

Research Article

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Solitary and compacton solutions of fractional KdV-like equations

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Abstract: In this paper, based on Jumarie's modified Riemann-Liouville derivative, we apply the fractional variational iteration method using He's polynomials to obtain solitary and compacton solutions of fractional KdV-like equations. The results show that the proposed method provides a very effective and reliable tool for solving fractional KdV-like equations, and the method can also be extended to many other fractional partial differential equations.

Keywords: Fractional variational iteration method; Homotopy perturbation method; Modified Riemann-Liouville derivative; Fractional partial differential equation

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1 Introduction

In recent years, theory and numerical analysis of fractional partial differential equations (FPDEs) have received considerable interest due to their numerous applications in the areas of physics, biology, fluid and continuum mechanics, and engineering [1–11]. For example, in [10], Devendra Kumar, Jagdev Singh and Sunil Kumar used the homotopy perturbation transform method to study the nonlinear fractional Zakharov-Kuznetsov equation arising in ion-acoustic waves; Devendra Kumar, Jagdev Singh and Sushila used the homotopy analysis transform method to

study the fractional biological population model in [11]. For better understanding of the complicated nonlinear physical phenomena, the solution of the fractional differential equation is much involved. In the past, various methods have been proposed to obtain solutions of FPDEs, such as homotopy perturbation method [12–14], homotopy perturbation Sumudu transform method [15, 16], Adomian decomposition method [17, 18], homotopy analysis method [19], fractional variational iteration method [20, 21], finite difference method [22], fractional sub-ODE method [23–26], and so on. Based on these methods, many fractional differential equations have been investigated.

Recently, Jumarie proposed a new definition of the fractional derivative which is a simple alternative definition to the Riemann-Liouville derivative. His definition has the advantages of both the standard Riemann-Liouville and the Caputo fractional derivatives, because the Jumarie derivative of a constant is equal to zero, an arbitrary continuous function needs not to be differentiable, and more importantly it removes singularity at the origin for all functions, for example, the exponentials functions and Mittag-Leffler functions.

In this paper, we will apply the the fractional variational iteration method using He's polynomials (FVIMHP) [27–29] and Jumarie's modified Riemann-Liouville derivative to get solitary and compacton solutions of the following KdV-like equations

- The time fractional K(2, 2) equation:

$$D_t^\alpha u + a(u^2)_x + b(u^2)_{xxx} = 0, \quad 0 < \alpha \leq 1.$$

- (2+1)-dimensional time fractional Z-K equation:

$$D_t^\alpha u + a(u^2)_x + b(u^2)_{xxx} + k(u^2)_{yyx} = 0, \quad 0 < \alpha \leq 1.$$

- (3+1)-dimensional time fractional Z-K equation:

$$D_t^\alpha u + a(u^2)_x + b(u^2)_{xxx} + k(u^2)_{yyx} + r(u^2)_{zzx} = 0, \quad 0 < \alpha \leq 1.$$

- (3+1)-dimensional time fractional K-P equation:

$$\left[D_t^\alpha u + \frac{a}{2}(u^2)_x + b(u(u_{xx})_x) \right]_x + u_{yy} + u_{zz} = 0, \quad 0 < \alpha \leq 1.$$

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The rest of this paper is organized as follows: In Section 2, we introduce some basic definitions of Jumarie's modified Riemann-Liouville derivative and give the main steps of the method here. In Section 3, we construct solitary and compacton solutions of fractional KdV-like equations by the proposed method. Some conclusions are given in Section 4.

2 Description of Modified Riemann-Liouville derivative and FVIMHP method

In this section, we first introduce some basic definitions of the fractional calculus theory.

Definition 2.1. The Riemann-Liouville fractional integral [30] is defined as

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - \xi)^{\alpha-1} f(\xi) d\xi, \quad \alpha > 0. \quad (1)$$

Definition 2.2. The Jumarie's modified Riemann-Liouville derivative [31–34] is defined as

$$D_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \frac{d}{dx} \int_0^x (x - \xi)^{-\alpha-1} (f(\xi) - f(0)) d\xi, & \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x - \xi)^{-\alpha} (f(\xi) - f(0)) d\xi, & 0 < \alpha < 1, \\ (f^{(n)}(x))^{(\alpha-n)}, & n \leq \alpha < n+1, \\ n \geq 1, & n \geq 1, \end{cases} \quad (2)$$

Definition 2.3. The integral with respect to $(dx)^\alpha$ is defined by Jumarie [31] as follows:

$$\int_0^x f(\xi) (d\xi)^\alpha = \alpha \int_0^x (x - \xi)^{\alpha-1} f(\xi) d\xi, \quad 0 < \alpha \leq 1. \quad (3)$$

We present the essential steps of the FVIMHP method as follows: Consider the following initial value problem

$$D_t^\alpha u(x, t) + L(u(x, t)) + N(u(x, t)) = f(x, t) \quad (4)$$

where D_t^α is the Jumarie's modified Riemann-Liouville derivative, L is the linear operator, N is the nonlinear operator.

Step 1. By using the fractional variational iteration method (FVIM), we can get the iteration formula as follows:

$$u_{m+1}(x, t) = u_m(x, t) + I_t^\alpha \left\{ \lambda(\tau) \left[D_t^\alpha u_m(x, \tau) + L(u_m(x, \tau)) + N(\tilde{u}_m(x, \tau)) - f(x, \tau) \right] \right\}, \quad (5)$$

where λ is the Lagrangian multiplier, the subscript m denotes the m -th order approximation, and $\tilde{u}_m$ is considered as a restricted variation.

