Research Article Open Access

Remus-Daniel Ene*, Vasile Marinca, and Valentin Bogdan Marinca

Viscous flow and heat transfer over an unsteady stretching surface

DOI 10.1515/phys-2016-0042
Received Jun 11, 2015; accepted Jul 18, 2016

Abstract: In this paper we have studied the flow and heat transfer of a horizontal sheet in a viscous fluid. The stretching rate and temperature of the sheet vary with time. The governing equations for momentum and thermal energy are reduced to ordinary differential equations by means of similarity transformation. These equations are solved approximately by means of the Optimal Homotopy Asymptotic Method (OHAM) which provides us with a convenient way to control the convergence of approximation solutions and adjust convergence rigorously when necessary. Some examples are given and the results obtained reveal that the proposed method is effective and easy to use.

Keywords: optimal homotopy asymptotic method (OHAM); film flow; heat transfer; unsteady stretching surface

PACS: 02.60.-x; 47.11.-j; 47.50.-d

1 Introduction

The flow and heat transfer in a viscous fluid over a stretching surface is important for engineers and applied mathematicians. Studies have been conducted which take into account the numerous industrial applications. Examples of such applications are crystal growing, continuous casting, polymer extrusion, manufacture and drawing of plastics and rubber sheets, wire drawing and so on. Sakiadis [1, 2], Crane [3], Tsou *et al.* [4], Gupta and Gupta [5], Maneschy *et al.* [6], Grubka and Bobba [7], Wang [8], Usha and Rukamani [9], Anderson *et al.* [10], Ali [11], Magyari *et al.* [12], Vajravelu [13], Magyari and Keller [14], Elbashbeshy and Bazid [15], Dandapat *et al.* [16], Ali and Magyari [17], Liu and Anderson [18], Chen [19], Dandapat *et al.* [20], Cortell [21] have studied different problems related to such applications.

Analytical solutions to nonlinear differential equations play an important role in the study of flow and heat transfer of different types of fluids, but it is difficult to find these solutions in the presence of strong nonlinearity. A few approaches have been proposed to find and develop approximate solutions of nonlinear differential equations. Perturbation methods have been applied to determine approximate solutions to weakly nonlinear problems [22]. But the use of perturbation theory in many problems is invalid for parameters beyond a certain specified range. Other procedures have been proposed such as the Adomian decomposition method [23], some linearization methods [24, 25], various modified Lindstedt-Poincare methods [26], variational iteration method [27], optimal homotopy perturbation method [28] and optimal homotopy asymptotic method [29–34].

In this study we propose an accurate approach to nonlinear differential equations of the flow and heat transfer in a viscous fluid, using an analytical technique, namely the optimal homotopy asymptotic method. Our procedure, which does not imply the presence of a small or large parameter in the equation or in the boundary/initial conditions, is based on the construction and determination of the linear operators and of the auxiliary functions, combined with a convenient way to optimally control the convergence of the solution. The efficiency of the proposed procedure is proven, while an accurate solution is explicitly analytically obtained in an iterative way after only one iteration. The validity of this method is demonstrated by comparing the results obtained with the numerical solution.

***Corresponding Author: Remus-Daniel Ene:** University Politehnica Timişoara, Department of Mathematics, Timişoara, 300006, Romania; Email: remus.ene@upt.ro
Vasile Marinca: University Politehnica Timişoara, Department of Mechanics and Vibration, Timişoara, 300222, Romania; Department of Electromechanics and Vibration, Center for Advanced and Fundamental Technical Research, Romania Academy, Timişoara, 300223, Romania
Valentin Bogdan Marinca: University Politehnica Timişoara, Department of Applied Electronics, Timişoara, 300223, Romania

2 Equations of motion

Consider an unsteady, two dimensional flow on a continuous stretching surface. If u and v are velocity components, T is the temperature and k is the thermal conductivity, then the governing time-dependent equations for the continuity, momentum and thermal energy are [8, 10, 15, 17, 18]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=k\frac{\partial^2 T}{\partial y^2} \tag{3}$$

The appropriate boundary conditions are:

$$u=\frac{u_0\frac{x}{l}}{1+\gamma t},\quad v=0, \qquad T=T_\infty+\frac{T_0}{(1+\gamma t)^c}\left(\frac{x}{l}\right)^n \quad \text{at } y=0 \tag{4}$$

$$u\to 0,\quad T\to T_\infty \qquad \text{at } y\to\infty \tag{5}$$

where u_0, T_0, T_∞, γ are positive constants, c and n are arbitrary and l is a reference length.
If $Re=\frac{u_0 l}{\nu}$ and $Pr=\frac{\nu}{k}$ are the Reynolds number and the Prandtl number respectively and if we choose a stream function $\Psi(x,y)$ such that:

$$u=\frac{\partial\Psi}{\partial y},\qquad v=-\frac{\partial\Psi}{\partial x} \tag{6}$$

then equation (1) of continuity is satisfied. The mathematical analysis of equations (2) and (3) is simplified by introducing the following similarity transformation:

$$\Psi=\frac{x}{l}\frac{f(\eta)}{\sqrt{Re}(1+\gamma t)^{1/2}} \tag{7}$$

$$\eta=\sqrt{Re}\frac{y}{l(1+\gamma t)^{1/2}} \tag{8}$$

$$T=T_\infty+T_0\left(\frac{x}{l}\right)^n\frac{\theta(\eta)}{(1+\gamma t)^c} \tag{9}$$

where T_0 is a reference temperature. In this way equations (6) can be written in the form:

$$u(x,y,t)=\frac{u_0}{l}\frac{x}{(1+\gamma t)}f'(\eta) \tag{10}$$

$$v(x,y,t)=-\frac{u_0}{\sqrt{Re}(1+\gamma t)^{1/2}}f(\eta) \tag{11}$$

where prime denotes differentiation with respect to η.

Substituting equations (7), (8), (9), (10) and (11) into equations (2) and (3), we obtain

$$f'''+ff''-f'^2+\Lambda\left(f'+\frac{1}{2}\eta f''\right)=0 \tag{12}$$

$$\frac{1}{Pr}\theta''+f\theta'-nf'\theta+\Lambda\left(c\theta+\frac{1}{2}\eta\theta'\right)=0. \tag{13}$$

Here $\Lambda=\frac{\gamma l}{u_0}$ is a dimensionless measure of the unsteadiness.

The dimensional boundary conditions (4) and (5) become:

$$u=\frac{u_0}{l}\frac{x}{(1+\gamma t)}f'(0) \quad \text{at } y=0 \tag{14}$$

$$v=-\frac{u_0}{\sqrt{Re}(1+\gamma t)^{1/2}}f(0) \quad \text{at } y=0 \tag{15}$$

$$T=T_\infty+T_0\left(\frac{x}{l}\right)^n\frac{\theta(0)}{(1+\gamma t)^c} \quad \text{at } y=0 \tag{16}$$

such that for the dimensionless functions f and θ, the boundary/initial conditions become:

$$f(0)=f_w,\quad f'(0)=1,\quad f'(\infty)=0 \tag{17}$$

$$\theta(0)=1,\quad \theta(\infty)=0. \tag{18}$$

In addition to the boundary conditions (17) and (18), the requirements

$$f'(\eta)\geq 0,\qquad \theta(\eta)\geq 0,\ \forall\,\eta\geq 0 \tag{19}$$

must also satisfied [17].

3 Basic ideas of optimal homotopy asymptotic method

Equations (12) (or (13)) with boundary conditions (17) (or (18)) can be written in a more general form:

$$N\left(\Phi(\eta)\right)=0 \tag{20}$$

where N is a given nonlinear differential operator depending on the unknown function $\Phi(\eta)$, subject to the initial/boundary conditions:

$$B\left(\Phi(\eta),\frac{d\Phi(\eta)}{d\eta}\right)=0. \tag{21}$$

It is clear that $\Phi(\eta) = f(\eta)$ or $\Phi(\eta) = \theta(\eta)$.

