n+1)!}$$

and $R(m)$ is given in (18).

Clearly, one can see that $\text{Diag}(k, M)$ is not equal to zero for all m . Hence the determinant given by

$$\prod_{m=1}^M \frac{1}{R(m)} a(m-1)$$

is not zero. Therefore, the set

$$\{\psi_{1,1}, \dots, \psi_{1,M}, \dots, \psi_{(2^{k-1}),1}, \dots, \psi_{(2^{k-1}),M}\} \subset L^2[0, 1]$$

is the set of linearly independent elements. Our claim is established.

Now, suppose that

$$Y = \text{Span}\{\psi_{1,1}, \dots, \psi_{1,M}, \dots, \psi_{(2^{k-1}),1}, \dots, \psi_{(2^{k-1}),M}\}$$

Let $f(t)$ be an arbitrary element of $L^2[0, 1]$ since Y is finite dimensional subspace (it is closed), then $f(t)$ has a unique best approximation out of Y , say $f^*(t)$ i.e

$$\forall y(t) \in Y \quad \|f(t) - f^*(t)\| \leq \|f(t) - y(t)\|$$

Thus, since $f^* \in Y$ there exist unique coefficients

$$c_{1,1}, \dots, c_{1,M}, c_{2,1}, \dots, c_{2,M}, \dots, c_{(2^{k-1}),1}, \dots, c_{(2^{k-1}),M}$$

such that

$$f(t) \approx f^*(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=1}^M c_{n,m} \psi_{n,m} \quad (20)$$

This implies that $\langle f(t) - f^*(t), f^*(t) \rangle = 0$.

$$\text{In particular } \langle \psi_{i,j}(t), f(t) - \sum_{n=1}^{2^{k-1}} \sum_{m=1}^M c_{n,m} \psi_{n,m} \rangle = 0$$

i.e.

$$\begin{aligned} & \langle \psi_{i,j}(t), f(t) \rangle - c_{1,1} \langle \psi_{i,j}(t), \psi_{1,1}(t) \rangle - \dots - \\ & c_{1,M} \langle \psi_{i,j}(t), \psi_{1,M}(t) \rangle - \dots - \\ & c_{2^{k-1},1} \langle \psi_{i,j}(t), \psi_{2^{k-1},1}(t) \rangle - \dots \\ & - c_{2^{k-1},M} \langle \psi_{i,j}(t), \psi_{2^{k-1},M}(t) \rangle = 0 \end{aligned}$$

$$i = 1, \dots, 2^{k-1}, \quad j = 1, \dots, M.$$

This is a non-homogeneous system of $(2^{k-1}M)$ linear equations. The determinant of the coefficients is given by Gram determinant (19) Since $f^*(t)$ exist and is unique, the system (21) has a unique solution, hence $\text{Gram}(\psi_{1,1}(t), \dots, \psi_{2^{k-1},M}(t))$ must be non zero. Therefore Cramer's rule now yields

$$c_{i,j} = \frac{\text{Gram}_{i,j}(\psi_{1,1}(t), \dots, \psi_{2^{k-1},M}(t))}{\text{Gram}(\psi_{1,1}(t), \dots, \psi_{2^{k-1},M}(t))} \quad (21)$$

where $\text{Gram}(\psi_{1,1}(t), \dots, \psi_{2^{k-1},M}(t))$ is given in (19) and $\text{Gram}_{i,j}(\psi_{1,1}(t), \dots, \psi_{2^{k-1},M}(t))$ is obtained from $\text{Gram}(\psi_{1,1}(t), \dots, \psi_{2^{k-1},M}(t))$ when we replace the $(i, j)^{\text{th}}$ column of $\text{Gram}((\psi_{1,1}(t), \dots, \psi_{2^{k-1},M}(t)))$ by the column with elements $\langle \psi_{1,1}(t), f(t) \rangle, \dots, \langle \psi_{1,2}(t), f(t) \rangle, \dots, \langle \psi_{2^{k-1},M}(t), f(t) \rangle$

3.2 Error Estimates

It is interesting to note that the error estimate can also be expressed in terms of Gram determinant as given in the following theorem [26]:

