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Approximate Solutions to the Nonlinear Klein-Gordon Equation in de Sitter Spacetime

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Abstract: We consider initial value problems for the nonlinear Klein-Gordon equation in de Sitter spacetime. We use the differential transform method for the solution of the initial value problem. In order to show the accuracy of results for the solutions, we use the variational iteration method with Adomian's polynomials for the nonlinearity. We show that the methods are effective and useful.

Keywords: de Sitter spacetime; Klein-Gordon equation; differential transform method; variational iteration method; Adomian's polynomials

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1 Introduction

In this article, we are interested in the initial value problem for the nonlinear Klein-Gordon equation in de Sitter spacetime,

$$\begin{aligned}&\phi_{tt} + nH\phi_t - e^{-2Ht}\Delta\phi + m^2\phi = |\phi|^{p-1}\phi, \quad (x,t) \in \mathbb{R}^n \times [0,\infty),\\ &\phi(x,0) = \varphi_0(x), \quad \phi_t(x,0) = \varphi_1(x), \quad x \in \mathbb{R}^n, \quad x \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where $m > 0$ represents physical mass, H is the Hubble constant and $p > 1$. The sign of H specifies the model of the universe. If $H < 0$, then it is called the anti de Sitter spacetime model while $H = 0$ determines the Minkowski spacetime model. On the other hand, when $H > 0$, then the so-called de Sitter spacetime model decribes exponential expansion of the universe.

The Klein-Gordon equation arises in relativistic physics such as cosmology and in general relativity, in particular in quantum field theory. We briefly explain how the equation in (1) is deduced.

The line element in de Sitter spacetime is given by

$$\begin{aligned}ds^2 = &-\left(1 - \frac{r^2}{R^2}\right) dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2\\ &+ r^2(d\theta^2 + \sin^2\theta d\phi^2),\end{aligned} \tag{2}$$

where R is the radius of the universe. By using the Lemaitre-Robertson transformation in [1],

$$\begin{aligned}&r' = \frac{r}{\sqrt{1 - r^2/R^2}} e^{-t/R}, \quad t' = t + \frac{R}{2}\ln\left(1 - \frac{r^2}{R^2}\right),\\ &\theta' = \theta, \quad \phi' = \phi,\end{aligned}$$

the line element has the following form

$$ds^2 = -dt'^2 + e^{2t'/R}\left(dr'^2 + r'^2 d\theta'^2 + r'^2\sin^2\theta' d\phi'^2\right). \tag{3}$$

Changing the coordinates as

$$\begin{aligned}&t = t', \quad x_1 = r'\sin\theta'\cos\phi', \quad x_2 = r'\sin\theta'\sin\phi',\\ &x_3 = r'\cos\theta',\end{aligned}$$

we get

$$ds^2 = -dt^2 + e^{2Ht}\left(dx_1^2 + dx_2^2 + dx_3^2\right), \tag{4}$$

where $H = 1/R$.

We may write the line element in general spatial dimensions as

$$ds^2 = -dt^2 + e^{2Ht}\left(dx_1^2 + \ldots + dx_n^2\right).$$

Thus the corresponding metric is

$$(g_{ik})_{0\le i,k\le n} := \mathrm{diag}(-1, e^{2Ht}, \ldots, e^{2Ht}).$$

Let $g := \det(g_{ik})_{0\le i,k\le n}$ and $(g^{ik})_{0\le i,k\le n}$ be the inverse matrix of $(g_{ik})_{0\le i,k\le n}$. Then the scalar field ϕ in de Sitter spacetime is described by the following equation:

$$\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_i}\left(\sqrt{|g|}g^{ik}\frac{\partial\phi}{\partial x_k}\right) = m^2\phi - V'(\phi),$$

where $x_0 := t$ and $V(\phi)$ is a potential function. More explicitly, we get

$$\phi_{tt} + nH\phi_t - e^{-2Ht}\Delta\phi + m^2\phi = -V'(\phi),$$

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$$(x, t) \in \mathbb{R}^n \times [0, \infty). \tag{5}$$

Setting $|\phi|^{p-1}\phi = -V'(\phi)$, we obtain the equation in (1).