Let $\Phi_0(\eta)$ be an initial approximation of $\Phi(\eta)$ and L an arbitrary linear operator such as:

$$L\Big(\Phi_0(\eta)\Big) = 0, \quad B\Big(\Phi_0(\eta), \frac{d\Phi_0(\eta)}{d\eta}\Big) = 0. \qquad (22)$$

We remark that this operator L is not unique.

If $p \in [0, 1]$ denotes an embedding parameter and F is a function, then we propose to construct a homotopy [29] - [33]:

$$\mathcal{H}\Big[L\Big(F(\eta, p)\Big), \ H(\eta, C_i), \ N\Big(F(\eta, p)\Big)\Big] \qquad (23)$$

with the following two properties:

$$\mathcal{H}\Big[L\Big(F(\eta, 0)\Big), \ H(\eta, C_i), \ N\Big(F(\eta, 0)\Big)\Big] = \qquad (24)$$
$$= L\Big(F(\eta, 0)\Big) = L\Big(\Phi_0(\eta)\Big)$$

$$\mathcal{H}\Big[L\Big(F(\eta, 1)\Big), \ H(\eta, C_i), \ N\Big(F(\eta, 1)\Big)\Big] \qquad (25)$$
$$= H(\eta, C_i)N\Big(\Phi(\eta)\Big)$$

where $H(\eta, C_i) \neq 0$, is an arbitrary auxiliary convergence-control function depending on variable η and on a number of arbitrary parameters C_1, C_2, ..., C_m which ensure the convergence of the approximate solution.

Let us consider the function F in the form:

$$F(x, p) = \Phi_0(\eta) + p\Phi_1(\eta, C_i) + p^2\Phi_2(\eta, C_i) + \dots \qquad (26)$$

By substituting equation (26) into the equation obtained by means of the homotopy (23), then:

$$\mathcal{H}\Big[L\Big(F(\eta, p)\Big), \ H(\eta, C_i), \ N\Big(F(\eta, p)\Big)\Big] = 0 \qquad (27)$$

and equating the coefficients of like powers of p, we obtain the governing equation of $\Phi_0(x)$ given by equation (22) and the governing equation of $\Phi_1(\eta, C_i)$, $\Phi_2(\eta, C_i)$ and so on. If the series (26) is convergent at $p = 1$, one has:

$$F(\eta, 1) = \Phi_0(\eta) + \Phi_1(\eta, C_i) + \Phi_2(\eta, C_i) + \dots \qquad (28)$$

But in particular we consider only the first-order approximate solution:

$$\overline{\Phi}(\eta, C_i) = \Phi_0(\eta) + \Phi_1(\eta, C_i), \quad i = 1, 2, \dots, m \qquad (29)$$

and the homotopy (23) in the form:

$$\mathcal{H}\Big[L\Big(F(\eta, p)\Big), \ H(\eta, C_i), \ N\Big(F(\eta, p)\Big)\Big] = L\Big(\Phi_0(\eta)\Big) + \qquad (30)$$
$$+ p\Big[L\Big(\Phi_1(\eta, C_i)\Big) - L\Big(\Phi_0(\eta)\Big) - H(\eta, C_i)N\Big(\Phi_0(\eta)\Big)\Big].$$

Equating only the coefficients of p^0 and p^1 into equation (30), we obtain the governing equation of $\Phi_0(\eta)$ given by equation (22) and the governing equation of $\Phi_1(\eta, C_i)$ *i.e.*:

$$L\Big(\Phi_1(\eta, C_i)\Big) = H(\eta, C_i)N\Big(\Phi_0(\eta)\Big), \qquad (31)$$
$$B\Big(\Phi_1(\eta, C_i), \frac{d\Phi_1(\eta, C_i)}{d\eta}\Big) = 0, \quad i = 1, 2, \dots, m.$$

It should be emphasized that $\Phi_0(\eta)$ and $\Phi_1(\eta, C_i)$ are governed by the linear equations (22) and (31) respectively, with boundary conditions that come from the original problem, which can be easily solved. The convergence of the approximate solution (29) depends upon the auxiliary convergence-control function $H(\eta, C_i)$. There are many possibilities to choose the auxiliary function $H(\eta, C_i)$. Basically, the shape of $H(\eta, C_i)$ must follow the terms appearing in equation (31). Therefore, we try to choose $H(\eta, C_i)$ so that in equation (31) the product $H(\eta, C_i)N\Big(\Phi_0(\eta)\Big)$ will be the same shape with $N\Big(\Phi_0(\eta)\Big)$. Now, by substituting equation (29) into equation (20), the following residual is given:

$$R(\eta, C_i) = N\Big(\overline{\Phi}(\eta, C_i)\Big). \qquad (32)$$

At this moment, the first-order approximate solution given by equation (29) depends on the parameters C_1, C_2, ..., C_m and these parameters can be optimally identified via various methods, such as the least square method, the Galerkin method, the Kantorowich method, the collocation method or by minimizing the square residual error:

$$J(C_1, C_2, \dots, C_m) = \int_a^b R^2(\eta, C_1, C_2, \dots, C_m)\, d\eta \qquad (33)$$

where a and b are two values depending on the given problem. The unknown parameters $C_1, C_2, \dots, C_m$ can be identified from the conditions:

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \dots = \frac{\partial J}{\partial C_m} = 0. \qquad (34)$$

With these parameters known (namely convergence-control parameters), the first-order approximate solution (29) is well-determined.

4 Application of OHAM to flow and heat transfer

We use the basic ideas of the OHAM by considering equation (12) with the boundary conditions given by equation

(17). We can choose the linear operator in the form:

$$L_f\left(\Phi(\eta)\right) = \Phi''' - K^2\Phi', \tag{35}$$

where $K > 0$ is an unknown parameter at this moment.

Here, we state that the linear operator is not unique. Equation (22) becomes:

$$\Phi_0''' - K^2\Phi_0' = 0, \quad \Phi_0(0) = f_w, \quad \Phi_0'(0) = 1, \quad \Phi_0'(\infty) = 0.$$

which has the following solution:

$$\Phi_0(\eta) = f_w + \frac{1 - e^{-K\eta}}{K}. \tag{36}$$

The nonlinear operator $N_f\left(\Phi(\eta)\right)$ is obtained from equation (12):

$$N_f\left(\Phi(\eta)\right) = \Phi'''(\eta) + \Phi(\eta)\Phi''(\eta) - \Phi'(\eta)^2 + \Lambda\left(\Phi'(\eta) + \frac{1}{2}\eta\Phi''(\eta)\right) = 0 \tag{37}$$

such that substituting equation (36) into equation (37), we obtain

$$N_f\left(\Phi_0(\eta)\right) = (\alpha\eta + \beta)e^{-K\eta} \tag{38}$$

where

$$\alpha = \frac{1}{2}K\Lambda; \qquad \beta = K^2 - 1 - Kf_w - \Lambda. \tag{39}$$

Having in view that in equation (38) there is an exponential function and that the auxiliary function $H_f(\eta, C_i)$ must follow the terms appearing in equation (38), then we can choose the function $H_f(\eta, C_i)$ in the following forms:

$$H_f(\eta, C_i) = C_1 + C_2\eta + (C_3 + C_4\eta)e^{-K\eta} + (C_5 + C_6\eta)e^{-2K\eta} \tag{40}$$

or

$$H_f^{\star}(\eta, C_i) = C_1 + (C_2 + C_3\eta + C_4\eta^2)e^{-K\eta} \tag{41}$$

or yet

$$H_f^{\star\star}(\eta, C_i) = C_1 + C_2\eta + C_3\eta^2 + (C_4 + C_5\eta)e^{-K\eta} + (C_6 + C_7\eta + C_8\eta^2)e^{-2K\eta} \tag{42}$$

and so on, where $C_1, C_2, \ldots$ are unknown parameters at this moment.