Theorem 3.1. Suppose that $H = L^2[0, 1]$ be the Hilbert space, and let Y be a closed subspace of H such that $Y = \text{Span}\{\psi_{1,1}, \dots, \psi_{1,M}, \dots, \psi_{(2^{k-1}),1}, \dots, \psi_{(2^{k-1}),M}\}$. Let $f(t)$ be an arbitrary element of H and $f^*(t)$ be the unique best approximation of $f(t)$ out of Y , then

$$\|f(t) - f^*(t)\|^2 = \frac{\text{Gram}(f(t), \psi_{1,1}(t), \dots, \psi_{2^{k-1},M}(t))}{\text{Gram}(\psi_{1,1}(t), \dots, \psi_{2^{k-1},M}(t))}$$

where,

$$\begin{aligned} \text{Gram}(f(t), \psi_{1,1}(t), \dots, \psi_{2^{k-1},M}(t)) = & \begin{vmatrix} \langle f(t), f(t) \rangle & \langle f(t), \psi_{1,1}(t) \rangle & \dots \\ \langle \psi_{1,1}(t), f(t) \rangle & \langle \psi_{1,1}(t), \psi_{1,1}(t) \rangle & \dots \\ \langle \psi_{1,2}(t), f(t) \rangle & \langle \psi_{1,2}(t), \psi_{1,1}(t) \rangle & \dots \\ \vdots & \vdots & \dots \\ \langle \psi_{2^{k-1},M}(t), f(t) \rangle & \langle \psi_{2^{k-1},M}(t), \psi_{1,1}(t) \rangle & \dots \\ \langle f(t), \psi_{2^{k-1},M}(t) \rangle & \vdots & \dots \\ \langle \psi_{1,1}(t), \psi_{2^{k-1},M}(t) \rangle & \vdots & \dots \\ \langle \psi_{1,2}(t), \psi_{2^{k-1},M}(t) \rangle & \vdots & \dots \\ \vdots & \vdots & \dots \\ \langle \psi_{2^{k-1},M}(t), \psi_{2^{k-1},M}(t) \rangle & \vdots & \dots \end{vmatrix} \end{aligned}$$

Proof. For the proof of this theorem see [26] □

4 Genocchi Wavelet-like Operational Matrix of Fractional Order Derivative

In this section, we derive the Genocchi wavelet-like operational matrix of the fractional derivative by first transforming the wavelets to Genocchi polynomials. We then make use of the Genocchi operational matrix of the fractional derivative to derive the Genocchi wavelet-like operational matrix of the fractional derivative.

4.1 Transformation matrix of the Genocchi wavelet-like to Genocchi polynomials

An arbitrary function $y(t) \in L^2[0, 1]$ can be expanded into Genocchi polynomials as

$$y(x) = \sum_{m=1}^M r_m G_m(x) = R^T \Psi'(x)$$

where the Genocchi coefficient vector R and the Genocchi vector $\Psi'(x)$ are given by

$$R = [r_0, r_1, \dots, r_M]^T \quad (22)$$

$$\Psi'(x) = [G_1(x), G_2(x), \dots, G_M(x)]^T \quad (23)$$

Similar to (21) we have

$$c_i = \frac{\text{Gram}_i(G_1(t), \dots, G_M(t))}{\text{Gram}(G_1(t), \dots, G_M(t))} \quad (24)$$

where:

$$\text{Gram}(G_1(t), \dots, G_M(t)) = \quad (25)$$

$$\begin{vmatrix} \langle G_1(t), G_1(t) \rangle & \langle G_1(t), G_2(t) \rangle & \dots & \langle G_1(t), G_M(t) \rangle \\ \langle G_2(t), G_1(t) \rangle & \langle G_2(t), G_2(t) \rangle & \dots & \langle G_2(t), G_M(t) \rangle \\ \vdots & \vdots & \dots & \vdots \\ \langle G_M(t), G_1(t) \rangle & \langle G_M(t), G_2(t) \rangle & \dots & \langle G_M(t), G_M(t) \rangle \end{vmatrix}$$

and $\text{Gram}_i(G_1(t), \dots, G_M(t))$ is obtained from $\text{Gram}(G_1(t), \dots, G_M(t))$ when we replace the i^{th} column of $\text{Gram}(G_1(t), \dots, G_M(t))$ by the column with elements $\langle G_1(t), y(t) \rangle, \langle G_2(t), y(t) \rangle, \dots, \langle G_M(t), y(t) \rangle$.