In Minkowski spacetime, the initial value problem for the semilinear Klein-Gordon equation

$$u_{tt} - \Delta u + m^2 u = |u|^\alpha u, \tag{6}$$

has been extensively investigated. The existence of global weak solutions has been obtained by Jörgens [2], Pecher [3], Brenner [4], Ginibre and Velo [5, 6]. On the other hand, the initial value problem for so-called Higgs boson equation

$$u_{tt} - \Delta u - m^2 u = -|u|^\alpha u, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}$$

in Minkowski spacetime, and

$$\phi_{tt} + nH\phi_t - e^{-2Ht}\Delta\phi - m^2\phi = -|\phi|^\alpha \phi, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}$$

in de Sitter spacetime have been studied by Yagdjian [7], and the necessary conditions have been derived for the existence of the global solution that the solution has a changing sign and is oscillating in time.

Turning back to the initial value problem (1), the small data global existence result is proved by Yagdjian [8] in Sobolev space $H^s(\mathbb{R}^n)$ for $s > n/2$ when $m \in (0, \sqrt{n^2 - 1}/2) \cup [n/2, \infty)$. In Nakamura [9], the existence of local and global solutions with power type nonlinear terms are shown by using the energy method in the case of large mass, *i.e.*, $m \geq n/2$.

Our first aim in this article is to give approximate solutions of (1) based on the initial data by using the differential transform method in de Sitter spacetime. This method was first considered by Zhou [10] for solving initial value problems in electrical circuit analysis. Jang, Chen and Liu [11] used the two dimensional differential transform for obtaining the analytic solutions of linear and nonlinear partial differential equations. In addition, Kurnaz, Oturanç and Kiris [12] generalized the transform method to the n dimensional case for solving partial differential equations.

In Minkowski spacetime (that is, $H = 0$), the initial value problem for the Klein-Gordon equation

$$u_{tt} - \Delta u + u = u^p \quad \mathbb{R}^n \times [0, \infty),$$

where $p \geq 2$ has been studied with the differential transform method by Kanth and Aruna [13] in one spatial dimension and by Do and Jang [14] in higher spatial dimension.

On the other hand, in order to illustrate our results, we use another method called variational iteration. This method which is iterative based on a correction functional with a Lagrange multiplier was first considered by He [15, 16]. It was applied to the Klein-Gordon equation by Yusufoglu [17] in Minkowski spacetime.

This paper is organized as follows. In Section 2, we give the definition of the differential transform and some basic properties of the transform. The basic concepts of the variational iteration method are given in Section 2.1. Section 3 is devoted to some numerical examples. We apply the methods to the linear and nonlinear Klein-Gordon equations in de Sitter spacetime to investigate the solutions. The results obtained by the differential transform method are compared with the variational iteration method. We give the conclusion in the last section.

2 Preliminaries

2.1 Differential Transform Method

We give the definition and some properties of differential transformations for solving (1). (See, e.g., [11–13].)

Let the function $u = u(x, t)$ be analytic in the domain D and let $(x_0, t_0) \in D$. Then the differential transform $U(k, h)$ of the function $u(x, t)$ which is the series expanded at $(x_0, t_0) \in D$ defined by

$$U(k, h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x, t)}{\partial x^k \partial t^h}\right]_{(x_0,t_0)}. \tag{7}$$

The differential inverse transform of $U(k, h)$ is defined by

$$u(x, t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k, h)(x - x_0)^k (t - t_0)^h. \tag{8}$$

The following fundamental properties of differential transformations are listed in [11–13]. Since the proofs are directly the result of (7), we give only their statements.

Theorem 2.1. *Let $c \in \mathbb{R}$ be a constant. Assume that $U(k, h)$, and $V(k, h)$ are the differential transforms of the functions $u(x, t)$ and $v(x, t)$ respectively. Then we have the following properties for linearity:*

1. *If $w(x, t) = u(x, t) \pm v(x, t)$ then, $W(k, h) = U(k, h) \pm V(k, h)$.*
2. *If $w(x, t) = cu(x, t)$ then, $W(k, h) = cU(k, h)$.*

Theorem 2.2. *Let $U(k, h)$ be the differential transform of the function $u(x, t)$. If $w(x, t) = \dfrac{\partial^{p+q}u(x, t)}{\partial x^p \partial t^q}$, then $W(k, h) = \dfrac{(k + p)!}{p!}\dfrac{(h + q)!}{q!}U(k + p, h + q)$.*

Theorem 2.3. *Let $U(k,h)$ and $V(k,h)$ be the differential transforms of the functions $u(x,t)$ and $v(x,t)$ respectively. If $w(x,t) = u(x,t)v(x,t)$, then we have the transformation* $W(k,h) = \sum_{p=0}^{k}\sum_{q=0}^{h} U(k-p,q)V(p,h-q)$.