If we choose only the expression (40) for $H_f(\eta, C_i)$, then by using equations (38), (40) and (31), we can obtain the equation in $\Phi_1(\eta, C_i)$:

$$\begin{aligned}\Phi_1''' - K^2\Phi_1' = {} & \left[\beta C_1 + (\alpha C_1 + \beta C_2)\eta + \alpha C_2\eta^2\right]e^{-K\eta} \\ & + \left[\beta C_3 + (\alpha C_3 + \beta C_4)\eta + \alpha C_4\eta^2\right]e^{-2K\eta} \\ & + \left[\beta C_5 + (\alpha C_5 + \beta C_6)\eta + \alpha C_6\eta^2\right]e^{-3K\eta}, \\ \Phi_1(0) = {} & \Phi_1'(0) = \Phi_1'(\infty) = 0.\end{aligned} \tag{43}$$

The solution of equation (43) can be found as:

$$\begin{aligned}\Phi_1(\eta) = {} & M_1 + \Big[N_1 + \Big(\frac{7\alpha C_2}{4K^4} + \frac{3\alpha C_1}{4K^3} + \frac{3\beta C_2}{4K^3} \\ & + \frac{\beta C_1}{2K^2}\Big)\eta + \Big(\frac{3\alpha C_2}{4K^3} + \frac{\alpha C_1}{4K^2} + \frac{\beta C_2}{4K^2}\Big)\eta^2 + \frac{\alpha C_2}{6K^2}\eta^3\Big]e^{-K\eta} \\ & + \Big[-\frac{85\alpha C_4}{108K^5} - \frac{11\alpha C_3}{36K^4} - \frac{11\beta C_4}{36K^4} - \frac{\beta C_3}{6K^3} - \Big(\frac{11\alpha C_4}{18K^4} \\ & + \frac{\alpha C_3}{6K^3} + \frac{\beta C_4}{6K^3}\Big)\eta - \frac{\alpha C_4}{6K^3}\eta^2\Big]e^{-2K\eta} + \Big(-\frac{115\alpha C_6}{1728K^5} \\ & - \frac{13\alpha C_5}{288K^4} - \frac{13\beta C_6}{288K^4} - \frac{\beta C_5}{24K^3}\Big)e^{-K\eta}\end{aligned} \tag{44}$$

where

$$\begin{aligned}M_1 = {} & -\frac{3\alpha + 2K\beta}{4K^4}C_1 - \frac{7\alpha + 3K\beta}{4K^5}C_2 - \frac{5\alpha + 6K\beta}{36K^4}C_3 - \\ & - \frac{19\alpha + 15K\beta}{108K^5}C_4 - \frac{7\alpha + 12K\beta}{144K^4}C_5 - \frac{37\alpha + 42K\beta}{864K^5}C_6 \\ N_1 = {} & \frac{3\alpha + 2K\beta}{4K^4}C_1 + \frac{7\alpha + 3K\beta}{4K^5}C_2 + \frac{4\alpha + 3K\beta}{9K^4}C_3 + \\ & + \frac{26\alpha + 12K\beta}{27K^5}C_4 + \frac{3\alpha + 4K\beta}{32K^4}C_5 + \frac{7\alpha + 6K\beta}{64K^5}C_6.\end{aligned} \tag{45}$$

The first-order approximate solution (29) for equations (12) and (17) is obtained from equations (36) and (45):

$$\bar{f}(\eta) = \bar{\Phi}(\eta) = \Phi_0(\eta) + \Phi_1(\eta). \tag{46}$$

In what follows, we consider equations (13) and (18). In this case, we choose the linear operator in the form:

$$L_\theta\left(\varphi(\eta)\right) = \varphi'' + K\varphi' \tag{47}$$

where the parameter K is defined in equation (35).

Equation (22) becomes:

$$\varphi_0'' + K\varphi_0' = 0, \quad \varphi_0(0) = 1, \quad \varphi_0(\infty) = 0. \tag{48}$$

Equation (48) has the solution

$$\varphi_0(\eta) = e^{-K\eta}. \tag{49}$$

The nonlinear operator $N_\theta\left(\varphi(\eta)\right)$ is obtained from equation (12):

$$N_\theta\left(\varphi(\eta)\right) = \frac{1}{Pr}\varphi'' + \Phi\varphi' - n\Phi'\varphi + \Lambda\left(c\varphi + \frac{1}{2}\eta\varphi'\right). \tag{50}$$

Substituting equation (49) into equation (50), we obtain:

$$N_\theta\left(\varphi_0(\eta)\right) = (m_1\eta + m_2)e^{-K\eta} + m_3e^{-2K\eta} \tag{51}$$

where

$$m_1 = -\frac{1}{2}K\Lambda; \quad m_2 = \frac{K^2}{Pr} - Kf_w - 1 + c\Lambda; \tag{52}$$
$$m_3 = 1 - n.$$

The auxiliary function $H_\theta(\eta, C_i)$ can be chosen in the forms:

$$H_\theta(\eta, C_i) = C_7 + C_8\eta + (C_9 + C_{10}\eta)e^{-K\eta} + (C_{11} + C_{12}\eta)e^{-2K\eta} \tag{53}$$

or

$$H_\theta^\star(\eta, C_i) = C_7 + C_8\eta + C_9\eta^2 + (C_{10} + C_{11}\eta)e^{-K\eta} + C_{13}e^{-2K\eta} \tag{54}$$

or yet

$$H_\theta^{\star\star}(\eta, C_i) = C_7 + (C_8 + C_9\eta)e^{-K\eta} + (C_{10} + C_{11}\eta)e^{-2K\eta} \tag{55}$$

and so on, where $C_7, C_8, \ldots$ are unknown parameters.

If we choose equation (53) for H_θ, then from equations (51), (53) and (31) we obtain the equation in $\varphi_1(\eta, C_i)$ as

$$\begin{aligned}\varphi_1'' + K\varphi_1' = {} & \left[m_2C_7 + (m_1C_7 + m_2C_8)\eta + m_1C_8\eta^2\right] \\ & \cdot e^{-K\eta} + \left[m_2C_9 + m_3C_7 + (m_1C_9 + m_2C_{10} \right. \\ & \left. + m_3C_8)\eta + m_1C_{10}\eta^2\right]e^{-2K\eta} + \left[m_3C_9 \right. \\ & + m_2C_{11} + (m_3C_{10} + m_1C_{11} + m_2C_{12})\eta \\ & \left. + m_1C_{12}\eta^2\right]e^{-3K\eta} + (m_3C_{11} + m_3C_{12}\eta)e^{-4K\eta},\end{aligned} \tag{56}$$
$$\phi_1(0) = \phi_1(\infty) = 0.$$

Solving equation (56), we obtain:

$$\begin{aligned}\varphi_1(\eta) = {} & \left[P_1 - \left(\frac{2m_1C_8}{K^3} + \frac{m_1C_7}{K^2} + \frac{m_2C_8}{K^2} + \frac{m_2C_7}{K}\right)\eta \right. \\ & \left. - \left(\frac{m_1C_8}{K^2} + \frac{m_1C_7}{2K} + \frac{m_2C_8}{2K}\right)\eta^2 - \frac{m_1C_8}{3K}\eta^3\right]e^{-K\eta} \\ & + \left[\frac{7m_1C_{10}}{4K^4} + \frac{3m_1C_9}{4K^3} + \frac{3m_2C_{10}}{4K^3} + \frac{3m_3C_8}{4K^3} \right. \\ & + \frac{m_2C_9}{2K^2} + \frac{m_3C_7}{2K^2} + \left(\frac{3m_1C_{10}}{2K^3} + \frac{m_1C_9}{2K^2} + \frac{m_2C_{10}}{2K^2} \right. \\ & \left.\left. + \frac{m_3C_8}{2K^2}\right)\eta + \frac{m_1C_{10}}{2K^2}\eta^2\right]e^{-2K\eta} \\ & + \left[\frac{5(m_3C_{10} + m_1C_{11} + m_2C_{12})}{36K^3} + \frac{m_3C_9 + m_2C_{11}}{6K^2} \right. \\ & + \left(\frac{m_3C_{10} + m_1C_{11} + m_2C_{12}}{6K^2} + \frac{5m_1C_{12}}{18K^3}\right)\eta \\ & \left. + \frac{m_1C_{12}}{6K^2}\eta^2\right]e^{-3K\eta} + \left(\frac{7m_3C_{11}}{12K^2} + \frac{7m_3C_{12}}{144K^3} \right. \\ & \left. + \frac{m_3C_{12}}{12K^2}\eta\right)e^{-4K\eta}, \quad \phi_1(0) = \phi_1(\infty) = 0,\end{aligned} \tag{57}$$

where

$$\begin{aligned}P_1 = {} & -\frac{m_3C_7}{2K^2} - \frac{3m_3C_8}{4K^3} - \frac{9m_1 + 6Km_2 + 2Km_3}{12K^3}C_9 \\ & - \frac{63m_1 + 27Km_2 + 5Km_3}{36K^4}C_{10} - \frac{5m_1 + K(m_2 + 3m_3)}{36K^3}C_{11} \\ & - \frac{20m_2 + 7m_3}{144K^3}C_{12}.\end{aligned} \tag{58}$$