The Genocchi wavelet-like basis may be expanded in to (M) -terms Genocchi polynomials as

$$\psi_{2^{k-1}(M) \times 1}(t) = \Phi_{2^{k-1}(M) \times (M)} \Psi'_{(M) \times 1}, \quad (26)$$

where Φ is the transformation matrix of the Genocchi wavelet-like basis to Genocchi polynomial.

The following lemma is also of great importance.

Lemma 4.1. Let $G_i(t)$ be the Genocchi polynomial then, $D^\alpha G_i(t) = 0$, for $i = 1, \dots, [\alpha] - 1$, $\alpha > 0$.

The proof of this Lemma is obvious, one can use (8), (9) and (10) on (13).

4.2 Genocchi operational matrix of fractional derivative

In the following theorem we derive the operational matrix of fractional order derivative for the Genocchi polynomials.

Theorem 4.2. Consider $\Psi'(x)$ the Genocchi vector given in (23) and let $\alpha > 0$. Then,

$$D^\alpha \Psi'(x)^T = P^\alpha \Psi'(x)^T, \quad (27)$$

where P^α is $M \times M$ operational matrix of fractional derivative of order α in Caputo sense and is defined as follows:

$$P^{(\alpha)} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ \sum_{k=[\alpha]}^{[\alpha]} \rho_{[\alpha],k,1} & \sum_{k=[\alpha]}^{[\alpha]} \rho_{[\alpha],k,2} & \dots & \sum_{k=[\alpha]}^{[\alpha]} \rho_{[\alpha],k,M} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{k=[\alpha]}^i \rho_{i,k,1} & \sum_{k=[\alpha]}^i \rho_{i,k,2} & \dots & \sum_{k=[\alpha]}^i \rho_{i,k,M} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{k=[\alpha]}^M \rho_{M,k,1} & \sum_{k=[\alpha]}^M \rho_{M,k,2} & \dots & \sum_{k=[\alpha]}^M \rho_{M,k,M} \end{bmatrix},$$

where $\rho_{i,k,j}$ is given by:

$$\rho_{i,k,j} = \frac{i! G_{i-k}}{(i-k)! \Gamma(k+1-\alpha)} c_j. \quad (28)$$

G_{i-k} is the Genocchi number and c_j can be obtained from (24).

Proof. From (13) we have

$$\begin{aligned} D^\alpha G_i(x) &= \sum_{k=1}^i \frac{i! G_{i-k}}{(i-k)! k!} D^\alpha x^k \\ &= \sum_{k=[\alpha]}^i \frac{i! G_{i-k}}{(i-k)! \Gamma(k+1-\alpha)} x^{k-\alpha} \end{aligned} \quad (29)$$

Let $f(x) = x^{k-\alpha}$, then if we approximate $f(x)$ by the truncated Genocchi series, we have $f(x) = \sum_{j=1}^M c_j G_j(x)$.

Therefore, substituting this in (29) we have:

$$\begin{aligned} D^\alpha G_i(x) &= \sum_{j=1}^M \left(\sum_{k=[\alpha]}^i \frac{i! G_{i-k}}{(i-k)! \Gamma(k+1-\alpha)} c_j \right) G_j(x) \\ &= \sum_{j=1}^M \left(\sum_{k=[\alpha]}^i \rho_{i,k,j} \right) G_j(x), \end{aligned} \quad (30)$$

where $\rho_{i,k,l}$ is as given in (28). Rewriting (30) in vector form, we have:

$$\begin{aligned} D^\alpha G_i(x) &= \\ \left[\sum_{k=[\alpha]}^i \rho_{[\alpha],k,1} & \sum_{k=[\alpha]}^i \rho_{[\alpha],k,2} \quad \dots \quad \sum_{k=[\alpha]}^i \rho_{[\alpha],k,M} \right] G(x) \end{aligned} \quad (31)$$

$$i = \lceil \alpha \rceil \cdots M.$$

Also according to Lemma 1, we can write

$$D^\alpha G_i(x) = [0, 0, \dots, 0] G(x) \quad i = 1, \dots, \lceil \alpha \rceil - 1 \quad (32)$$