Theorem 2.4. *Let $a, b \in \mathbb{R}$ be constants. If $w(x,t) = e^{ax+bt}$, then $W(k,h) = \frac{a^k b^h}{k!h!}$.*

2.2 Variational Iteration Method

In this subsection, basic concepts of the variational iteration method are given for the general nonlinear differential equation

$$Lu(x,t) + Nu(x,t) = g(x,t) \tag{9}$$

where L is a linear operator, N is a nonlinear operator and $g(x,t)$ is a given analytic function. By [15], the correction functional for (9) is written as

$$u_{i+1}(x,t) = u_i(x,t) + \int_0^t \lambda \left(Lu_i(x,\tau) + N\tilde{u}_i(x,\tau) - g(x,\tau)\right) d\tau, \quad i \geq 0, \tag{10}$$

where λ is a Lagrange multiplier and $\tilde{u}_i$ is a restricted variation which is $\delta\tilde{u}_i = 0$. The Lagrange multiplier λ is obtained via integration by parts from the restricted variation of the correction functional $\delta u_{i+1} = 0$. (See, *e.g.*, [15, 16, 18].)

3 Applications

In this section, the differential transform method is applied to solve the linear and nonlinear Klein-Gordon equations in de Sitter spacetime. To illustrate the accuracy of the results, we compare them with the results obtained by using the variational iteration method. We have used Mathematica 10 for the results. However, we notice that the computations in the nonlinear term for the variational iterational method become complicated. In order to overcome the difficulty arising in calculating, we apply the variational iteration method with Adomian's polynomials for the nonlinear part proposed in [19, 20]. For simplicity, we take $H = 1$ and $m = 1$.

Example 1. We first consider the initial value problem for the linear Klein-Gordon equation in de Sitter spacetime,

$$\phi_{tt} + \phi_t - e^{-2t}\Delta\phi + \phi = 0, \quad (x,t) \in \mathbb{R}\times[0,\infty),$$

$$\phi(x,0) = e^{-x}, \quad \phi_t(x,0) = 0, \quad x \in \mathbb{R}. \tag{11}$$

If we take the differential transform of the equation in (11), by using Theorem 2.1, Theorem 2.2 and Theorem 2.3, we get

$$(h+1)(h+2)\Phi(k,h+2) + (h+1)\Phi(k,h+1) - \sum_{s=0}^{h}\frac{(-2)^{h-s}}{(h-s)!}\left[(k+1)(k+2)\Phi(k+2,s)\right] + \Phi(k,h) = 0. \tag{12}$$

Hence we have

$$\Phi(k,h+2) = -\frac{1}{(h+2)}\Phi(k,h+1) - \frac{1}{(h+1)(h+2)}\Phi(k,h) + \frac{1}{(h+1)(h+2)}\sum_{s=0}^{h}\frac{(-2)^{h-s}}{(h-s)!}[(k+1)(k+2)\Phi(k+2,s)], \tag{13}$$

for $h = 0, 1, 2, \ldots$. From Theorem 2.4, the transforms of the initial conditions in (11) are

$$\Phi(k,0) = \frac{(-1)^k}{k!}, \qquad \Phi(k,1) = 0. \tag{14}$$

Substituting (14) into (13), we obtain the closed form of the solution as

$$\begin{aligned}\phi(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\Phi(k,h)x^k t^h \\ &= \left(1 - \frac{t^3}{3} + \frac{t^4}{4} - \frac{7t^5}{60} + \frac{23t^6}{360} - \frac{109t^7}{2520} + \frac{113t^8}{4032} - \frac{2939t^9}{181440} + \ldots\right) \\ &\quad\times\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots\right) \\ &= \left(1 - \frac{t^3}{3} + \frac{t^4}{4} - \frac{7t^5}{60} + \frac{23t^6}{360} - \frac{109t^7}{2520} + \frac{113t^8}{4032} - \frac{2939t^9}{181440} + \ldots\right)e^{-x}.\end{aligned} \tag{15}$$

On the other hand, if we apply the variational iteration method, we construct the correction functional as

$$\phi_{i+1}(x,t) = \phi_i(x,t) + \int_0^t \lambda\left(\phi_{i\tau\tau}(x,\tau) + \phi_{i\tau}(x,\tau) - e^{-2\tau}\tilde{\phi}_{ixx}(x,\tau) + \phi_i(x,\tau)\right)d\tau. \tag{16}$$