In this way, the first-order approximate solution (29) for equations (13) and (18) becomes

$$\overline{\theta}(\eta) = \overline{\varphi}(\eta) = \varphi_0(\eta) + \varphi_1(\eta, C_i). \tag{59}$$

5 Numerical examples

In order to prove the accuracy of the obtained results, we will determine the convergence-control parameters K and C_i which appear in equations (46) and (59) by means of the least square method. In this way, the convergence-control parameters are optimally determined and the first-order approximate solutions known for different values of the known parameters f_w, Λ, Pr, n and c. In what follows, we illustrate the accuracy of the OHAM by comparing previously obtained approximate solutions with the numerical integration results computed by means of the shooting method combined with fourth-order Runge-Kutta method using Wolfram Mathematica 6.0 software. For some values of the parameters f_w, Λ, Pr, n and c we will determine the approximate solutions.

Example 5.1.a In the first case we consider $f_w = -1$, $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 0.7$. For equation (46), following the procedure described above, the following convergence-control parameters are obtained:

$$\begin{aligned}& C_1 = -0.0881661632, \quad C_2 = 0.0159074525, \\ & C_3 = 101.5499315816, \quad C_4 = -16.3157319695, \\ & C_5 = -99.5951678657, \quad C_6 = -64.5910051875 \\ & K = 0.7591636981\end{aligned}$$

and consequently the first-order approximate solution (46) can be written in the form:

$$\begin{aligned}\overline{f}(\eta) = {} & 0.4921333156 + (-0.5668554881 \\ & + 0.0071536463\eta - 0.0087518103\eta^2 \\ & + 0.0017461596\eta^3)e^{-0.7591636981\eta} \\ & + (-0.3980837570 - 7.4190413787\eta \\ & + 2.3591440003\eta^2)e^{-1.5183273963\eta} \\ & + (-0.5271940704 + 6.1764503488\eta \\ & + 2.3348551362\eta^2)e^{-2.2774910945\eta}\end{aligned} \tag{60}$$

Table 1: Comparison between OHAM results given by equation (60) and numerical results for $f_w = -1$, $\Lambda = 1$.

η	f_{numeric}	$\bar{f}_{\text{OHAM}}$, Eq. (60)	relative error = $\lvert f_{\text{numeric}} - \bar{f}_{\text{OHAM}}\rvert$
0	−1	−0.9999999999	$1.88 \cdot 10^{-15}$
1	−0.1497942276	−0.1501421749	$3.47 \cdot 10^{-4}$
2	0.3108643384	0.3106655367	$1.98 \cdot 10^{-4}$
3	0.4604991620	0.4600705386	$4.28 \cdot 10^{-4}$
4	0.4887865463	0.4894195278	$6.32 \cdot 10^{-4}$
5	0.4919308455	0.4920394273	$1.08 \cdot 10^{-4}$
6	0.4921388939	0.4918417250	$2.97 \cdot 10^{-4}$
7	0.4921471111	0.4919782168	$1.68 \cdot 10^{-4}$
8	0.4921472622	0.4922151064	$6.78 \cdot 10^{-5}$
9	0.4921472290	0.4923440748	$1.96 \cdot 10^{-4}$
10	0.4921472001	0.4923640175	$2.16 \cdot 10^{-4}$

Table 2: Comparison between OHAM results given by equation (61) and numerical results for $f_w = -1$, $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 0.7$.

η	θ_{numeric}	$\bar{\theta}_{\text{OHAM}}$, Eq. (61)	relative error = $\lvert\theta_{\text{numeric}} - \bar{\theta}_{\text{OHAM}}\rvert$
0	1	0.9999999999	$8.88 \cdot 10^{-16}$
1	0.5325816311	0.5344078544	$1.82 \cdot 10^{-3}$
2	0.2137609331	0.2123010871	$1.45 \cdot 10^{-3}$
3	0.0624485224	0.0627998626	$3.51 \cdot 10^{-4}$
4	0.0129724736	0.0141235139	$1.15 \cdot 10^{-3}$
5	0.0019027817	0.0018865990	$1.61 \cdot 10^{-5}$
6	0.0001968219	−0.0002871987	$4.84 \cdot 10^{-4}$
7	0.0000144329	−0.0002516924	$2.66 \cdot 10^{-4}$
8	$8.27 \cdot 10^{-7}$	0.0000468974	$4.60 \cdot 10^{-5}$
9	$1.08 \cdot 10^{-7}$	0.0002281868	$2.28 \cdot 10^{-4}$
10	$7.47 \cdot 10^{-8}$	0.0002807824	$2.80 \cdot 10^{-4}$

Now, for equation (59), the convergence-control parameters are:

$$C_7 = 0.0363993085, \quad C_8 = 0.0363993085,$$
$$C_9 = -7.1448075448, \quad C_{10} = 4.3237724702,$$
$$C_{11} = 42.1871800319, \quad C_{12} = 18.0805214975$$

and therefore the first-order approximate solution (59) becomes:

$$\begin{aligned}\bar{\theta}(\eta) = {}& (0.3541683003 + 0.2415876957\eta \\ & - 0.0823906873\eta^2 + \\ & + 0.0060665514\eta^3)e^{-0.7591636981\eta} \\ & + (-2.6848440528 + 6.3660521866\eta \\ & - 1.4238603876\eta^2)e^{-1.5183273963\eta} + \\ & + (3.3306757524 - 3.3281240073\eta \\ & - 1.9846972772\eta^2)e^{-2.2774910945\eta}\end{aligned} \tag{61}$$

In Tables 1 and 2 we present a comparison between the first-order approximate solutions given by equations (60) and (61) respectively, with numerical results for some values of variable η and the corresponding relative errors.

Example 5.1.b In this case, we consider $f_w = -1$, $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 2$. The solution $\bar{f}(\eta)$ is given by equation (60). The convergence-control parameters for equation (59) are:

$$C_7 = 0.3992391297, \quad C_8 = -0.0398233823,$$
$$C_9 = 13.9195482545, \quad C_{10} = -9.8140323466,$$
$$C_{11} = -37.1866029355, \quad C_{12} = -48.0634410813$$

such that the first-order approximate solution (59) becomes:

$$\begin{aligned}\bar{\theta}(\eta) = {}& (-2.1207041376 - 0.0561685488\eta \\ & + 0.0879369070\eta^2 - 0.0066372303\eta^3) \\ & \cdot e^{-0.7591636981\eta} + e^{-1.5183273963\eta}(7.9716346529- \\ & - 9.1921128520\eta + 3.2318564396\eta^2)e^{-0.7591636981\eta} \\ & + (-4.8509305153 + 8.0572360536\eta \\ & + 5.2759197606\eta^2)e^{-2.2774910945\eta}\end{aligned} \tag{62}$$

In Table 3 we present a comparison between the first-order approximate solutions given by equation (62) with numerical results and corresponding relative errors.