Thus, combining (31) and (32) leads to the desired result. $\square$

4.3 Genocchi wavelet-like operational matrix of fractional order derivative

Now, we derive Genocchi wavelet-like operational matrix of fractional order derivative.

Let

$$D^\alpha \psi(x) = H^{(\alpha)} \psi(x) \quad (33)$$

where $H^{(\alpha)}$ is the Genocchi wavelet-like operational matrix of fractional derivative and ψ is the Genocchi wavelet-like vector. Following the same method as in [17] we obtain the Genocchi wavelet-like operational matrix of fractional derivative $H^{(\alpha)}$ to be given by;

$$H^{(\alpha)} = \Phi P^{(\alpha)} \Phi^{-1} \quad (34)$$

Before we see the application of this operational matrix in solving NFDEs, we refer to the work [27, 28] for the existence and uniqueness of the solution for systems of FDEs.

5 Collocation Method based on Genocchi Wavelet-like Operational Matrix of Fractional Derivative

In this section, we use the collocation method based on Genocchi wavelet-like operational matrix of fractional derivatives to numerically solve the NFDEs (1). To do this, we first approximate $y_j(t)$ for $j = 1, 2, \dots, n$, by Genocchi wavelet-like basis as follows:

$$y_j(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=1}^M c_{n,m} \psi_{n,m} = \mathbf{C}_j^T \psi(t)^T \quad j = 1, 2, \dots, n. \quad (35)$$

Where,

$\mathbf{C}_j = [c_{1,1}^j, \dots, c_{1,M}^j, \dots, c_{(2^{k-1}),1}^j, \dots, c_{(2^{k-1}),M}^j]^T$ is unknown vector and

$\psi(t) = [\psi_{1,1}, \dots, \psi_{1,M}, \dots, \psi_{(2^{k-1}),1}, \dots, \psi_{(2^{k-1}),M}]^T$ is the wavelet-like vector.

Now employing (33) on (35), we have

$$D^\alpha y_j(t) \simeq \mathbf{C}_j H^{(\alpha)} \psi(t)^T, \quad j = 1, 2, \dots, n. \quad (36)$$

Therefore, substituting (35) and (36) in (1), we have

$$\mathbf{C}_j H^{(\alpha)} \psi(t)^T = f_j \left(t, \mathbf{C}_1 \psi(t)^T, \mathbf{C}_2 \psi(t)^T, \dots, \mathbf{C}_n \psi(t)^T \right) \quad j = 1, 2, \dots, n. \quad (37)$$

From the initial conditions we have

$$\mathbf{C}_j \psi(0)^T = d_j \quad j = 1, 2, \dots, n. \quad (38)$$

To find the solution of (1), we collocate (37) at the collocation points $t_i = \frac{i}{N-1}$, $i = 1, 2, \dots, N-1$ to obtain

$$\mathbf{C}_j H^{(\alpha)} \psi(t_i)^T = f_j \left(t_i, \mathbf{C}_1 \psi(t_i)^T, \mathbf{C}_2 \psi(t_i)^T, \dots, \mathbf{C}_n \psi(t_i)^T \right) \quad i = 1, 2, \dots, N-1, \quad j = 1, 2, \dots, n. \quad (39)$$

Thus, (39) are $n(N-1)$ algebraic equations. These equations together with (38) make $n(N)$ algebraic equations which can be solved using Newton's iterative method. Consequently $y_j(t)$ given in (35) can be calculated.

6 Numerical Examples

In this section, some numerical examples are given to illustrate the applicability and accuracy of the proposed method. All the numerical computations are carried out using Maple 18.

