

Research Article

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On the Construction of a Surface Family with Common Geodesic in Galilean Space G_3

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Abstract: In this paper, we investigate the parametric representation for a family of surfaces through a given geodesic curve G_3 . We provide necessary and sufficient conditions for this curve to be an isogeodesic curve on the parametric surfaces using Frenet frame in Galilean space. Also, for the sake of visualizing of this study, we plot an example for this surfaces family.

Keywords: Geodesic curve, Isoparametric curve, Parametric surface, Galilean space.

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1 Introduction

Geodesics have a very important role in surface theory and physics. One of the primary reasons why they are so important for physics is that any mass point is not acted on by any forces but is constrained to remain on a fixed surfaces moves on a geodesic line of the surface. Also geodesics play a role in the Lagrange equations of the first kind. Geodesics are so important is that they generalize (locally) the shortest path between points in the space. Geodesics have been commonly studied in Riemannian geometry, more generally metric geometry and general relativity. More precisely, a curve in a surface is said to be geodesic if its geodesic curvature is equally zero. In other words the normal vector of a curve is everywhere parallel to the normal of the surface. Geodesics are helpful in many areas, for example; computer vision, industrial applications and image processing. Moreover, surface with common geodesic is one of the most important research

topics in Differential Geometry. Some more studies and results about surfaces in G_3 have been given in [1–4].

There are nine related plane geometries including Euclidean geometry, hyperbolic geometry and elliptic geometry. The Galilean geometry is one of these geometries whose motions are the Galilean transformations of classical kinematics [5]. There has been a lot of studying about the Differential geometry of the Galilean space G_3 in [6–9].

The purpose of this paper is to introduce the parametric representation of surface through a given isogeodesic curve in Galilean space G_3 . We derive the necessary and sufficient conditions for the given curve as the geodesic and isoparametric on the parametric surface. Also, we define the family of parametric surfaces with common geodesic curve in Galilean space G_3 . Finally, for the sake of visualizing of this study, we demonstrate an example for this family of surfaces.

We present this paper from the Galilean point of view. The results can be easily transferred to the Pseudo-Galilean geometry with minor changes.

2 Preliminaries

The Galilean space G_3 is a Cayley-Klein space equipped with the projective metric of signature $(0, 0, +, +)$, given in [10]. The absolute figure of the Galilean space consists of an ordered triple $\{\omega, f, I\}$ in which ω is the ideal (absolute) plane, f is the line (absolute line) in ω and I is the fixed elliptic involution of points of f .

Definition 2.1. A vector $\mathbf{x} = (x_1, x_2, x_3)$ is called a non-isotropic if $x_1 \neq 0$. All unit isotropic vectors are of the form $\mathbf{x} = (1, x_2, x_3)$. For isotropic vectors $x_1 = 0$ holds.

Definition 2.2. The Galilean scalar product between two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ vectors in G_3 , is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \begin{cases} x_1 y_1, & \text{if } x_1 \neq 0 \text{ or } y_1 \neq 0 \\ x_2 y_2 + x_3 y_3, & \text{if } x_1 = 0 \text{ and } y_1 = 0 \end{cases},$$

in [11].

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Definition 2.3. Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ be vectors in $\mathbf{G}_3$, the cross product of the vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as follows

$$\mathbf{x} \wedge \mathbf{y} = \begin{vmatrix} 0 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (0, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1),$$

in [11].

An admissible curve r of the class $\mathbf{C}^r$ ($r > 3$) in $\mathbf{G}_3$, and parametrized by the invariant parameter u , is given by

$$r(u) = (u, f(u), g(u)).$$

For an admissible curve, the associated invariant moving trihedron satisfies the following equation

$$\begin{aligned} t(u) &= r'(u) = (1, f'(u), g'(u)), \\ n(u) &= \frac{r''(u)}{\kappa(u)} = \frac{1}{\kappa(u)} (0, f''(u), g''(u)), \\ b(u) &= \frac{1}{\kappa(u)} (0, -g''(u), f''(u)), \end{aligned}$$

where t , n and b are called the vectors of the tangent, principal normal and binormal of $r(u)$, respectively, and the curvature $\kappa(u)$ and the torsion $\tau(u)$ of the curve r can be given by, respectively,

$$\begin{aligned} \kappa(u) &= \sqrt{f''(u)^2 + g''(u)^2}, \\ \tau(u) &= \frac{\det(r'(u), r''(u), r'''(u))}{\kappa^2(u)}, \end{aligned}$$

Frenet formulas may be written as

$$\begin{aligned} t' &= \kappa n, \\ n' &= \tau b, \\ b' &= -\tau n, \end{aligned}$$

in [12].