Example 5.2.a For $f_w = 0$, $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 0.7$, the convergence-control parameters for equation (46) are:

$$C_1 = -1.2640611927, \quad C_2 = 0.1680009020,$$
$$C_3 = -34.0575215187, \quad C_4 = 30.7898356526,$$
$$C_5 = 37.1281425060, \quad C_6 = 13.8590545976,$$
$$K = 1.1203766872$$

and therefore, the first-order approximate solution (46) can be written in the form:

$$\begin{aligned}\bar{f}(\eta) = {}& 0.9662722752 + (1.3563995648 \\ & + 0.0351604322\eta - 0.1157605059\eta^2 \\ & + 0.0124958644\eta^3)e^{-1.1203766872\eta} \\ & + (-2.5490820161 - 1.7111113870\eta \\ & - 2.0440805123\eta^2)e^{-2.2407533744\eta} + (0.2264101761- \\ & - 0.7552406743\eta - 0.2300192808\eta^2)e^{-3.3611300616\eta}\end{aligned} \tag{63}$$

For equation (59), the convergence-control parameters are:

$$C_7 = -1.6947892627, \quad C_8 = 0.2632100295,$$
$$C_9 = -1.2815579214, \quad C_{10} = 2.6268699384,$$

$$C_{11} = 15.8897673738, \quad C_{12} = 9.5966071873$$

and the first-order approximate solution (59) is:

$$\begin{aligned}\overline{\theta}(\eta) = {} & (0.2987405005 + 1.1383970916\eta \\ & - 0.4581387031\eta^2 + 0.0438683382\eta^3) \\ & \cdot e^{-1.1203766872\eta} + (-0.1000278783+ \\ & + 1.4818560845\eta - 0.5861577557\eta^2) \\ & \cdot e^{-2.2407533744\eta} + (0.8012873778- \\ & - 0.5959066165\eta - 0.7137932043\eta^2) \\ & \cdot e^{-3.3611300616\eta}\end{aligned} \tag{64}$$

In Tables 4 and 5 we present a comparison between the first-order approximate solutions given by equations (63) and (64) respectively, with numerical results and corresponding relative errors.

Table 3: Comparison between OHAM results given by equation (62) and numerical results for $f_w = -1$, $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 2$.

η	θ_{numeric}	$\overline{\theta}_{\text{OHAM}}$, Eq. (62)	relative error = $\lvert\theta_{\text{numeric}} - \overline{\theta}_{\text{OHAM}}\rvert$
0	1	0.9999999999	$1.77 \cdot 10^{-15}$
1	0.3341908857	0.3295767993	$4.61 \cdot 10^{-3}$
2	0.0347540513	0.0372485103	$2.49 \cdot 10^{-3}$
3	0.0011305745	−0.0002320988	$1.36 \cdot 10^{-3}$
4	0.0000127262	−0.0002852788	$2.98 \cdot 10^{-4}$
5	$-5.44 \cdot 10^{-8}$	0.0002991504	$2.99 \cdot 10^{-4}$
6	$-9.15 \cdot 10^{-8}$	0.0002884740	$2.88 \cdot 10^{-4}$
7	$-7.96 \cdot 10^{-8}$	0.0001372878	$1.37 \cdot 10^{-4}$
8	$-7.05 \cdot 10^{-8}$	−0.0000295212	$2.94 \cdot 10^{-5}$
9	$-6.53 \cdot 10^{-8}$	−0.0001505702	$1.50 \cdot 10^{-4}$
10	$-5.99 \cdot 10^{-8}$	−0.0002044072	$2.04 \cdot 10^{-4}$

Table 4: Comparison between OHAM results given by equation (63) and numerical results for $f_w = 0$, $\Lambda = 1$.

η	f_{numeric}	$\overline{f}_{\text{OHAM}}$, Eq. (63)	relative error = $\lvert f_{\text{numeric}} - \overline{f}_{\text{OHAM}}\rvert$
0	$-5.50 \cdot 10^{-21}$	$4.44 \cdot 10^{-16}$	$4.44 \cdot 10^{-16}$
1	0.6894348341	0.6894914970	$5.66 \cdot 10^{-5}$
2	0.9167696529	0.9166682157	$1.01 \cdot 10^{-4}$
3	0.9608821303	0.9609858144	$1.03 \cdot 10^{-4}$
4	0.9659196704	0.9659030730	$1.65 \cdot 10^{-5}$
5	0.9662619960	0.9661631962	$9.87 \cdot 10^{-5}$
6	0.9662759513	0.9662663279	$9.62 \cdot 10^{-6}$
7	0.9662762950	0.9663395358	$6.32 \cdot 10^{-5}$
8	0.9662763018	0.9663501433	$7.38 \cdot 10^{-5}$
9	0.9662763032	0.9663306688	$5.43 \cdot 10^{-5}$
10	0.9662763043	0.9663080318	$3.17 \cdot 10^{-5}$

Table 5: Comparison between OHAM results given by equation (64) and numerical results for $f_w = 0$, $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 0.7$.

η	θ_{numeric}	$\overline{\theta}_{\text{OHAM}}$, Eq. (64)	relative error = $\lvert\theta_{\text{numeric}} - \overline{\theta}_{\text{OHAM}}\rvert$
0	1	1.00	$2.22 \cdot 10^{-16}$
1	0.4003127445	0.4006174190	$3.04 \cdot 10^{-4}$
2	0.1184290061	0.1183366534	$9.23 \cdot 10^{-5}$
3	0.0250358122	0.0254649591	$4.29 \cdot 10^{-4}$
4	0.0037386526	0.0032572108	$4.81 \cdot 10^{-4}$
5	0.0003935287	−0.0000242886	$4.17 \cdot 10^{-4}$
6	0.0000291963	0.0001165652	$8.73 \cdot 10^{-5}$
7	$1.53 \cdot 10^{-6}$	0.0003370019	$3.35 \cdot 10^{-4}$
8	$6.39 \cdot 10^{-8}$	0.0003255701	$3.25 \cdot 10^{-4}$
9	$8.43 \cdot 10^{-9}$	0.0002261157	$2.26 \cdot 10^{-4}$
10	$6.39 \cdot 10^{-9}$	0.0001326393	$1.32 \cdot 10^{-4}$

Example 5.2.b For $f_w = 0$, $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 2$ the firs-order approximate solution (46) is given by equation (63). The convergence-control parameters for equation (59) are determined as:

$$C_7 = 0.4652101281, \quad C_8 = -0.0724620728,$$
$$C_9 = 14.4924736065, \quad C_{10} = -12.1643720147,$$
$$C_{11} = 0.7277740132, \quad C_{12} = -56.0450834796$$

such that the first-order approximate solution (59) may be written as:

$$\begin{aligned}\overline{\theta}(\eta) = {} & (-1.4030799687 + 0.1042610535\eta \\ & + 0.0880913412\eta^2 - 0.0120770121\eta^3) \\ & \cdot e^{-1.1203766872\eta} + (3.1476673319- \\ & - 3.8089261057\eta + 2.7143486992\eta^2) \\ & \cdot e^{-2.2407533744\eta} + (-0.7445873632+ \\ & + 5.1973912018\eta + 4.1686190694\eta^2)e^{-3.3611300616\eta}\end{aligned} \tag{65}$$

In Table 6 we present a comparison between the first-order approximate solutions given by equation (65) with numerical results. The corresponding relative errors are also presented.

Example 5.3.a We consider $f_w = 1$, $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 0.7$. The convergence-control parameters for equation (46) are given by:

$$C_1 = 0.6287723857, \quad C_2 = -0.1379103919,$$
$$C_3 = -64.6127509553, \quad C_4 = 52.1862259014,$$
$$C_5 = 65.7049797786, \quad C_6 = 66.9471031457,$$
$$K = 1.6976766716.$$

Table 6: Comparison between OHAM results given by equation (65) and numerical results for $f_w = 0$, $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 2$.