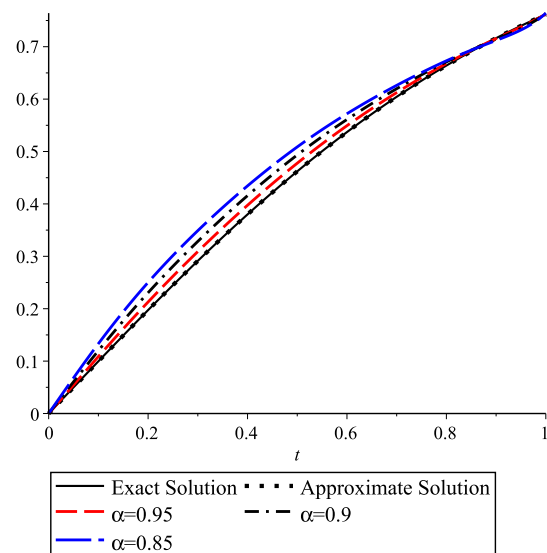


Figure 1: Exact and numerical results obtained when $\alpha = 1, 0.95, 0.9$ and 0.85 for example 2.

Table 1: Comparison of the L^2 and L^∞ errors obtained by the present method and that in [29] for numerical solution $y(t)$ for Example 1.

L^2 Error		L^∞ Error	
B-spline[29]	present method $k = 1$	B-spline[29]	present method $k = 1$
$4.0\text{E-}4$ ($J = 3$)	$3.6\text{E-}5$ ($M = 3$)	$1.0\text{E-}3$ ($J = 3$)	$5.3\text{E-}3$ ($M = 3$)
$9.5\text{E-}5$ ($J = 4$)	$2.8\text{E-}6$ ($M = 4$)	$2.6\text{E-}4$ ($J = 4$)	$1.2\text{E-}5$ ($M = 4$)
$2.3\text{E-}5$ ($J = 5$)	$6.2\text{E-}6$ ($M = 5$)	$7.1\text{E-}5$ ($J = 5$)	$2.7\text{E-}5$ ($M = 5$)
$5.8\text{E-}6$ ($J = 6$)	$1.3\text{E-}7$ ($M = 6$)	$1.5\text{E-}5$ ($J = 6$)	$1.6\text{E-}6$ ($M = 6$)
$1.5\text{E-}6$ ($J = 7$)	$1.9\text{E-}7$ ($M = 7$)	$4.2\text{E-}6$ ($J = 7$)	$1.8\text{E-}7$ ($M = 7$)

Table 2: Comparison of the numerical results obtained by the present method with different values of α and that obtains using second kind chebyshev wavelet (SKCW) in [30] for Example 2.

t	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1$		exact solution
	SKCW [30]	Present Method	SKCW [30]	Present Method	SKCW [30]	Present Method	
0.2	0.436737	0.4204949	0.309886	0.2945296	0.197358	0.1973753	0.1973753
0.4	0.553802	0.5442814	0.481638	0.4708147	0.379946	0.3799490	0.3799490
0.6	0.621026	0.6150123	0.597790	0.5906088	0.537048	0.5370496	0.5370496
0.8	0.666016	0.6630723	0.678835	0.6742144	0.664009	0.6640368	0.6640368

Table 3: Comparison of the numerical solution $y_1(t)$ and $y_2(t)$, exact solutions and absolute errors obtained by present method for example 3.