Step 2. By means of the homotopy perturbation method (HPM), Eq. (5) becomes the following form:

$$\sum_{m=0}^{\infty} p^m u_m(x, t) = u_0(x, t) + p \left\{ \sum_{m=1}^{\infty} p^m u_m(x, t) + I_t^\alpha \left[\lambda(\tau) t \left(\sum_{m=0}^{\infty} p^m D_t^\alpha u_m(x, \tau) + \sum_{m=0}^{\infty} p^m L(u_m(x, \tau)) + \sum_{m=0}^{\infty} p^m N(\tilde{u}_m(x, \tau)) - f(x, \tau) \right) \right] \right\}, \quad (6)$$

where $p \in [0, 1]$ is an imbedding parameter, and u_0 is an initial approximation of Eq. (4).

Step 3. Comparing the coefficients of the same order of p on both sides of Eq. (6), we can obtain $u_m (m = 0, 1, 2, \dots)$. According to the HPM, the solution of Eq. (4) can be expressed as:

$$u = \sum_{m=0}^{\infty} u_m. \quad (7)$$

3 Application of the proposed method

In this section, we apply the FVIMHP method to obtain solitary and compacton solutions of the time fractional KdV-like equations.

3.1 Time fractional K(2, 2) equation

We consider the following time fractional K(2, 2) equation:

$$D_t^\alpha u + a(u^2)_x + b(u^2)_{xxx} = 0, \quad 0 < \alpha \leq 1, \quad (8)$$

with the initial condition

$$u(x, 0) = \frac{4c}{3a} \cosh^2\left(\frac{1}{4}\sqrt{-\frac{a}{b}}x\right). \quad (9)$$

According to the FVIMHP method given in Section 2, we can get the iteration formula of Eq. (8) in the following form:

$$\sum_{m=0}^{\infty} p^m u_m = u_0 + p \left\{ \sum_{m=1}^{\infty} p^m u_m - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left[\sum_{m=0}^{\infty} p^m D_\tau^\alpha u \right] d\tau \right\}$$

$$+ a \left(\left(\sum_{m=0}^{\infty} p^m u_{m \text{right}} \right)^2 \right)_x + b \left(\left(\sum_{m=0}^{\infty} p^m u_m \right)^2 \right)_{xxx} \Big] (d\tau)^\alpha \Big\}. \quad (10)$$

By the above iteration formula and initial approximation $u_0(x, t) = u(x, 0) = \frac{4c}{3a} \cosh^2(\frac{1}{4}\sqrt{-\frac{a}{b}}x) = \frac{2c}{3a} [\cosh(\frac{1}{2}\sqrt{-\frac{a}{b}}x) + 1]$, we can obtain

$$\begin{aligned} p^1: \quad u_1(x, y, t) &= -\frac{2c}{3a} \left(\frac{c}{2} \sqrt{-\frac{a}{b}} \right) \\ &\quad \cdot \sinh \left(\frac{1}{2} \sqrt{-\frac{a}{b}} x \right) \frac{t^\alpha}{\Gamma(1+\alpha)}, \\ p^2: \quad u_2(x, y, t) &= \frac{2c}{3a} \left(\frac{c}{2} \sqrt{-\frac{a}{b}} \right)^2 \\ &\quad \cdot \cosh \left(\frac{1}{2} \sqrt{-\frac{a}{b}} x \right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \\ p^3: \quad u_3(x, y, t) &= -\frac{2c}{3a} \left(\frac{c}{2} \sqrt{-\frac{a}{b}} \right)^3 \\ &\quad \cdot \sinh \left(\frac{1}{2} \sqrt{-\frac{a}{b}} x \right) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \\ p^4: \quad u_4(x, y, t) &= \frac{2c}{3a} \left(\frac{c}{2} \sqrt{-\frac{a}{b}} \right)^4 \\ &\quad \cdot \cosh \left(\frac{1}{2} \sqrt{-\frac{a}{b}} x \right) \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}, \\ &\vdots \end{aligned}$$

So we have the solitary solution of Eq. (8):

$$\begin{aligned} u(x, t) &= \frac{2c}{3a} \left\{ \cosh \left(\frac{1}{2} \sqrt{-\frac{a}{b}} x \right) \left[1 + \left(\frac{c}{2} \sqrt{-\frac{a}{b}} \right)^2 \right. \right. \\ &\quad \cdot \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(\frac{c}{2} \sqrt{-\frac{a}{b}} \right)^4 \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \dots \Big] \\ &\quad - \sinh \left(\frac{1}{2} \sqrt{-\frac{a}{b}} x \right) \left[\left(\frac{c}{2} \sqrt{-\frac{a}{b}} \right) \frac{t^\alpha}{\Gamma(1+\alpha)} \right. \\ &\quad \left. \left. + \left(\frac{c}{2} \sqrt{-\frac{a}{b}} \right)^3 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \dots \right] + 1 \right\} \\ &= \frac{2c}{3a} \left[\cosh \left(\frac{1}{2} \sqrt{-\frac{a}{b}} x \right) \cosh \left(\frac{1}{2} \sqrt{-\frac{a}{b}} ct^\alpha, \alpha \right) \right. \\ &\quad \left. - \sinh \left(\frac{1}{2} \sqrt{-\frac{a}{b}} x \right) \sinh \left(\frac{1}{2} \sqrt{-\frac{a}{b}} ct^\alpha, \alpha \right) + 1 \right], \end{aligned} \quad (11)$$

where the functions $\sinh(z, \alpha)$ and $\cosh(z, \alpha)$ are defined as follows:

$$\sinh(z, \alpha) = \frac{E_\alpha(z) - E_\alpha(-z)}{2}, \quad \cosh(z, \alpha) = \frac{E_\alpha(z) + E_\alpha(-z)}{2}.$$

Here $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}$ ($\alpha > 0$) is the Mittag-Leffler function.