η	θ_{numeric}	$\overline{\theta}_{\text{OHAM}}$, Eq. (65)	relative error = $\lvert\theta_{\text{numeric}} - \overline{\theta}_{\text{OHAM}}\rvert$
0	1	1	0
1	0.1197281855	0.1187072441	$1.02 \cdot 10^{-3}$
2	0.0042249398	0.0041010249	$1.23 \cdot 10^{-4}$
3	$5.09 \cdot 10^{-5}$	$-6.04 \cdot 10^{-6}$	$5.69 \cdot 10^{-5}$
4	$2.41 \cdot 10^{-7}$	0.0001841967	$1.83 \cdot 10^{-4}$
5	$1.67 \cdot 10^{-8}$	0.0000163574	$1.63 \cdot 10^{-5}$
6	$1.42 \cdot 10^{-8}$	−0.0001452859	$1.45 \cdot 10^{-4}$
7	$1.26 \cdot 10^{-8}$	−0.0001791057	$1.79 \cdot 10^{-4}$
8	$1.15 \cdot 10^{-8}$	−0.0001403309	$1.40 \cdot 10^{-4}$
9	$1.09 \cdot 10^{-8}$	−0.0000887807	$8.87 \cdot 10^{-5}$
10	$1.05 \cdot 10^{-8}$	−0.0000493843	$4.93 \cdot 10^{-5}$

such that the first-order approximate solution (46) may be written as:

$$\overline{f}(\eta) = 1.6119245343 + (-0.1095284867 - 0.0145745495\eta + 0.0381089526\eta^2 - 0.0067695650\eta^3)e^{-1.6976766716\eta} + (-0.6841717456 + 0.0590171180\eta - 1.5089146740\eta^2)e^{-3.3953533432\eta} + (0.1817756979 - 0.6276022634\eta - 0.4839278208\eta^2)e^{-5.0930300148\eta} \tag{66}$$

The convergence-control parameters for equation (59), are:

$$C_7 = -2.4317156290, \quad C_8 = 0.4199773974,$$
$$C_9 = 3.2538989273, \quad C_{10} = 2.8819189890,$$
$$C_{11} = 2.6307320394, \quad C_{12} = 0.0503403055$$

and the first-order approximate solution (59) becomes:

$$\overline{\theta}(\eta) = (0.0191058146 + 1.8994242629\eta - 0.7216780879\eta^2 + 0.0699962329\eta^3) \cdot e^{-1.6976766716\eta} + (0.9928667494 + 0.3712879543\eta - 0.4243916166\eta^2) \cdot e^{-3.3953533432\eta} + (-0.0119725641 - 0.1259716711\eta - 0.0024710391\eta^2) \cdot e^{-5.0930300148\eta} \tag{67}$$

In Tables 7 and 8 we present a comparison between the first-order approximate solutions given by equations (66) and (67) respectively, with numerical results and corresponding relative errors.

Table 7: Comparison between OHAM results given by equation (66) and numerical results for $f_w = 1$, $\Lambda = 1$.

η	f_{numeric}	$\overline{f}_{\text{OHAM}}$, Eq. (66)	relative error = $\lvert f_{\text{numeric}} - \overline{f}_{\text{OHAM}}\rvert$
0	1	1.00	$2.22 \cdot 10^{-16}$
1	1.5177074192	1.5176780223	$2.93 \cdot 10^{-5}$
2	1.6030516967	1.6030350430	$1.66 \cdot 10^{-5}$
3	1.6114161917	1.6114348208	$1.86 \cdot 10^{-5}$
4	1.6119056438	1.6119031833	$2.46 \cdot 10^{-6}$
5	1.6119228465	1.6119073012	$1.55 \cdot 10^{-5}$
6	1.6119232066	1.6119136285	$9.57 \cdot 10^{-6}$
7	1.6119232084	1.6119199331	$3.27 \cdot 10^{-6}$
8	1.6119232063	1.6119229505	$2.55 \cdot 10^{-7}$
9	1.6119232045	1.6119240509	$8.46 \cdot 10^{-7}$
10	1.6119232031	1.6119243982	$1.19 \cdot 10^{-6}$

Table 8: Comparison between OHAM results given by equation (67) and numerical results for $f_w = 1$, $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 0.7$.

η	θ_{numeric}	$\overline{\theta}_{\text{OHAM}}$, Eq. (67)	relative error = $\lvert\theta_{\text{numeric}} - \overline{\theta}_{\text{OHAM}}\rvert$
0	1	1	0
1	0.2625870984	0.2626175913	$3.04 \cdot 10^{-5}$
2	0.0500304293	0.0500306615	$2.32 \cdot 10^{-7}$
3	0.0067456425	0.0067634155	$1.77 \cdot 10^{-5}$
4	0.0006411532	0.0006125219	$2.86 \cdot 10^{-5}$
5	0.0000429270	0.0000457402	$2.81 \cdot 10^{-6}$
6	$2.01 \cdot 10^{-6}$	$2.08 \cdot 10^{-5}$	$1.88 \cdot 10^{-5}$
7	$5.14 \cdot 10^{-8}$	$1.35 \cdot 10^{-5}$	$1.34 \cdot 10^{-5}$
8	$-1.29 \cdot 10^{-8}$	$6.14 \cdot 10^{-6}$	$6.16 \cdot 10^{-6}$
9	$-1.33 \cdot 10^{-8}$	$2.24 \cdot 10^{-6}$	$2.25 \cdot 10^{-6}$
10	$-1.22 \cdot 10^{-8}$	$7.13 \cdot 10^{-7}$	$7.25 \cdot 10^{-7}$

Example 5.3.b For $f_w = 1$, $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 2$ the first-order approximate solution for $\overline{f}(\eta)$ is given by equation (66).

For equation (59) the convergence-control parameters are given by:

$$C_7 = -0.3023817542, \quad C_8 = 0.0519503347,$$
$$C_9 = -27.1653468182, \quad C_{10} = 23.1898068679,$$
$$C_{11} = -15.3543205956, \quad C_{12} = 31.0226487950.$$

such that the first-order approximate solution (59) may be written as:

$$\overline{\theta}(\eta) = (0.9205703595 - 0.1921602046\eta - 0.0487183578\eta^2 + 0.0086583891\eta^3) \cdot e^{-1.6976766716\eta} + (0.2637846962 + 0.9429053366\eta - 3.4149327807\eta^2) \tag{68}$$

$$\cdot e^{-3.3953533432\eta} + (-0.1843550558-$$
$$-2.0986586159\eta - 1.5227992326\eta^2)$$
$$\cdot e^{-5.0930300148\eta}$$

In Table 9 we present a comparison between the first-order approximate solutions given by equations (68) with numerical results. The corresponding relative errors are presented.

Table 9: Comparison between OHAM results given by equation (68) and numerical results for $f_w = 1$, $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 2$.

η	θ_{numeric}	$\overline{\theta}_{\text{OHAM}}$, Eq. (68)	relative error = $\lvert\theta_{\text{numeric}} - \overline{\theta}_{\text{OHAM}}\rvert$
0	1	1.00	$2.22 \cdot 10^{-16}$
1	0.0288461240	0.0286378502	$2.08 \cdot 10^{-4}$
2	0.0002627867	0.0004342087	$1.71 \cdot 10^{-4}$
3	$1.00 \cdot 10^{-6}$	$-1.91 \cdot 10^{-4}$	$1.91 \cdot 10^{-4}$
4	$1.21 \cdot 10^{-7}$	$-1.46 \cdot 10^{-4}$	$1.46 \cdot 10^{-4}$
5	$1.05 \cdot 10^{-7}$	$-3.96 \cdot 10^{-5}$	$3.97 \cdot 10^{-5}$
6	$9.39 \cdot 10^{-8}$	$-4.54 \cdot 10^{-6}$	$4.63 \cdot 10^{-6}$
7	$8.68 \cdot 10^{-8}$	$1.08 \cdot 10^{-6}$	$9.96 \cdot 10^{-7}$
8	$7.95 \cdot 10^{-8}$	$8.82 \cdot 10^{-7}$	$8.02 \cdot 10^{-7}$
9	$7.44 \cdot 10^{-8}$	$3.60 \cdot 10^{-7}$	$2.85 \cdot 10^{-7}$
10	$6.91 \cdot 10^{-8}$	$1.18 \cdot 10^{-7}$	$4.89 \cdot 10^{-8}$

In Figs 1 and 2 are plotted the profiles of $\overline{f}(\eta)$ and velocity profile $\overline{f}'(\eta)$ respectively for different values of f_w. It is clear that the solution $\overline{f}(\eta)$ increases with an increase of f_w and the velocity decreases with an increase of f_w. The condition $\overline{f}'(\eta) > 0$ for $\eta > 0$ is satisfied.