t	Exact $y_1(t)$	Exact $y_2(t)$	Approx. $y_1(t)$	Approx. $y_2(t)$	Abs. Err $y_1(t)$	Abs. Err $y_2(t)$
0.1	1.0512711	0.1105171	1.0512711	0.1105171	$3.74\text{E-}14$	$5.37\text{E-}10$
0.2	1.1051709	0.2442806	1.1051709	0.2442806	$3.83\text{E-}14$	$5.78\text{E-}10$
0.3	1.1618342	0.4049576	1.1618342	0.4049576	$4.05\text{E-}14$	$6.42\text{E-}10$
0.4	1.2214028	0.5967299	1.2214028	0.5967298	$4.25\text{E-}14$	$7.09\text{E-}10$
0.5	1.2840254	0.8243606	1.2840254	0.8243606	$4.47\text{E-}14$	$7.84\text{E-}10$
0.6	1.3498588	1.0932713	1.3498588	1.0932713	$4.69\text{E-}14$	$8.65\text{E-}10$
0.7	1.4190675	1.4096269	1.4190675	1.4096269	$4.94\text{E-}14$	$9.57\text{E-}10$
0.8	1.4918247	1.7804327	1.4918247	1.7804327	$5.18\text{E-}14$	$1.06\text{E-}09$
0.9	1.5683122	2.2136428	1.5683122	2.2136428	$5.54\text{E-}14$	$1.18\text{E-}09$
1.0	1.6487213	2.7182818	1.6487213	2.7182818	$2.18\text{E-}14$	$7.92\text{E-}10$

Example 1. We first consider the following FDE solved using B-spline operational matrix in [29]

$$4(t+1)D^{\frac{5}{2}}y(t) + 4D^{\frac{3}{2}}y(t) + \frac{1}{\sqrt{t+1}}y(t) = \sqrt{t} + \sqrt{\pi} \quad (40)$$

subject to, $y(0) = \sqrt{\pi}$, $y'(0) = \frac{\sqrt{\pi}}{2}$, $y(1) = \sqrt{2\pi}$

This example is solved using B-spline operational matrix in [29], its exact solution is known to be $y(t) = \sqrt{\pi(t+1)}$. We consider this problem when $M = 3, 4, 5, 6, 7$ and $k = 1$. The L^2 and L^∞ errors of the results obtained are compared with that obtained using B-spline operational method [29] as shown in Table 1. From this table one can observe that,

we are able to obtain a more accurate result than that obtained using the B spline operational method in [29].

Example 2. We consider the following Riccati equation solved in [30–32]

$$D^\alpha y(t) = -y^2(t) + 1, \alpha \in (0, 1], t > 0 \quad (41)$$

subject to, $y(0) = 0$

The exact solution when $\alpha = 1$ is known to be $y(t) = \frac{e^{2t}-1}{e^{2t}+1}$. Our numerical solution is in very good agreement with the exact solution when $\alpha = 1$. We solve this problem when

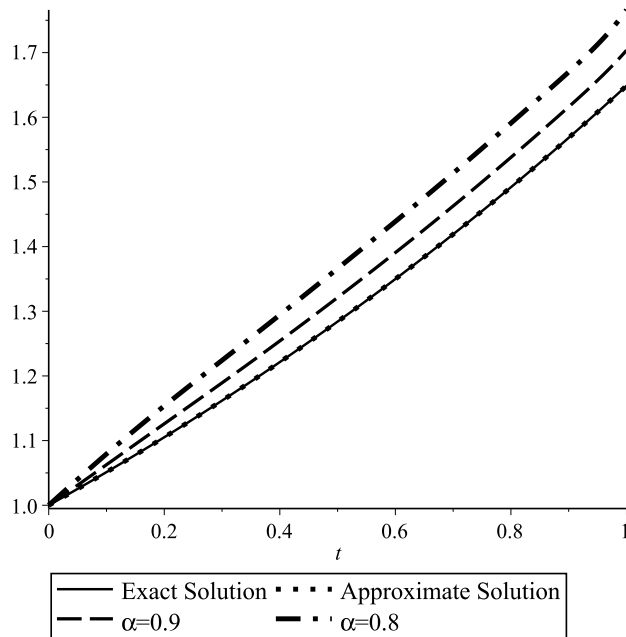


Figure 2: Comparison of our solution $y_1(t)$, when $\alpha = 0.8, 0.9$ and 1 for Example 3.