We show some properties of the approximate solution (11) obtained by the proposed method in Fig. 1. The plot (a) shows that approximate solution (11) is in good agreement with the exact solution. The plot (b) shows the fifth-order approximate solution (11) when $\alpha = 0.89$. The plot (c) shows the approximate solution (11) for different values of α .

If we select the initial approximation $u(x, 0) = -\frac{4c}{3a} \sinh^2(\frac{1}{4}\sqrt{-\frac{a}{b}}x)$, we can get the solitary solution as follows:

$$\begin{aligned} u(x, t) &= -\frac{2c}{3a} \left[\cosh \left(\frac{1}{2} \sqrt{-\frac{a}{b}} x \right) \cosh \left(\frac{1}{2} \sqrt{-\frac{a}{b}} ct^\alpha, \alpha \right) \right. \\ &\quad \left. - \sinh \left(\frac{1}{2} \sqrt{-\frac{a}{b}} x \right) \sinh \left(\frac{1}{2} \sqrt{-\frac{a}{b}} ct^\alpha, \alpha \right) - 1 \right]. \end{aligned} \quad (12)$$

In order to search compacton solutions of Eq. (8), we select the following initial approximations

$$u(x, 0) = \begin{cases} \frac{4c}{3a} \sin^2(\frac{1}{4}\sqrt{\frac{a}{b}}x), & |x| < \frac{\pi}{\mu}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$u(x, 0) = \begin{cases} \frac{4c}{3a} \cos^2(\frac{1}{4}\sqrt{\frac{a}{b}}x), & |x| < \frac{\pi}{2\mu}, \\ 0, & \text{otherwise} \end{cases}$$

Using the FVIMHP in the same manner, we could get the compacton solutions as follows:

$$u(x, t) = \begin{cases} \frac{2c}{3a} \left[1 - \cos(\frac{1}{2}\sqrt{\frac{a}{b}}x) \cos(\frac{1}{2}\sqrt{\frac{a}{b}}ct^\alpha, \alpha) - \right. \\ \left. \sin(\frac{1}{2}\sqrt{\frac{a}{b}}x) \sin(\frac{1}{2}\sqrt{\frac{a}{b}}ct^\alpha, \alpha) \right], & |x - ct^\alpha| < \frac{\pi}{\mu}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$u(x, t) = \begin{cases} \frac{2c}{3a} \left[1 + \cos(\frac{1}{2}\sqrt{\frac{a}{b}}x) \cos(\frac{1}{2}\sqrt{\frac{a}{b}}ct^\alpha, \alpha) + \right. \\ \left. \sin(\frac{1}{2}\sqrt{\frac{a}{b}}x) \sin(\frac{1}{2}\sqrt{\frac{a}{b}}ct^\alpha, \alpha) \right], & |x - ct^\alpha| < \frac{\pi}{2\mu}, \\ 0, & \text{otherwise} \end{cases}$$

where $\mu = \frac{1}{4}\sqrt{\frac{a}{b}}$.

3.2 (2+1)-dimensional time fractional Z-K equation

Consider the following (2+1)-dimensional time fractional Z-K equation:

$$D_t^\alpha u + a(u^2)_x + b(u^2)_{xxx} + k(u^2)_{yyx} = 0, \quad 0 < \alpha \leq 1. \quad (13)$$

with the initial condition

$$u(x, y, 0) = \frac{4c}{3a} \cosh^2\left(\frac{1}{4}\sqrt{-\frac{a}{b+k}}(x+y)\right). \quad (14)$$

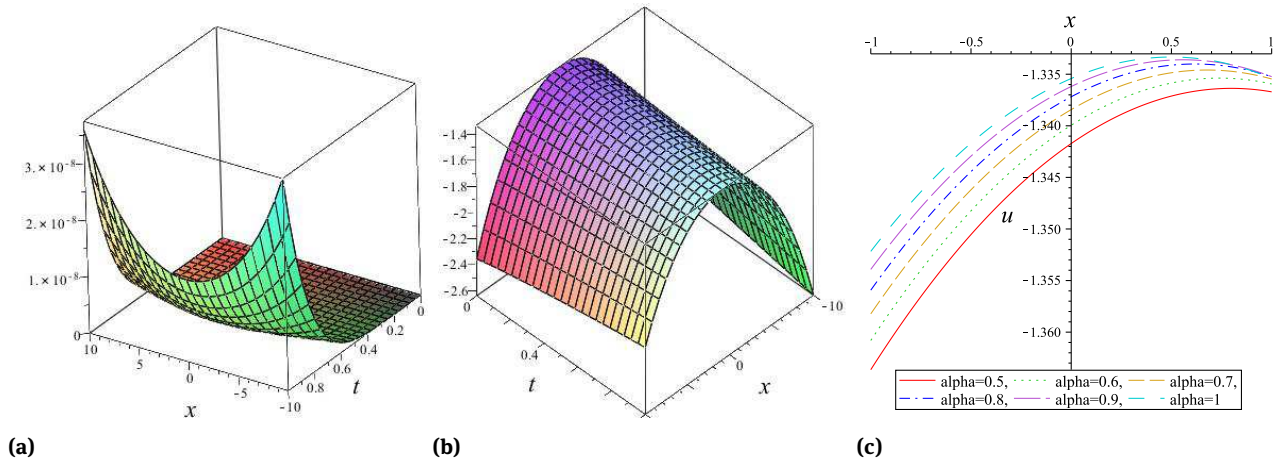


Figure 1: (a) the absolute error between the fifth-order approximate solution (11) and exact solution when $\alpha = 1$ (b) the fifth-order approximate solution (11) when $\alpha = 0.89$ (c) the fifth-order approximate solution (11) for different values of α when $t = 0.5$; $c = 1$; $a = -1$; $b = 10$