In order to make (16) stationary, and noticing that $\delta\tilde{\phi}_i = 0$, we get

$$\delta\phi_{i+1}(x,t) = \delta\phi_i(x,t) + \delta\int_0^t \lambda\left(\phi_{i\tau\tau}(x,\tau) + \phi_{i\tau}(x,\tau)\right.$$

$$-e^{-2\tau}\tilde{\phi}_{ixx}(x,\tau)+\phi_i(x,\tau)\Big)\,d\tau. \tag{17}$$

By using integration by parts, we have the following conditions

$$\lambda'' - \lambda' + \lambda = 0, \tag{18}$$

$$1 + \lambda - \lambda'\big|_{\tau=t} = 0, \tag{19}$$

$$\lambda\big|_{\tau=t} = 0. \tag{20}$$

Therefore the Lagrange multiplier has the following form:

$$\lambda(\tau) = \frac{2\sin[\sqrt{3}(\tau-t)/2]}{\sqrt{3}}e^{(\tau-t)/2}. \tag{21}$$

Hence we obtain the iterative formula

$$\begin{aligned}\phi_{i+1}(x,t) = \phi_i(x,t) + \int_0^t \frac{2\sin[\sqrt{3}(\tau-t)/2]}{\sqrt{3}}e^{(\tau-t)/2}\\ \times\Big(\phi_{i\tau\tau}(x,\tau)+\phi_{i\tau}(x,\tau)-e^{-2\tau}\phi_{ixx}(x,\tau)\\ +\phi_i(x,\tau)\Big)\,d\tau,\end{aligned} \tag{22}$$

for $i \geq 0$ where we set the first step

$$\phi_0(x,t) = \phi(x,0) + t\phi_t(x,0) = e^{-x}. \tag{23}$$

Using the iteration formula (22), we obtain

$$\begin{aligned}\phi_1(x,t) &= \frac{1}{3}e^{-2t-x} + \frac{2}{3}e^{-x-t/2}\Big(\cos(\sqrt{3}t/2)\\ &\quad +\sqrt{3}\sin(\sqrt{3}t/2)\Big),\\ \phi_2(x,t) &= \frac{7e^{-4t-x}}{273} + \frac{e^{-5t/2-x}}{273}\Big(\left(65+201e^{2t}\right)\cos(\sqrt{3}t/2)\\ &\quad +\sqrt{3}\left(13+181e^{2t}\right)\sin(\sqrt{3}t/2)\Big),\\ \phi_3(x,t) &= \frac{532e^{-6t-x}}{643188} + \frac{e^{-9t/2-x}}{643188}\Big(\left(9269+159030e^{2t}\right.\\ &\quad \left.+474357e^{4t}\right)\cos(\sqrt{3}t/2)+\sqrt{3}\,(-403\\ &\quad +27094e^{2t}+426413e^{4t}\right)\sin(\sqrt{3}t/2)\Big),\end{aligned} \tag{24}$$

and so on. A closed form solution is not obtainable for the initial value problem (11). Therefore this approximation can only be used for numerical purposes. In order to illustrate our results, we use another method called the projected differential method. This method which is a series solution with respect to the variable t at t_0 was introduced in [14]. Since it is similar to the differential transform method, we omit the statements. The comparison between the sixth iteration solution of the variational iteration method, the differential transform method and the projected differential transform method are given in Table 1.

Example 2. We consider the initial value problem for the nonlinear Klein-Gordon equation in de Sitter spacetime,

$$\begin{aligned}&\phi_{tt}+\phi_t-e^{-2t}\Delta\phi+\phi=|\phi|^2\phi, \quad (x,t)\in\mathbb{R}\times[0,\infty),\\ &\phi(x,0)=e^{-x}, \quad \phi_t(x,0)=0, \quad x\in\mathbb{R}.\end{aligned} \tag{25}$$

If we take the differential transform of the equation in (25), by using Theorem 2.1, Theorem 2.2 and Theorem 2.3, we get

$$\begin{aligned}&(h+1)(h+2)\Phi(k,h+2)+(h+1)\Phi(k,h+1)\\ &-\sum_{s=0}^{h}\frac{(-2)^{h-s}}{(h-s)!}\left[(k+1)(k+2)\Phi(k+2,s)\right]+\Phi(k,h)\\ &=\sum_{w=0}^{k}\sum_{v=0}^{k-w}\sum_{s=0}^{h}\sum_{m=0}^{h-s}\Phi(w,h-s-m)\Phi(w,s)\Phi(k-w-v,m).\end{aligned} \tag{26}$$