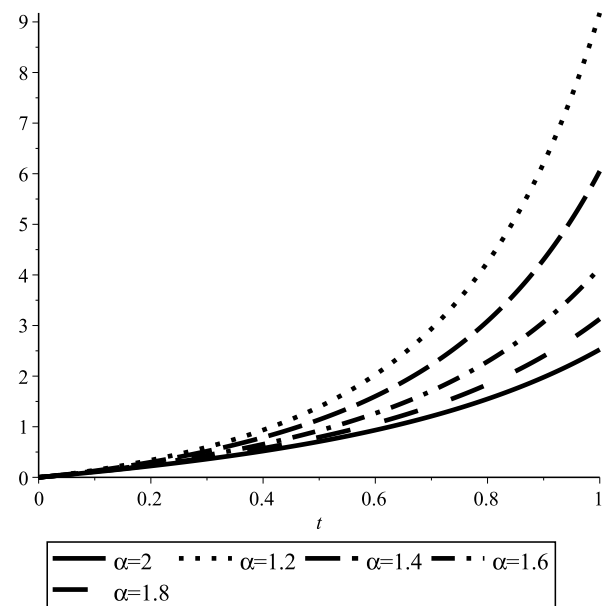


Figure 4: Comparison of our solution $y_1(t)$, when $\alpha = 1.2, 1.4, 1.6, 1.8, 2.0$ and $\beta = 2.2, 2.4, 2.6, 2.8, 3.0$ for example 4.

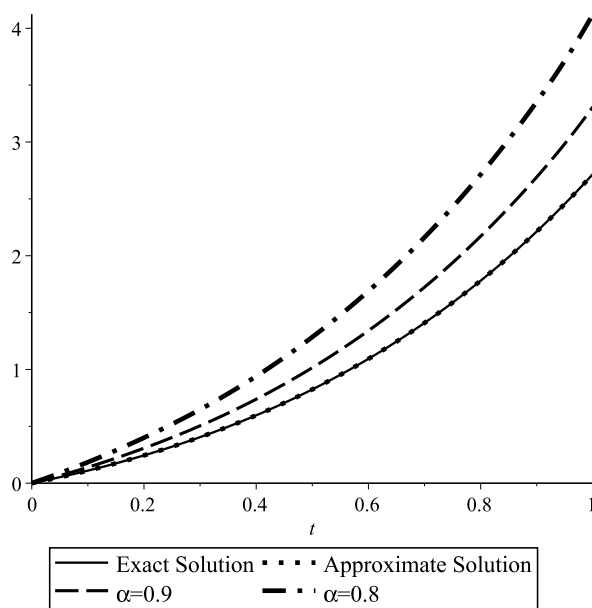


Figure 3: Comparison of our solution $y_2(t)$, when $\alpha = 0.8, 0.9$ and 1 for Example 3.

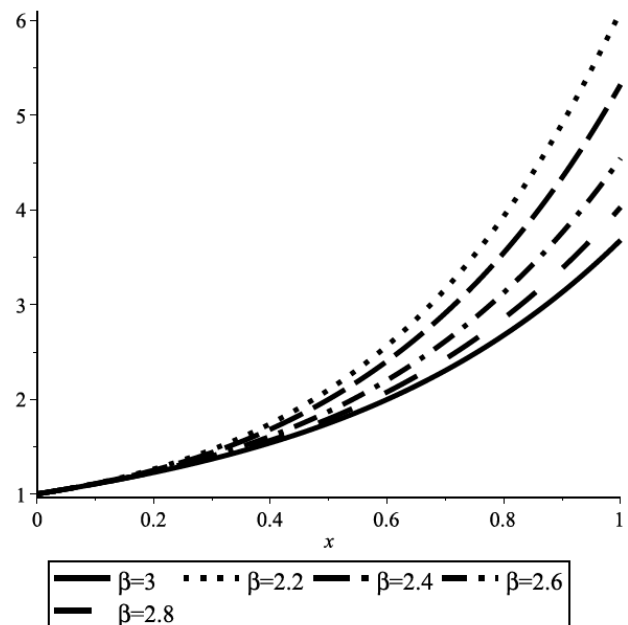


Figure 5: Comparison of our solution $y_2(t)$, when $\alpha = 1.2, 1.4, 1.6, 1.8, 2.0$ and $\beta = 2.2, 2.4, 2.6, 2.8, 3.0$ for example 4.