According to the method given in Section 2, its iteration formula can be constructed in the following form:

$$\sum_{m=0}^{\infty} p^m u_m = u_0 + p \left\{ \sum_{m=1}^{\infty} p^m u_m - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left[\sum_{m=0}^{\infty} p^m D_{\tau}^{\alpha} u_m + a \left(\left(\sum_{m=0}^{\infty} p^m u_m \right)^2 \right)_x + b \left(\left(\sum_{m=0}^{\infty} p^m u_m \right)^2 \right)_{xxx} + k \left(\left(\sum_{m=0}^{\infty} p^m u_m \right)^2 \right)_{yyx} \right] (d\tau)^{\alpha} \right\}. \quad (15)$$

By initial approximation $u_0(x, y, t) = u(x, y, 0) = \frac{4c}{3a} \cosh^2\left(\frac{1}{4}\sqrt{-\frac{a}{b+k}}(x+y)\right)$, we can obtain

$$\begin{aligned} p^1: \quad u_1(x, y, t) &= -\frac{2c}{3a} \left(\frac{c}{2} \sqrt{-\frac{a}{b+k}} \right) \cdot \sinh\left(\frac{1}{2}\sqrt{-\frac{a}{b+k}}(x+y)\right) \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \\ p^2: \quad u_2(x, y, t) &= \frac{2c}{3a} \left(\frac{c}{2} \sqrt{-\frac{a}{b+k}} \right)^2 \cdot \cosh\left(\frac{1}{2}\sqrt{-\frac{a}{b+k}}(x+y)\right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \\ p^3: \quad u_3(x, y, t) &= -\frac{2c}{3a} \left(\frac{c}{2} \sqrt{-\frac{a}{b+k}} \right)^3 \cdot \sinh\left(\frac{1}{2}\sqrt{-\frac{a}{b+k}}(x+y)\right) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \\ p^4: \quad u_4(x, y, t) &= \frac{2c}{3a} \left(\frac{c}{2} \sqrt{-\frac{a}{b+k}} \right)^4 \cdot \cosh\left(\frac{1}{2}\sqrt{-\frac{a}{b+k}}(x+y)\right) \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}, \end{aligned}$$

Consequently, we have the following solitary solution of Eq. (13)

$$u(x, y, t) = \frac{2c}{3a} \left[\cosh\left(\frac{1}{2}\sqrt{-\frac{a}{b+k}}(x+y)\right) \cdot \cosh\left(\frac{1}{2}\sqrt{-\frac{a}{b+k}}ct^{\alpha}, \alpha\right) - \sinh\left(\frac{1}{2}\sqrt{-\frac{a}{b+k}}(x+y)\right) \cdot \sinh\left(\frac{1}{2}\sqrt{-\frac{a}{b+k}}ct^{\alpha}, \alpha\right) + 1 \right], \quad (16)$$

We show some properties of the approximate solution (16) obtained by the proposed method in Fig. 2. The plot (a) shows that the approximate solution (16) is in good agreement with the exact solution when $\alpha = \gamma = 1$. The plot (b) shows the fifth-order approximate solution (16) when $x = 2$, $\alpha = 0.75$. The plot (c) shows the approximate solution (16) for different values of α .

If we choose $u(x, y, 0) = -\frac{4c}{3a} \sinh^2\left(\frac{1}{4}\sqrt{-\frac{a}{b+k}}(x+y)\right)$, then we can obtain the solitary solutions of Eq. (13) as follows:

$$u(x, y, t) = -\frac{2c}{3a} \left[\cosh\left(\frac{1}{2}\sqrt{-\frac{a}{b+k}}(x+y)\right) \cdot \cosh\left(\frac{1}{2}\sqrt{-\frac{a}{b+k}}ct^{\alpha}, \alpha\right) - \sinh\left(\frac{1}{2}\sqrt{-\frac{a}{b+k}}(x+y)\right) \cdot \sinh\left(\frac{1}{2}\sqrt{-\frac{a}{b+k}}ct^{\alpha}, \alpha\right) - 1 \right].$$