In Figs. 3–7 are plotted the temperature profiles given for two values of Prandtl number $Pr = 0.7$ and $Pr = 2$ respectively and different values of f_w. From Figs 3 and 4

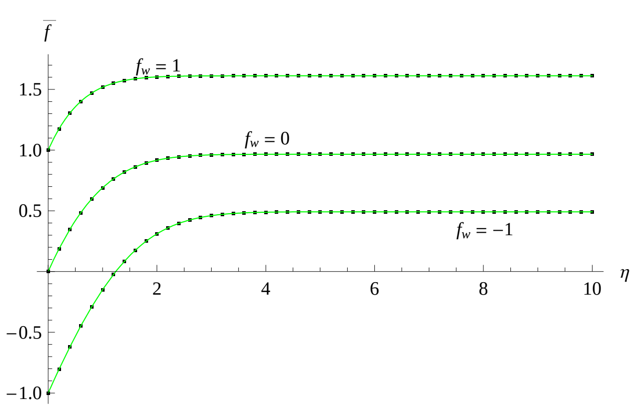

Figure 1: Solutions $\overline{f}_{OHAM}(\eta)$ given by (60), (63) and (66) for different values of f_w
— numerical solution; ··· OHAM solution.

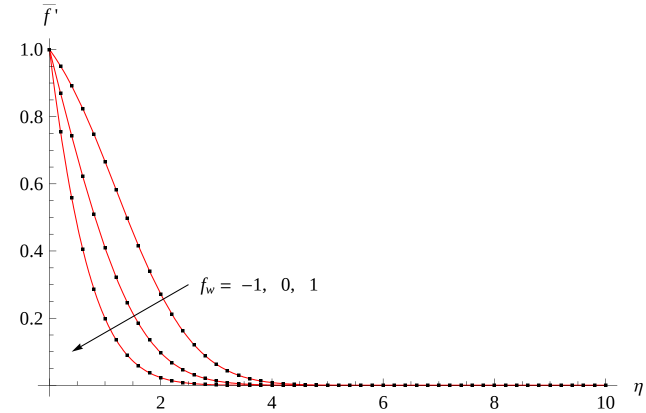

Figure 2: Solutions $\overline{f}'_{OHAM}(\eta)$ obtained from (60), (63) and (66) for different values of f_w
— numerical solution; ··· OHAM solution.

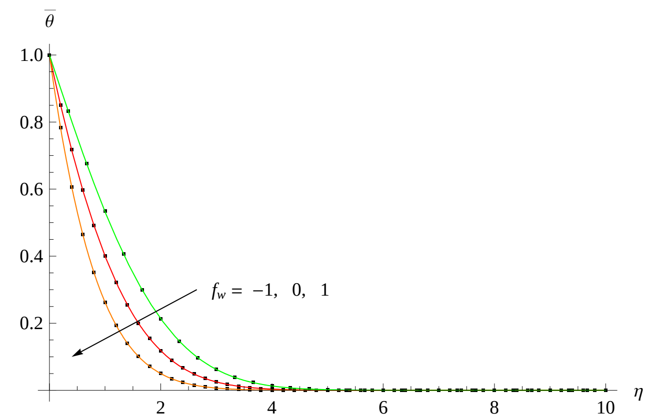

Figure 3: Plots of $\overline{\theta}_{OHAM}(\eta)$ given by Eqs. (61), (64) and (67) for $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 0.7$ and three values of f_w
— numerical solution; ··· OHAM solution.

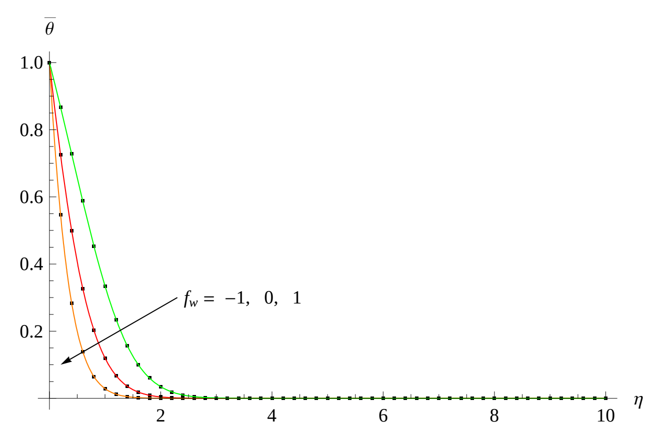

Figure 4: Plots of $\overline{\theta}_{OHAM}(\eta)$ given by Eqs. (62), (65) and (68) for $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 2$ and three values of f_w
— numerical solution; ··· OHAM solution.

it is observed that the temperature $\overline{\theta}(\eta)$ decreases with an increase of the f_w for any values of parameter Pr.

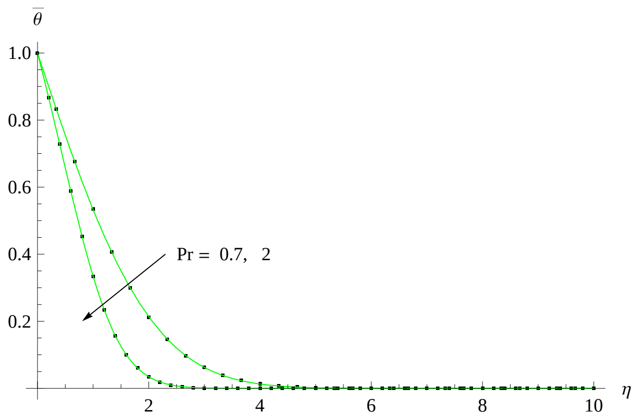

Figure 5: Plots of $\overline{\theta}_{OHAM}(\eta)$ given by Eqs. (61) and (62) for $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $f_w = -1$ and two values of Pr
— numerical solution; ··· OHAM solution.

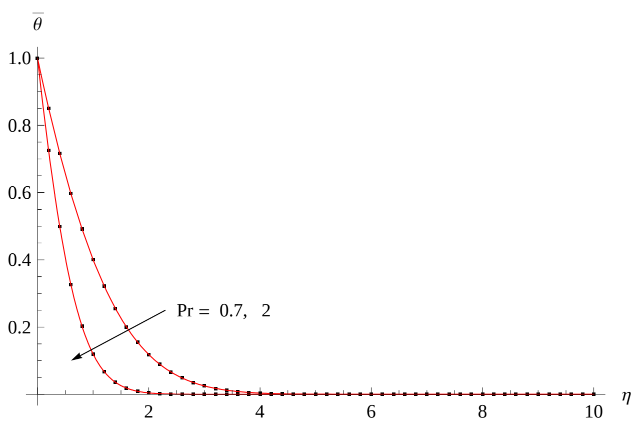

Figure 6: Plots of $\overline{\theta}_{OHAM}(\eta)$ given by Eqs. (64) and (65) for $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $f_w = 0$ and two values of Pr
— numerical solution; ··· OHAM solution.

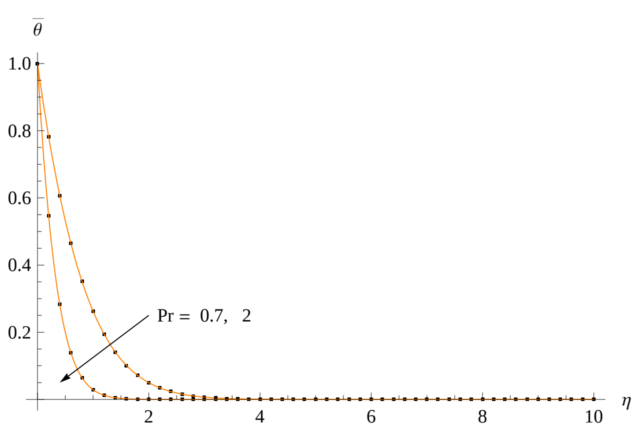

Figure 7: Plots of $\overline{\theta}_{OHAM}(\eta)$ given by Eqs. (67) and (68) for $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $f_w = 1$ and two values of Pr
— numerical solution; ··· OHAM solution.