Definition 2.4. Let M is a surface in $\mathbf{G}_3$, the equation of a surface in $\mathbf{G}_3$ can be expressed as the parametrization

$$\phi(u, v) = (\phi_1(u, v), \phi_2(u, v), \phi_3(u, v)), \quad u, v \in \mathbb{R},$$

where $\phi_1(u, v)$, $\phi_2(u, v)$ and $\phi_3(u, v) \in \mathbf{C}^3$, in [13].

Also, the isotropic normal vector field is given by

$$\eta(u, v) = \phi_u \times \phi_v,$$

where ϕ_u and ϕ_v are partial differentiations with respect to u and v , respectively.

3 Surfaces with Common Geodesic Curve in Galilean Space $\mathbf{G}_3$

Let $\phi = \phi(u, v)$ be a parametric surface on the arc-length parametrized curve $r(u)$ in $\mathbf{G}_3$. The surface is defined by

$$\phi(u, v) = r(u) + [x(u, v)t(u) + y(u, v)n(u) + z(u, v)b(u)], \quad (1)$$

$$L_1 \leq u \leq L_2 \text{ and } T_1 \leq v \leq T_2, \quad (2)$$

where $x(u, v)$, $y(u, v)$ and $z(u, v)$ are C^1 functions $\{t(u), n(u), b(u)\}$ is the frame associated with the curve $r(u)$ in $\mathbf{G}_3$.

The normal $\eta(u, v)$ of the surface is given by

$$\eta(u, v) = \phi_u \times \phi_v, \quad (3)$$

from (1)

$$\begin{aligned} \phi_u &= (1 + x_u)t + (kx + y_u - \tau z)n + (\tau y + z_u)b, \\ \phi_v &= x_v t + y_v n + z_v b, \end{aligned}$$

Taking account (3), the normal vector $\eta(u, v)$ can be expressed as

$$\begin{aligned} \eta(u, v) &= [-(1 + x_u)z_v + (\tau y + z_u)x_v]n \\ &\quad + [(1 + x_u)y_v - (kx + y_u - \tau z)x_v]b, \end{aligned}$$

Let $r(u)$ be a curve on a surface $\phi(u, v)$ in $\mathbf{G}_3$. If $r(u)$ is isoparametric curve on this surface, then there exists a parameter $v = v_0$ such that $r(u) = \phi(u, v_0)$, that is

$$x(u, v_0) = y(u, v_0) = z(u, v_0) = 0. \quad (4)$$

From (4), we get

$$\begin{aligned} \eta(u, v_0) &= [-(1 + x_u)z_v + z_u x_v]n \\ &\quad + [(1 + x_u)y_v - y_u x_v]b. \end{aligned}$$

According to [14], the curve $r(u)$ on the surface $\phi(u, v)$ is geodesic if and only if the normal vector $n(u)$ of the curve $r(u)$ is everywhere parallel to the normal vector $\eta(u, v_0)$ of the surface $\phi(u, v)$. Then, $n(u) \parallel \eta(u, v_0)$ if and only if

$$\begin{aligned} -(1 + x_u)z_v + z_u x_v &\neq 0, \\ (1 + x_u)y_v - y_u x_v &= 0. \end{aligned} \quad (5)$$

Thus, the necessary and sufficient conditions for the surface ϕ to have the curve $r(u)$ in $\mathbf{G}_3$ as an isoparametric and geodesic can be given with the following theorem.

Theorem 3.1. Let ϕ be a surface having a curve $r(u)$ in the 3-dimensional Galilean space with parametrization (1). The curve $r(u)$ is isogeodesic on a surface $\phi(u, v)$ if and only if the following conditions are satisfied:

$$\begin{aligned} x(u, v_0) &= y(u, v_0) = z(u, v_0) = 0, \\ -(1 + x_u)z_v + z_u x_v &\neq 0, \\ (1 + x_u)y_v - y_u x_v &= 0. \end{aligned}$$

We call the set of surfaces given by (1) and satisfying (4) and (5) the family of surfaces with common isogeodesic in $\mathbf{G}_3$. Any surface $\phi(u, v)$ defined by (1) and satisfying (4) and (5) is a member of the family.