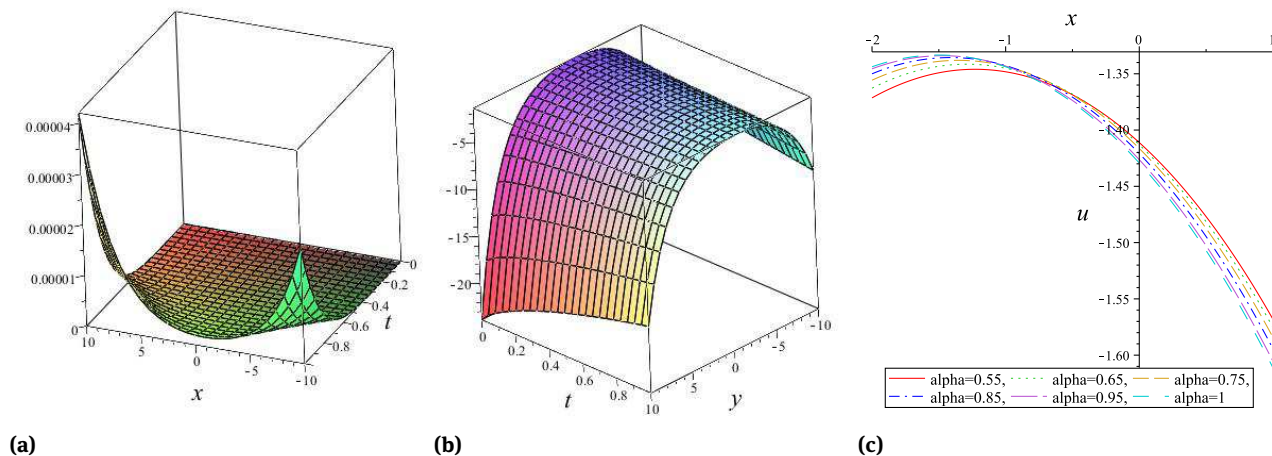


Figure 2: (a) the absolute error between the fifth-order approximate solution (16) and exact solution when $\alpha = y = 1$ (b) the fifth-order approximate solution (16) when $x = 2, \alpha = 0.75$ (c) the fifth-order approximate solution (16) for different values of α when $t = 0.5, y = 2, c = 1; a = -1; b = k = 1$

In order to construct compacton solutions of Eq. (13), we select initial approximations as follows:

$$u(x, y, 0) = \begin{cases} \frac{4c}{3a} \sin^2\left(\frac{1}{4}\sqrt{\frac{a}{b+k}}(x+y)\right), & |x+y| < \frac{\pi}{\mu}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$u(x, y, 0) = \begin{cases} \frac{4c}{3a} \cos^2\left(\frac{1}{4}\sqrt{\frac{a}{b+k}}(x+y)\right), & |x+y| < \frac{\pi}{2\mu}, \\ 0, & \text{otherwise} \end{cases}$$

By using the FVIMHP method, we can get the following compacton solutions of Eq. (13)

$$u(x, y, t) = \begin{cases} \frac{2c}{3a} \left[1 - \cos\left(\frac{1}{2}\sqrt{\frac{a}{b+k}}(x+y)\right) \cos\left(\frac{1}{2}\sqrt{\frac{a}{b+k}}ct^\alpha, \alpha\right) \right. \\ \left. - \sin\left(\frac{1}{2}\sqrt{\frac{a}{b+k}}(x+y)\right) \sin\left(\frac{1}{2}\sqrt{\frac{a}{b+k}}ct^\alpha, \alpha\right) \right], & |x+y-ct^\alpha| < \frac{\pi}{\mu}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$u(x, y, t) = \begin{cases} \frac{2c}{3a} \left[1 + \cos\left(\frac{1}{2}\sqrt{\frac{a}{b+k}}(x+y)\right) \cos\left(\frac{1}{2}\sqrt{\frac{a}{b+k}}ct^\alpha, \alpha\right) \right. \\ \left. + \sin\left(\frac{1}{2}\sqrt{\frac{a}{b+k}}(x+y)\right) \sin\left(\frac{1}{2}\sqrt{\frac{a}{b+k}}ct^\alpha, \alpha\right) \right], & |x+y-ct^\alpha| < \frac{\pi}{2\mu}, \\ 0, & \text{otherwise} \end{cases}$$

where $\mu = \frac{1}{4}\sqrt{\frac{a}{b+k}}$.

3.3 (3+1)-dimensional time fractional Z-K equation

We consider the following (2+1)-dimensional time fractional Z-K equation:

$$D_t^\alpha u + a(u^2)_x + b(u^2)_{xxx} + k(u^2)_{yyx} + r(u^2)_{zzx} = 0, \quad 0 < \alpha \leq 1. \quad (17)$$

with the initial condition

$$u(x, y, z, 0) = \frac{4c}{3a} \cosh^2\left(\frac{1}{4}\sqrt{-\frac{a}{b+k+r}}(x+y+z)\right). \quad (18)$$

According to the FVIMHP method, its iteration formula can be constructed in the following form:

$$\begin{aligned} \sum_{m=0}^{\infty} p^m u_m = u_0 + p \left\{ \sum_{m=1}^{\infty} p^m u_m - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left[\sum_{m=0}^{\infty} p^m \right. \right. \\ \cdot D_t^\alpha u_m + a \left(\left(\sum_{m=0}^{\infty} p^m u_m \right)^2 \right)_x \\ + b \left(\left(\sum_{m=0}^{\infty} p^m u_m \right)^2 \right)_{xxx} \\ + k \left(\left(\sum_{m=0}^{\infty} p^m u_m \right)^2 \right)_{yyx} \\ \left. \left. + r \left(\left(\sum_{m=0}^{\infty} p^m u_m \right)^2 \right)_{zzx} \right] (dt)^\alpha \right\}. \quad (19) \end{aligned}$$

By initial approximation $u_0(x, y, z, t) = u(x, y, z, 0) = \frac{4c}{3a} \cosh^2\left(\frac{1}{4} \sqrt{-\frac{a}{b+k+r}}(x+y)\right)$, we can obtain

$$\begin{aligned} p^1: \quad u_1(x, y, t) &= -\frac{2c}{3a} \left(\frac{c}{2} \sqrt{-\frac{a}{b+k+r}} \right) \\ &\quad \cdot \sinh \left(\frac{1}{2} \sqrt{-\frac{a}{b+k+r}}(x+y+z) \right) \frac{t^\alpha}{\Gamma(1+\alpha)}, \\ p^2: \quad u_2(x, y, t) &= \frac{2c}{3a} \left(\frac{c}{2} \sqrt{-\frac{a}{b+k+r}} \right)^2 \\ &\quad \cdot \cosh \left(\frac{1}{2} \sqrt{-\frac{a}{b+k+r}}(x+y+z) \right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \\ p^3: \quad u_3(x, y, t) &= -\frac{2c}{3a} \left(\frac{c}{2} \sqrt{-\frac{a}{b+k+r}} \right)^3 \\ &\quad \cdot \sinh \left(\frac{1}{2} \sqrt{-\frac{a}{b+k+r}}(x+y+z) \right) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \\ p^4: \quad u_4(x, y, t) &= \frac{2c}{3a} \left(\frac{c}{2} \sqrt{-\frac{a}{b+k+r}} \right)^4 \\ &\quad \cdot \cosh \left(\frac{1}{2} \sqrt{-\frac{a}{b+k+r}}(x+y+z) \right) \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}, \\ &\vdots \end{aligned}$$