From Figs. 5–7 we can conclude that the temperature decreases with the Prandtl number and different values of f_w.

From Tables 1–9 we can summarize that the results obtained by means of OHAM are accurate in comparison with the numerical results.

6 Conclusions

In this work, the Optimal Homotopy Asymptotic Method (OHAM) is employed to propose analytical approximate solutions to the flow and heat transfer in a viscous fluid over an unsteady stretching surface. For three values of the suction/injection parameter f_w, the method provides solutions which are compared with numerical solutions computed by means of the shooting method combined with Runge-Kutta method and using Wolfram Mathematica 6.0 software. An analytical expressions for the heat transfer for two values of the Prandtl number are obtained. The solution $\overline{f}(\eta)$ increases with an increase of f_w and velocity decreases with an increase of f_w. The temperature $\overline{\theta}(\eta)$ decreases monotonically with the Prandtl number and with the distance η from the stretching surface.

Our procedure is valid even if the nonlinear equations of the motion do not contain any small or large parameters. The proposed approach is mainly based on a new construction of the approximate solutions and especially with regard to the involvement of the convergence-control parameters via the auxiliary functions. These parameters lead to an excellent agreement of the approximate solutions with numerical results. This technique is effective, explicit and accurate for nonlinear approximations rapidly converging to the exact solution after only one iteration. Also, OHAM provides a simple but rigorous way to control and adjust the convergence of the solution by means of some convergence-control parameters. Our construction of homotopy is different from other approaches, especially with regard to the linear operator L and to the auxiliary convergent-control function H_f and H_θ which ensure a fast convergence of the solutions.

It is worth mentioning that the proposed method is straightforward, concise and can be applied to other nonlinear problems.

Conflict of Interests: The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] Sakiadis B. C., Boundary-layer behavior on continuous solid surface, I. Boundary layer equations for two-dimensional and axisymmetric flow, A. I. Ch. E. J., 1961, 7, 26–28.

[2] Sakiadis B. C., Boundary-layer behavior on continuous solid surface, II. Boundary layer behavior on continuous flat surface, A. I. Ch. E. J., 1961, 7, 221-225.

[3] Crane L. J., Flow past a stretching plate, Z. Angew. Math. Phys., 1970, 21, 645-647.

[4] Tsou F. K., Sparrow E. M., Goldstein R. J., Flow and heat transfer in the boundary layer on a continuous moving surface, Int. J. Heat Mass Transfer, 1967, 10, 219-235.

[5] Gupta P. S., Gupta A. S., Heat and mass transfer on a stretching sheet with suction or blowing, Can. J. Chem. Eng., 1977, 55, 744-746.

[6] Maneschy C. E., Massoudi M., Rajagopal K. R., Flow of a second-grade fluid over a porous elastic sheet due to stretching, J. Math. Phys. Sci., 1993, 27, 353-367.

[7] Grubka L. J., Bobba K. M., Heat transfer characteristics of a continuous stretching surface with variable temperature, J. Heat Mass Transfer, 1985, 107, 248-250.

[8] Wang C. Y., Liquid film on an unsteady stretching surface, Quart. Appl. Math., 1990, XLVIII, 601-610.

[9] Usha R., Rukmani S., The axisymmetric motion of a liquid film on an unsteady stretching surface, ASME, J. Fluids Eng., 1995, 117, 81-85.

[10] Anderson H. J., Aaresth J. B., Dandapat D. S., Heat transfer in a liquid film on an unsteady stretching surface, Int. J. Heat Mass Transfer, 2000, 43, 69-74.

[11] Ali M. E., On thermal boundary layer on a power-law stretched surface with suction or injection, Int. J. Heat Fluid Flow, 1995, 16, 280-290.

[12] Magyari E., Ali M. E., Keller B., Heat and mass transfer characteristics of the self-similar boundary-layer flows induced by continuous surface stretched with rapidly decreasing velocities, Heat Mass Transfer, 2001, 38, 65-74.

[13] Vajravelu K., Viscous flow over a nonlinear stretching sheet, Appl. Math. Comput., 2001, 124, 281-288.

[14] Magyari E., Keller B., A direct method to calculate the heat transfer coefficient of steady similar boundary layer flows induced by continuous moving surfaces, Int. J. Therm. Sci., 2005, 44, 245-254.

[15] Elbashbeshy E. M., Bazid M. A. A., Heat transfer an unsteady stretching surface, Heat Mass Transfer, 2004, 41, 1-4.

[16] Dandapat B. S., Kitamura A., Santra B., Transient film profile on thin liquid film flow on a stretching surface, Z. Angew. Math. Phys., 2006, 57, 623-635.

[17] Ali M. E., Magyari E., Unsteady fluid and heat flow induced by a submerged stretching surface while its steady motion is slowed down gradually, Heat Mass Transfer, 2007, 50, 188-195.

[18] Liu I.- Chung, Anderson H. I., Heat transfer in a liquid film on an unsteady stretching sheet, Int. J. Thermal Sciences, 2008, 47, 766-772.

[19] Chen C. H., Magneto-hydrodynamic mixed convection of a power-law fluid past a stretching surface in the presence of thermal radiation and internal heat generation/absorbtion, Int. J. Non-Linear Mechanics, 2009, 44, 596-603.

[20] Dandapat B. S., Santra B., Singh S. K, Thin film flow over a non-linear stretching sheet in presence of uniform transverse magnetic field, Z. Angew. Math. Phys., 2010, 61, 685-695.

[21] Cortell R., Heat and fluid flow due to non-linearly stretching surfaces, Appl. Math. and Comput., 2011, 217, 7564-7572.

[22] Nayfeh A., Problems in Perturbation, Wiley New York, 1985.

[23] Adomian G., A review of the decomposition method in applied mathematics, J. Math. Anal. Appl., 1988, 135, 501-544.

[24] Beléndez A., Pascual C., Neipp C., Beléndez T., Hernández A., An equivalent linearization method for conservative nonlinear oscillators, Int. J. Non-Linear Numer. Simulat., 2001, 9, 9-19.

[25] Ramos J. I., Linearized Galerkin and artificial parameter techniques for determination of periodic solutions of nonlinear oscillators, Appl. Math. Comput., 2008, 196, 483-493.

[26] He J. N., Modified L-P methods for some strongly nonlinear oscillations. Part I expansion of a constant, Int. J. Non-Linear Mechanics, 2002, 37, 309-314.

[27] He J. N., Variational iteration method - Some recent results and new interpretations, J. Comput. and Appl. Math., 2007, 207, 3-17.

[28] Herişanu N., Marinca V., Optimal homotopy perturbation method for non-conservative dynamical system of a rotating electrical machine, Z. Naturforsch, 2012, 670, 509-516.

[29] Marinca V., Herişanu N., Nemeş I., Optimal homotopy asymptotic method with application to thin film flow, Cent. Eur. J. Phys., 2008, 6, 648-653.

[30] Marinca V., Herişanu N., Bota C., Marinca B., An optimal homotopy asymptotic method applied to the steady flow of a fourth grade fluid past a porous plate, Applied Mathematics Letters, 2009, 22, 245-251.

[31] Marinca V., Herişanu N., Determination of periodic solutions for the motion of particle on a rotating parabola by means of the optimal homotopy asymptotic method, J. of Sound and Vibration, 2010, 329, 1450-1459.

[32] Marinca V., Herişanu N., An optimal homotopy asymptotic approach applied to nonlinear MHD Jeffery-Hamel flow, Mathematical Problems in Engineering, 2011, Article ID 169056, 16 pages.

[33] Marinca V., Herişanu N., Nonlinear Dynamical Systems in Engineering - Some Approximate Approaches, Springer Verlag, Heidelberg, 2011.

[34] Marinca V., Herişanu N., The Optimal Homotopy Asymptotic Method: Engineering Applications, Springer Verlag, Heidelberg, 2015.