Hence we have

$$\begin{aligned}\Phi(k,h+2) &= -\frac{1}{(h+2)}\Phi(k,h+1)-\frac{1}{(h+1)(h+2)}\Phi(k,h)\\ &+\frac{1}{(h+1)(h+2)}\sum_{s=0}^{h}\frac{(-2)^{h-s}}{(h-s)!}[(k+1)(k+2)\Phi(k+2,s)]\\ &+\frac{1}{(h+1)(h+2)}\sum_{w=0}^{k}\sum_{v=0}^{k-w}\sum_{s=0}^{h}\sum_{m=0}^{h-s}\Phi(w,h-s-m)\Phi(w,s)\\ &\cdot\Phi(k-w-v,m),\end{aligned} \tag{27}$$

for $h = 0, 1, 2, \ldots$. From Theorem 2.4, the transforms of the initial conditions in (25) are

$$\Phi(k,0)=\frac{(-1)^k}{k!}, \qquad \Phi(k,1)=0. \tag{28}$$

Substituting (28) into (27), we obtain the closed form of the solution as

$$\begin{aligned}\phi(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\Phi(k,h)x^k t^h &= 1+\frac{t^2}{2}-\frac{t^3}{2}+\frac{3t^4}{4}-\frac{97t^5}{120}\\ &-x-\frac{3t^2x}{2}-\frac{5t^3x}{6}-2t^4x+\frac{55t^5x}{24}+\frac{x^2}{2}+\frac{9t^2x^2}{4}\\ &-\frac{11t^3x^2}{12}-\frac{x^3}{6}-\frac{9t^2x^3}{4}+\frac{29t^3x^3}{36}+\frac{x^4}{24}-\frac{83t^3x^4}{144}\\ &+\ldots.\end{aligned} \tag{29}$$

On the other hand, if we apply the variational iteration method to (25), we have the following the correction functional as

$$\phi_{i+1}(x,t)=\phi_i(x,t)+\int_0^t\lambda\big(\phi_{i\tau\tau}(x,\tau)+\phi_{i\tau}(x,\tau)$$

$$- e^{-2\tau}\tilde{\phi}_{ixx}(x,\tau) + \phi_i(x,\tau) - |\phi_i|^2\phi_i\big)d\tau. \quad (30)$$

In order to make (30) stationary, and noticing that $\delta\tilde{\phi}_i = 0$, we get

$$\delta\phi_{i+1}(x,t) = \delta\phi_i(x,t) + \delta\int_0^t \lambda\big(\phi_{i\tau\tau}(x,\tau) + \phi_{i\tau}(x,\tau) - e^{-2\tau}\tilde{\phi}_{ixx}(x,\tau) + \phi_i(x,\tau) - \tilde{\phi}_i{}^3(x,\tau)\big)d\tau. \quad (31)$$

Due to the stationary condition for the nonlinear part, we have the same Lagrange multiplier with (21). Hence we obtain the iterative formula

$$\phi_{i+1}(x,t) = \phi_i(x,t) + \int_0^t \frac{2\sin[\sqrt{3}(\tau-t)/2]}{\sqrt{3}}e^{(\tau-t)/2} \cdot \big(\phi_{i\tau\tau}(x,\tau) + \phi_{i\tau}(x,\tau) - e^{-2\tau}\phi_{ixx}(x,\tau) + \phi_i(x,\tau) - \phi_i^3(x,\tau)\big)d\tau, \quad (32)$$

for $i \geq 0$. The nonlinear part $N(\phi) = |\phi|^2\phi$ in (25) can be expressed by the Adomian's polynomials as follows

$$N(\phi) = \sum_{i=0}^{\infty} A_i. \quad (33)$$

The polynomials A_i are defined in [21] by

$$\begin{aligned} A_0 &= N(\phi_0),\\ A_1 &= \phi_1 N'(\phi_0),\\ A_2 &= \phi_2 N(\phi_0) + \frac{1}{2}\phi_1^2 N''(\phi_0),\\ A_3 &= \phi_3 N'(\phi_0) + \phi_1\phi_2 N''(\phi_0) + \frac{1}{3!}\phi_1^3 N'''(\phi_0),\\ &\dots, \end{aligned} \quad (34)$$