Consequently, we have the following solitary solution of Eq. (17)

$$\begin{aligned} u(x, y, z, t) &= \frac{2c}{3a} \left[\cosh \left(\frac{1}{2} \sqrt{-\frac{a}{b+k+r}}(x+y+z) \right) \right. \\ &\quad \cdot \cosh \left(\frac{1}{2} \sqrt{-\frac{a}{b+k+r}} ct^\alpha, \alpha \right) \\ &\quad - \sinh \left(\frac{1}{2} \sqrt{-\frac{a}{b+k+r}}(x+y+z) \right) \\ &\quad \left. \cdot \sinh \left(\frac{1}{2} \sqrt{-\frac{a}{b+k+r}} ct^\alpha, \alpha \right) + 1 \right]. \quad (20) \end{aligned}$$

We show some properties of the approximate solution (20) obtained by the proposed method in Fig. 3. The plot (a) shows that our approximate solution (20) is in good agreement with the exact solution. The plot (b) shows the fifth-order approximate solution expressed by (20) when α, x, y are fixed. The plot (c) shows the approximate solution (20) for different values of α when t, y, z are fixed.

If we choose $u(x, y, z, 0) = -\frac{4c}{3a} \sinh^2\left(\frac{1}{4} \sqrt{-\frac{a}{b+k+r}}(x+y+z)\right)$, we can get the solitary solution of Eq. (17) as follows:

$$\begin{aligned} u(x, y, z, t) &= -\frac{2c}{3a} \left[\cosh \left(\frac{1}{2} \sqrt{-\frac{a}{b+k+r}}(x+y+z) \right) \right. \\ &\quad \cdot \cosh \left(\frac{1}{2} \sqrt{-\frac{a}{b+k+r}} ct^\alpha, \alpha \right) \\ &\quad - \sinh \left(\frac{1}{2} \sqrt{-\frac{a}{b+k+r}}(x+y+z) \right) \\ &\quad \left. \sinh \left(\frac{1}{2} \sqrt{-\frac{a}{b+k+r}} ct^\alpha, \alpha \right) - 1 \right]. \end{aligned}$$

In order to construct compacton solutions of Eq. (17), we select initial approximations as follows:

$$u(x, y, z, 0) = \begin{cases} \frac{4c}{3a} \sin^2\left(\frac{1}{4} \sqrt{\frac{a}{b+k+r}}(x+y+z)\right), & |x+y+z| < \frac{\pi}{\mu}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$u(x, y, z, 0) = \begin{cases} \frac{4c}{3a} \cos^2\left(\frac{1}{4} \sqrt{\frac{a}{b+k+r}}(x+y+z)\right), & |x+y+z| < \frac{\pi}{2\mu}, \\ 0, & \text{otherwise} \end{cases}$$

By using the FVIMHP method, we can get the compacton solutions of Eq. (17) as follows

$$u(x, y, z, t) = \begin{cases} \frac{2c}{3a} \left[1 - \cos\left(\frac{1}{2} \sqrt{\frac{a}{b+k+r}}(x+y+z)\right) \cos\left(\frac{1}{2} \sqrt{\frac{a}{b+k+r}} ct^\alpha, \alpha \right) \right. \\ \quad \left. - \sin\left(\frac{1}{2} \sqrt{\frac{a}{b+k+r}}(x+y+z)\right) \sin\left(\frac{1}{2} \sqrt{\frac{a}{b+k+r}} ct^\alpha, \alpha \right) \right], & |x+y+z - ct^\alpha| < \frac{\pi}{\mu}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$u(x, y, z, t) = \begin{cases} \frac{2c}{3a} \left[1 + \cos\left(\frac{1}{2} \sqrt{\frac{a}{b+k+r}}(x+y+z)\right) \cos\left(\frac{1}{2} \sqrt{\frac{a}{b+k+r}} ct^\alpha, \alpha \right) \right. \\ \quad \left. + \sin\left(\frac{1}{2} \sqrt{\frac{a}{b+k+r}}(x+y+z)\right) \sin\left(\frac{1}{2} \sqrt{\frac{a}{b+k+r}} ct^\alpha, \alpha \right) \right], & |x+y+z - ct^\alpha| < \frac{\pi}{2\mu}, \\ 0, & \text{otherwise} \end{cases}$$

where $\mu = \frac{1}{4} \sqrt{\frac{a}{b+k+r}}$.