The functions $x(u, v)$, $y(u, v)$ and $z(u, v)$ can be chosen in two different forms:

Case 1. If we take

$$\begin{aligned} x(u, v) &= \sum_{i=1}^p a_{1i} l(u)^i x(v)^i, \\ y(u, v) &= \sum_{i=1}^p a_{2i} m(u)^i y(v)^i, \\ z(u, v) &= \sum_{i=1}^p a_{3i} n(u)^i z(v)^i, \end{aligned} \quad (6)$$

then, the sufficient condition for which the curve $r(u)$ is an isogeodesic curve on the surface $\phi(u, v)$ can be given as

$$x(v_0) = y(v_0) = z(v_0) = 0, \quad (7)$$

$$a_{21} = 0 \text{ or } m(u) = 0 \text{ or } \frac{dy(v_0)}{dv} = 0,$$

$$a_{31} \neq 0 \text{ and } n(u) \neq 0, \text{ and } \frac{dz(v_0)}{dv} \neq 0,$$

where $l(u)$, $m(u)$, $n(u)$, $x(v)$, $y(v)$ and $z(v)$ are C^1 functions, $a_{ij} \in \mathbb{R}$ ($i = 1, 2, 3; j = 1, 2, \dots, p$) and $l(u)$, $m(u)$ and $n(u)$ are not identically zero.

For the case when the functions $x(u, v)$, $y(u, v)$ and $z(u, v)$ depend only on the parameter v , the family of surfaces with common geodesic becomes

$$\phi(u, v) = r(u) + x(v)t(u) + y(v)n(u) + z(v)b(u).$$

Case 2. If we take

$$\begin{aligned} x(u, v) &= \mathbf{f}(l(u)x(v)), \\ y(u, v) &= \mathbf{g}(m(u)y(v)), \\ z(u, v) &= \mathbf{h}(n(u)z(v)), \end{aligned} \quad (8)$$

then, the sufficient condition for which the curve $r(u)$ is an isogeodesic curve on the surface $\phi(u, v)$ can be expressed as

$$x(v_0) = y(v_0) = z(v_0) = 0 \text{ and } \mathbf{f}(0) = \mathbf{g}(0) = \mathbf{h}(0) = 0,$$

$$m(u) = 0 \text{ or } \mathbf{g}'(0) = 0 \text{ or } \frac{dy(v_0)}{dv} = 0,$$

$$n(u) \neq 0 \text{ and } \mathbf{h}'(0) \neq 0 \text{ and } \frac{dz(v_0)}{dv} \neq 0,$$

where $l(u)$, $m(u)$, $n(u)$, $x(v)$, $y(v)$, $z(v)$, $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ are C^1 functions and $l(u)$, $m(u)$ and $n(u)$ are not identically zero.

So, we get the functions in (6) and (8) which are general for expressing surfaces with a given curve as an isogeodesic curve in $\mathbf{G}_3$. Also, different types of these functions can be chosen according to Theorem 3.1.

Example 3.1. Let r be a parametrized by

$$r(u) = (u, \sin u, \cos u).$$

It is easy to calculate that

$$t = (1, \cos u, -\sin u),$$

$$n = (0, -\sin u, -\cos u),$$

$$b = (0, \cos u, -\sin u),$$

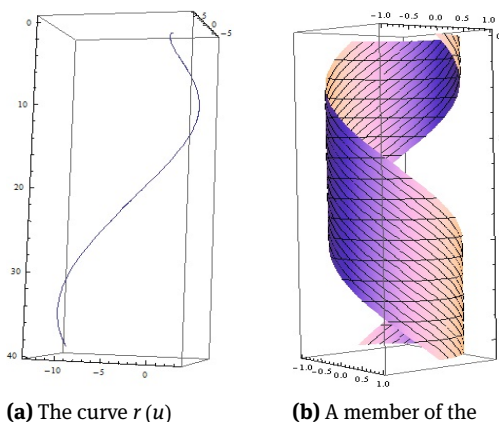
where $\kappa = 1$ is the curvature and $\tau = 1$ is the torsion of the curve in $\mathbf{G}_3$.

Then, we obtain the surfaces family with the common isogeodesic. If we take

$$x(u, v) = 0, y(u, v) = 1 - \cos v \text{ and } z(u, v) = \sin v,$$

and $v_0 = 0$ such that Equation (7) is satisfied. Thus, a member of this family is obtained by

$$\begin{aligned} \phi(u, v) &= [u, \sin u - (1 - \cos v)\sin u + \sin v \cos u, \\ &\quad \cos u - (1 - \cos v)\cos u - \sin v \sin u]. \end{aligned}$$



(a) The curve $r(u)$

(b) A member of the family of surfaces having $r(u)$ as a geodesic curve.

Figure 1: The representation of the curve and a member of surfaces.

4 Conclusions

We showed the parametric representation for a family of surfaces through a given geodesic curve G_3 . We gave a theorem related to the curve $r(u)$ is isogeodesic on a surface $\phi(u, v)$. Consequently, an example for this surfaces family was plotted.

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