where we set

$$\phi_0(x,t) = \phi(x,0) + t\phi_t(x,0) = e^{-x}. \quad (35)$$

The Adomian's method defines the series solution $\phi = \phi(x,t)$ by

$$\phi(x,t) = \sum_{i=0}^{\infty}\phi_i(x,t). \quad (36)$$

Substituting (33) and (36) into (32), the components ϕ_i are obtained by

$$\phi_{i+1}(x,t) = \int_0^t \frac{2\sin[\sqrt{3}(\tau-t)/2]}{\sqrt{3}}e^{(\tau-t)/2}\Big(\sum_{j=0}^{i}\phi_{j\tau\tau}(x,\tau) + \sum_{j=0}^{i}\phi_{j\tau}(x,\tau) - e^{-2\tau}\sum_{j=0}^{i}\phi_{jxx}(x,\tau) + \sum_{j=0}^{i}\phi_j(x,\tau) - \sum_{j=0}^{i}A_j\Big)d\tau, \quad (37)$$

for $i \geq 0$. From the iteration formula (37), we obtain

$$\phi_1(x,t) = \frac{\sqrt{3}}{6}e^{-2t+3x}\left(3e^{2t} + e^{2x} - 3e^{2(t+x)}\right) + \frac{\sqrt{3}}{6}e^{-t/2+3x} \times \left((-3+3e^{2x})\cos(\sqrt{3}t/2) + (-3+6e^{2x})\sin(\sqrt{3}t/2)\right),$$

$$\begin{aligned} \phi_2(x,t) = {} & \frac{1}{52}e^{-4t-x} + \frac{1}{6}e^{-2t-3x}\left(15 + (-3\sqrt{3})e^{2x}\right)\\ & - \frac{1}{4}\left(-e^{-5x} + e^{-3x}\right)\times\left(9 + (-3+2\sqrt{3})e^{2x}\right)\\ & + \left[\frac{195}{1092}e^{-5t/2-x} + \left(\frac{-489}{1092} + \frac{\sqrt{3}}{3}\right)e^{-t/2-x}\right.\\ & + \frac{39}{2184}e^{-5t/2-3x}(-81 + e^{2t}(109 - 28\sqrt{3} - 84t))\\ & \left. + \frac{1638}{2184}e^{-t/2-5x}\times(t-3)\right]\cos(\sqrt{3}t/2)\\ & + \frac{39\sqrt{3}}{2184}e^{-5t/2-3x}\left(9 + 2e^{2x}\right)\sin(\sqrt{3}t/2)\\ & - \left[\frac{1}{728}\left(-728 + 366\sqrt{3}\right)e^{-t/2-x}\right.\\ & + \frac{\sqrt{3}}{12}(5+3t)e^{-t/2-5x}\\ & \left. - \frac{39}{2184}(-28 + 135\sqrt{3} + 28\sqrt{3}t)e^{-t/2-3x}\right]\\ & \cdot\sin(\sqrt{3}t/2), \end{aligned} \quad (38)$$

and so on. A closed form solution is not obtainable for the initial value problem (25). Therefore we can only use this approximation for numerical values of the solution. The comparison between the fourth iteration solution of the variational iteration method, the differential transform method and the projected differential transform method are given in Table 2.

4 Conclusion

In this contribution, we have considered the Klein-Gordon equations in de Sitter spacetime. The lack of results for the global solutions of such nonlinear equations motivate us to approach the solutions approximately. Therefore, differential transforms and variational iteration methods were used. To overcome the computational difficulty arising from the nonlinear term, we have used Adomian's polynomials with the variational iterational method. Since the analytical solutions of these initial value problems are not obtainable from these approaches, we deal with the numerical results. As shown in Table 1 and Table 2, we get

Table 1: Comparison between the value ϕ for the solution of the linear Klein-Gordon equation for differential transform method (DTM), variational iteration method (VIM) and projected differential transform method (PDTM) at values of (x, t).