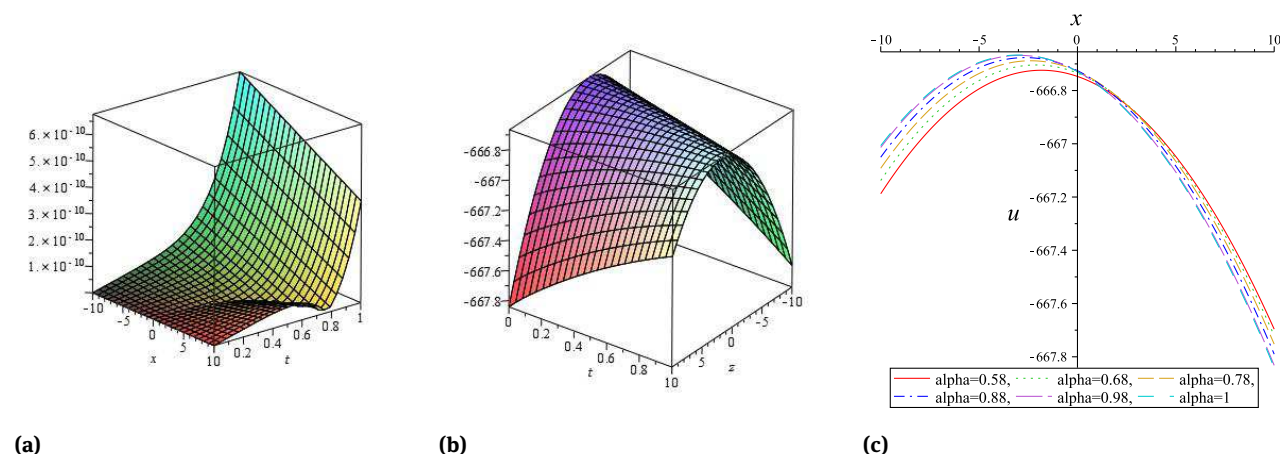


Figure 3: (a) the absolute error between the fifth-order approximate solution (20) and exact solution when $\alpha = 1$, $y = 0.1$, $z = 0.2$ (b) the fifth-order approximate solution (20) when $x = 2$, $y = 1$, $\alpha = 0.94$ (c) the fifth-order approximate solution (20) for different values of α when $t = 0.6$, $y = 1$, $z = 5$, $c = 5$; $a = -0.01$; $b = 10$; $k = 20$; $r = 30$

3.4 (3+1)-dimensional time fractional K-P equation

We consider the following (3+1)-dimensional time fractional K-P equation:

$$\left[D_t^\alpha u + \frac{a}{2} (u^2)_x + b(u(u_{xx}))_x \right]_x + u_{yy} + u_{zz} = 0, \quad 0 < \alpha \leq 1. \quad (21)$$

with the initial condition

$$u(x, y, z, 0) = \frac{4(c-2)}{a} \cosh^2 \left(\frac{1}{2} \sqrt{-\frac{a}{2b}} (x + y + z) \right). \quad (22)$$

According to the FVMHP method, its iteration formula can be constructed in the following form:

$$\begin{aligned} \sum_{m=0}^{\infty} p^m u_m = u_0 + p \left\{ \sum_{m=1}^{\infty} p^m u_m - \frac{1}{\Gamma(1+\alpha)} \right. \\ \int_0^t \left[\sum_{m=0}^{\infty} p^m D_t^\alpha u_m + \frac{a}{2} \left(\left(\sum_{m=0}^{\infty} p^m u_m \right)^2 \right)_x \right. \\ \left. + b \left(\sum_{m=0}^{\infty} p^m u_m \left(\sum_{m=0}^{\infty} p^m u_m \right)_{xx} \right)_x \right] \\ \left. + \sum_{m=0}^{\infty} p^m (u_{myy} + u_{mzz})(d\tau)^\alpha \right\}, \quad (23) \end{aligned}$$

By initial approximation $u_0(x, y, z, t) = u(x, y, z, 0) = \frac{4(c-2)}{a} \cosh^2 \left(\frac{1}{2} \sqrt{-\frac{a}{2b}} (x + y + z) \right)$, we can obtain

$$p^1 : u_1(x, y, t) = -\frac{2(c-2)}{a} \left(\frac{c}{2} \sqrt{-\frac{a}{2b}} \right)$$

$$\begin{aligned} & \cdot \sinh \left(\sqrt{-\frac{a}{2b}} (x + y + z) \right) \frac{t^\alpha}{\Gamma(1+\alpha)}, \\ p^2 : u_2(x, y, t) &= \frac{2(c-2)}{a} \left(\frac{c}{2} \sqrt{-\frac{a}{2b}} \right)^2 \\ & \cdot \cosh \left(\sqrt{-\frac{a}{2b}} (x + y + z) \right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \\ p^3 : u_3(x, y, t) &= -\frac{2(c-2)}{a} \left(\frac{c}{2} \sqrt{-\frac{a}{2b}} \right)^3 \\ & \cdot \sinh \left(\sqrt{-\frac{a}{2b}} (x + y + z) \right) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \\ p^4 : u_4(x, y, t) &= \frac{2(c-2)}{a} \left(\frac{c}{2} \sqrt{-\frac{a}{2b}} \right)^4 \\ & \cdot \cosh \left(\sqrt{-\frac{a}{2b}} (x + y + z) \right) \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}, \\ & \vdots \end{aligned}$$

Consequently, we have the following solitary solution of Eq. (21)

$$\begin{aligned} u(x, y, z, t) &= \frac{2(c-2)}{a} \left[\cosh \left(\sqrt{-\frac{a}{2b}} (x + y + z) \right) \right. \\ & \cdot \cosh \left(\sqrt{-\frac{a}{2b}} ct^\alpha, \alpha \right) \\ & - \sinh \left(\sqrt{-\frac{a}{2b}} (x + y + z) \right) \\ & \left. \cdot \sinh \left(\sqrt{-\frac{a}{2b}} ct^\alpha, \alpha \right) + 1 \right]. \quad (24) \end{aligned}$$

We show some properties of the approximate solution (24) obtained by the proposed method in Fig. 4. The plot (a) shows the absolute error between the approximate