$\alpha = 0.95, 0.9, 0.85$, our results (shown in figure 1) show a similar result obtained in [31].

We also solve the example when $\alpha = 0.5$ and 0.75 , the numerical results are compared with that given in [30] as shown in Table 2.

Example 3. Here, we consider the following system of NFDE [33, 34].

$$\begin{aligned} D^\alpha y_1(t) &= \frac{y_1(t)}{2} \\ D^\alpha y_2(t) &= (y_1(t))^2 + y_2(t) \end{aligned} \quad (42)$$

subject to, $y_1(0) = 1, \quad y_2(0) = 0$

Table 4: Errors for different values of M when $k = 1$, $\alpha = 2$, $\beta = 3$ obtained by present method for example 4.

t	$y_1(t) (l_{1,9} - l_{1,10})$	$y_2(t) (l_{1,9} - l_{1,10})$	$y_1(t) (l_{1,10} - l_{1,11})$	$y_2(t) (l_{1,10} - l_{1,11})$
0.1	2.15835E-05	2.20000E-07	3.54090E-06	3.10000E-08
0.2	5.24145E-05	1.19300E-06	8.32190E-06	1.61000E-07
0.3	8.29739E-05	2.92700E-06	1.31082E-05	3.89000E-07
0.4	1.14481E-04	5.35600E-06	1.80340E-05	7.05000E-07
0.5	1.46896E-04	8.40700E-06	2.31086E-05	1.09600E-06
0.6	1.80494E-04	1.19810E-05	2.83838E-05	1.54600E-06
0.7	2.15461E-04	1.59670E-05	3.38690E-05	2.03200E-06
0.8	2.51648E-04	2.02270E-05	3.96210E-05	2.54400E-06
0.9	2.90102E-04	2.47820E-05	4.55490E-05	3.03800E-06
1.0	3.19375E-04	3.14880E-05	5.31230E-05	3.24200E-06

The exact solution of this system when $\alpha = 1$ is known to be $y_1(t) = e^{\frac{t}{2}}$ and $y_2(t) = te^t$. The example is solved by our method with $M = 10$ and $k = 1$. The numerical results and absolute errors for $y_1(t)$ and $y_2(t)$ are, respectively, shown in Table 3.

We consider this example when $\alpha = 0.8, 0.9$ and the results are compared with the exact solution when $\alpha = 1$ as shown in figures 2 and 3, the figures affirm that when α approaches 1, our results approach the exact solution

Example 4. Consider the following NSFDE [35].

$$\begin{aligned} D^\alpha y_1(t) &= y_1(t) + y_2^2(t) \quad 1 < \alpha \leq 2 \\ D^\beta y_2(t) &= y_1(t) + 5y_2(t) \quad 2 < \beta \leq 3 \end{aligned} \quad (43)$$

subject to, $y_1(0) = 0$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2'(0) = 1$, $y_2''(0) = 1$

The exact solutions of this system are unknown. We solve this example by our method with $\alpha = 2$ and $\beta = 3$ at the levels $M = 9$, $k = 1$, $M = 10$, $k = 1$ and $M = 11$, $k = 1$ ($l_{1,9}$, $l_{1,10}$ and $l_{1,11}$). The error estimation here is referred to the l^{th} level estimate given by $|l_{k,M} - l_{k,M+1}|$. For this example the error estimates are displayed on Table 4.

In figure 4 and 5, we show the behavior of the solutions of this system at different values of α and β i.e when $\alpha = 1.2, 1.4, 1.6, 1.8, 2.0$ and $\beta = 2.2, 2.4, 2.6, 2.8, 3.0$

7 Conclusion

In this paper, a new operational matrix based on the Genocchi wavelet-like basis is derived and applied together with the collocation method to numerically solve the NSFDEs. The comparison of the results shows that the

present method is a simple and effective mathematical tool for finding the numerical solutions of NSFDEs. The advantage of this operational matrix over others is that it has less computational complexity because every operational matrix of differentiation involves more numbers of zeros and thus, reduces the run time and provide the solution at high accuracy.

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