	x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
t=0.1	DTM	0.904557	0.818477	0.740589	0.670113	0.606343	0.548642	0.496432	0.44919	0.406444	0.367766
	VIM	0.904557	0.818477	0.740589	0.670113	0.606343	0.548642	0.496432	0.44919	0.406444	0.367766
	PDTM	0.904557	0.818477	0.740589	0.670113	0.606343	0.548642	0.496432	0.44919	0.406444	0.367766
t=0.2	DTM	0.902756	0.816847	0.739114	0.668778	0.605135	0.547549	0.495443	0.448295	0.405634	0.367033
	VIM	0.902756	0.816847	0.739114	0.668778	0.605135	0.547549	0.495443	0.448295	0.405634	0.367033
	PDTM	0.902756	0.816847	0.739114	0.668778	0.605135	0.547549	0.495443	0.448295	0.405634	0.367033
t=0.3	DTM	0.898305	0.81282	0.73547	0.66548	0.602152	0.544849	0.493	0.446085	0.403634	0.365223
	VIM	0.898305	0.81282	0.73547	0.66548	0.602152	0.544849	0.493	0.446085	0.403634	0.365223
	PDTM	0.898305	0.81282	0.73547	0.66548	0.602152	0.544849	0.493	0.446085	0.403634	0.365223
t=0.4	DTM	0.89043	0.805695	0.729023	0.659647	0.596873	0.540073	0.488679	0.442175	0.400096	0.362022
	VIM	0.89043	0.805695	0.729023	0.659647	0.596873	0.540073	0.488679	0.442175	0.400096	0.362022
	PDTM	0.89043	0.805695	0.729023	0.659647	0.596873	0.540073	0.488679	0.442175	0.400096	0.362022
t=0.5	DTM	0.881532	0.797643	0.721738	0.653055	0.590909	0.534676	0.483795	0.437756	0.396098	0.358404
	VIM	0.878649	0.795035	0.719377	0.650919	0.588976	0.532928	0.482213	0.436324	0.394802	0.357232
	PDTM	0.878649	0.795035	0.719377	0.650919	0.588976	0.532928	0.482213	0.436324	0.394802	0.357232

Table 2: Comparison between the value ϕ for the solution of the nonlinear Klein-Gordon equation for differential transform method (DTM), variational iteration method with Adomian's polynomials (VIM-A) and projected differential transform method (PDTM) at values of (x, t).

	x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
t=0.1	DTM	0.908169	0.821152	0.74257	0.67158	0.607429	0.549447	0.497028	0.449631	0.406771	0.368008
	VIM-A	0.908168	0.821151	0.742569	0.671579	0.607429	0.549446	0.497028	0.449631	0.406771	0.368008
	PDTM	0.908148	0.821136	0.742558	0.671571	0.607423	0.549442	0.497024	0.449629	0.406769	0.368006
t=0.2	DTM	0.917019	0.827398	0.74692	0.674555	0.609412	0.550715	0.497787	0.450031	0.40692	0.367985
	VIM-A	0.917011	0.827392	0.746916	0.674553	0.60941	0.550714	0.497786	0.450031	0.406919	0.367985
	PDTM	0.916725	0.827181	0.74676	0.674437	0.609324	0.55065	0.497739	0.449995	0.406893	0.367966
t=0.3	DTM	0.930261	0.836413	0.752899	0.678363	0.611678	0.551896	0.498214	0.449943	0.40649	0.367338
	VIM-A	0.93023	0.836392	0.752885	0.678353	0.611671	0.551891	0.498211	0.449941	0.406489	0.367337
	PDTM	0.928957	0.835452	0.752191	0.67784	0.611291	0.55161	0.498003	0.449787	0.406375	0.367252
t=0.4	DTM	0.947313	0.847574	0.759892	0.682422	0.613689	0.552497	0.497862	0.448966	0.405119	0.365739
	VIM-A	0.947226	0.847517	0.759854	0.682397	0.613672	0.552486	0.497854	0.448961	0.405116	0.365737
	PDTM	0.943679	0.844907	0.757931	0.680979	0.612625	0.551712	0.497282	0.448537	0.404803	0.365505
t=0.5	DTM	0.967871	0.86049	0.767485	0.68633	0.615072	0.552178	0.496425	0.446823	0.402562	0.36297
	VIM-A	0.967736	0.860403	0.767429	0.686294	0.615049	0.552163	0.496415	0.446817	0.402559	0.362968
	PDTM	0.960062	0.854782	0.763302	0.683259	0.612814	0.550515	0.495198	0.445918	0.401894	0.362476

similar results for the solutions of the linear and nonlinear Klein-Gordon equations. Hence the numerical results reveal that the proposed methods are accurate and effective for the approximate solutions.

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