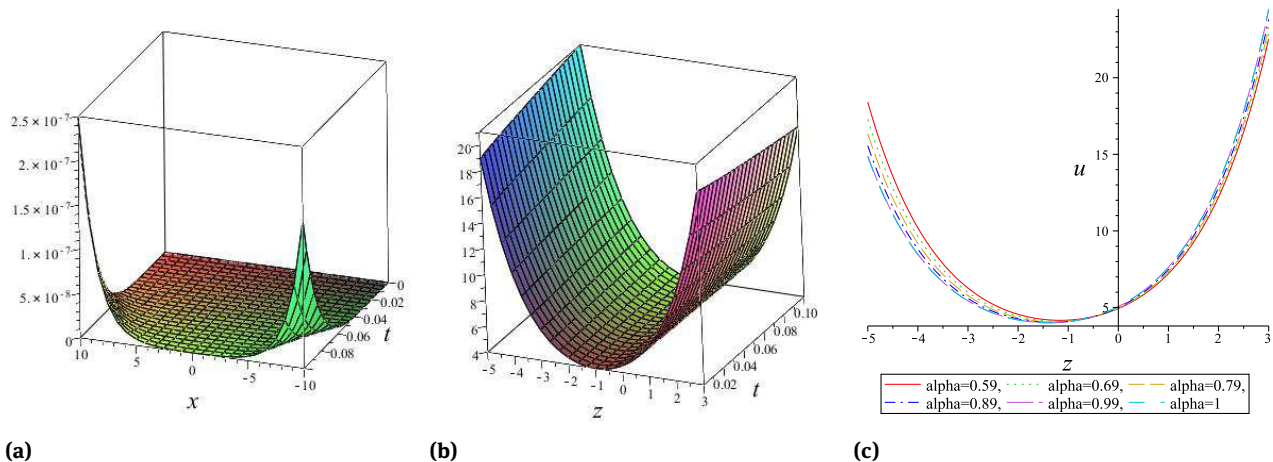


Figure 4: (a) the absolute error between the fifth-order approximate solution (24) and exact solution when $\alpha = 1, y = 0.1, z = 0.2$ (b) the fifth-order approximate solution (24) when $x = 1, y = 0, \alpha = 0.82$ (c) the fifth-order approximate solution (24) for different values of α when $t = 0.6, x = 1, y = 1, c = 1; a = -1; b = 1$

solution (24) and exact solution. The plot (b) shows the approximate solution expressed by (24) when α is fixed. And the plot (c) shows the approximate solution (24) for different values of α when time is fixed.

If we choose $u(x, y, z, 0) = -\frac{4(c-2)}{a} \sinh^2(\frac{1}{2} \sqrt{-\frac{a}{2b}}(x+y+z))$, we can get the solitary solution of Eq. (22) as follows:

$$u(x, y, z, t) = -\frac{2(c-2)}{a} \left[\cosh \left(\sqrt{-\frac{a}{2b}}(x+y+z) \right) \cdot \cosh \left(\sqrt{-\frac{a}{2b}} ct^\alpha, \alpha \right) - \sinh \left(\sqrt{-\frac{a}{2b}}(x+y+z) \right) \sinh \left(\sqrt{-\frac{a}{2b}} ct^\alpha, \alpha \right) - 1 \right].$$

In order to construct compacton solutions of Eq. (22), we select initial approximations as follows:

$$u(x, y, z, 0) = \begin{cases} \frac{4(c-2)}{a} \sin^2(\frac{1}{2} \sqrt{\frac{a}{2b}}(x+y+z)), & |x+y+z| < \frac{\pi}{\mu}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$u(x, y, z, 0) = \begin{cases} \frac{4(c-2)}{a} \cos^2(\frac{1}{2} \sqrt{\frac{a}{2b}}(x+y+z)), & |x+y+z| < \frac{\pi}{2\mu}, \\ 0, & \text{otherwise} \end{cases}$$

By using the FVIMHP method, we can get the following compacton solutions of Eq. (22)

$$u(x, y, z, t) = \begin{cases} \frac{2(c-2)}{a} [1 - \cos(\sqrt{\frac{a}{2b}}(x+y+z)) \cdot \cos(\frac{1}{2} \sqrt{\frac{a}{2b}} ct^\alpha, \alpha)] \\ - \sin(\sqrt{\frac{a}{2b}}(x+y+z)) \sin(\sqrt{\frac{a}{2b}} ct^\alpha, \alpha), \\ |x+y+z - ct^\alpha| < \frac{\pi}{\mu}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$u(x, y, z, t) = \begin{cases} \frac{2(c-2)}{a} [1 + \cos(\sqrt{\frac{a}{2b}}(x+y+z)) \cdot \cos(\sqrt{\frac{a}{2b}} ct^\alpha, \alpha) \\ + \sin(\sqrt{\frac{a}{2b}}(x+y+z)) \sin(\sqrt{\frac{a}{2b}} ct^\alpha, \alpha)], \\ |x+y+z - ct^\alpha| < \frac{\pi}{2\mu}, \\ 0, & \text{otherwise} \end{cases}$$

where $\mu = \frac{1}{2} \sqrt{\frac{a}{2b}}$.

4 Conclusion

In this paper, we apply the FVIMHP method to obtain the solitary and compacton solutions of fractional KdV-like equations. The numerical results given in Section 3 demonstrate the good accuracy of the proposed method. The results show that the FVIMHP method is direct, effective, and can be useful in dealing with many other fractional partial differential equations. Our results confirm that the method takes all the advantages of the variational iteration method, homotopy perturbation method, and Jumarie's modified Riemann-Liouville derivative. The comparison made with the exact solutions, enables us to see the accuracy of the FVIMHP method clearly. It is worth mentioning that the FVIMHP method is capable of reducing the volume of the computational work. In our future studies, we will solve many other nonlinear fractional partial differential equations by this method.

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