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On uninorms and nullnorms on direct product of bounded lattices

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Abstract: We will study uninorms on the unit square endowed with the natural partial order defined coordinatewise. We will show that we can choose arbitrary pairs of incomparable elements, (a, e) and construct a uninorm whose neutral element is e and annihilator is a. As a special case we construct uninorms which are at the same time also nullnorms (or, expressed another way, we construct proper nullnorms with neutral element). We will also generalize this result to the direct product of two bounded lattices. I.e., we will show that it is possible to construct nullnorms with a neutral element on the direct product of two bounded lattices.

Keywords: bounded lattice; nullnorm on bounded lattice; uninorm on bounded lattice

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1 Introduction and Preliminaries

Associative, commutative and monotone (binary) operations on the unit interval, due to their associativity, can be straightforwardly extended to n-ary operations for arbitrary $n \in \mathbb{N}$. This means they are special types of aggregation functions. As such they have proven their importance in various fields of applications, *e.g.*, neuron nets, fuzzy decision making and fuzzy modelling. Studying their behaviour is also interesting from a theoretical point of view. It is important for researchers to have many families of such operations to hand. Associative, commutative and monotone operations have recently been studied also on bounded lattices (see, *e.g.*, [2, 7, 10, 11]).

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1.1 Associative Commutative and Monotone Operations on [0, 1]

Currently, we distinguish several types of associative commutative and monotone (isotone) operations. In fact, we will consider only increasing operations. We say that an operation $\star\colon [0,1]^2 \to [0,1]$ possesses:

(NE) a neutral element $e \in [0,1]$ if for every $x \in [0,1]$

$$x \star e = e \star x = x,$$

(AE) an absorbing element (called also annihilator) $a \in [0,1]$ if for every $x \in [0,1]$

$$x \star a = a \star x = a,$$

(IE) an idempotent element $i \in [0,1]$ if $i \star i = i$

Lemma 1. *Let $\star\colon [0,1]^2 \to [0,1]$ be an associative commutative and monotone operation. Then $\star$ has an idempotent element i which is also an absorbing element.*
If $\star$ has a neutral element e then 0, 1 and e are idempotent elements. Further, $a = 0 \star 1$ is the absorbing element of $\star$.

Schweizer and Sklar [13] introduced the notion of a triangular norm (t-norm for brevity).

Definition 1 ([13]). *An operation $T\colon [0,1]^2 \to [0,1]$ is a t-norm if it is associative, commutative, monotone, and 1 is its neutral element.*

T-norms and t-conorms are dual to each other. If $T\colon [0,1]^2 \to [0,1]$ is a t-norm, then

$$S(x,y) = 1 - T(1-x, 1-y)$$

is the dual t-conorm to T. For details on t-norms and t-conorms see, *e.g.*, [12].

Another type of operation was introduced by Dombi in [8, 9] under the name *aggregative operator*. In current terminology, aggregative operators are representable uninorms. They are defined via a continuous, strictly increasing function $g : [0,1] \to [-\infty, \infty]$ with $g(0) = -\infty$ and $g(1) = \infty$ by the following formula

$$A(x,y) = g^{-1}(g(x) + g(y)),$$

and the value $-\infty + \infty$ is defined either as $-\infty$ (conjuctive case) or ∞ (disjuctive case).

General associative commutative and monotone operations on $[0, 1]$ were studied also by Czogała and Drewniak [5].

As a generalization of both t-norms and t-conorms Yager and Rybalov [14] proposed the notion of uninorm.

Definition 2 ([14]). *An operation $U\colon [0,1]^2 \to [0,1]$ is a uninorm if it is associative, commutative, monotone, and if it possesses a neutral element $e \in [0, 1]$.*

We say that a uninorm U is *proper* if its neutral element $e \in]0, 1[$.

Every uninorm has an absorbing element. We distinguish two types of uninorms according to the value of absorbing element, describing them as *conjunctive uninorms* if the absorbing element is 0, and *disjunctive uninorms* if the absorbing element is 1.

Lemma 2. *Let $U\colon [0,1]^2 \to [0,1]$ be a uninorm whose neutral element is e. Then its dual operation*

$$U^d(x, y) = 1 - U(1 - x, 1 - y)$$

is a uninorm whose neutral element is $1 - e$. Moreover, U is conjunctive if and only if U^d is disjunctive.

Nullnorms as operations occur when studying the functional equation by Frank and Alsina

$$U(x, y) + V(x, y) = x + y,$$

where U is a special uninorm and therefore the other operation, V, is a nullnorm.

Definition 3 ([3]). *An operation $N\colon [0,1]^2 \to [0,1]$ is a nullnorm if it is associative, commutative, monotone and with an absorbing element $a \in [0, 1]$ and moreover*

$$\begin{aligned} (\forall x \le a) \quad & N(x, 0) = x, \\ (\forall x \ge a) \quad & N(x, 1) = x. \end{aligned}$$

We say that a nullnorm N is *proper* if its absorbing element $a \in]0, 1[$.

Lemma 3. *Let $N\colon [0,1]^2 \to [0,1]$ be a proper nullnorm. Then N has no neutral element.*

More detailed information on proper uninorms and proper nullnorms one can found in [4].

1.2 Associative, Commutative and Monotone Operations on Bounded Lattices

We will skip a detailed introduction to bounded lattices referring to the monograph [1]. Let us just recall that on every lattice $(L, \le_L)$ there exists a partial order $\le_L$ and this order induces two binary operations: *meet*, $\wedge$, and *join*, $\vee$. Meet of x and y is the greatest lower bound of x, y. Join of x and y is the lowest upper bound of x, y.

On every bounded lattice $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ we can define *t-norms*, *t-conorms* and *proper uninorms* as associative commutative and monotone operations having $\mathbf{1}$ ($\mathbf{0}$, an element $\mathbf{e} \in L$, respectively) as neutral element. As examples of t-norms and t-conorms we present the following.

$$\begin{aligned} T_{\perp}(x, y) &= \begin{cases} x \wedge y & \text{if } x = \mathbf{1} \text{ or } y = \mathbf{1}, \\ \mathbf{0} & \text{otherwise} \end{cases} \\ T_M(x, y) &= x \wedge y, \\ S_{\top}(x, y) &= \begin{cases} x \vee y & \text{if } x = \mathbf{0} \text{ or } y = \mathbf{0}, \\ \mathbf{1} & \text{otherwise,} \end{cases} \\ S_M(x, y) &= x \vee y. \end{aligned}$$

T_M and $T_{\perp}$ are the greatest and the least t-norm, respectively. S_M and $S_{\top}$ are the least and the greatest t-conorm, respectively.

Remark 1. Let $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ be a bounded lattice. Assume that $a \in L$, $b \in L$ are arbitrary elements such that $a \le_L b$ and $a \neq b$. Then $([a, b], \tilde{\wedge}, \tilde{\vee}, a, b)$ is also a bounded lattice with the lattice-theoretical operations inherited from the lattice $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$.

As it was shown by Karaçal and Mesiar in [11], on every bounded lattice it is possible to construct a proper uninorm.

Similarly, we can also define proper nullnorms on general bounded lattices.

Definition 4. *Let $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ be a bounded lattice. An operation $N\colon L^2 \to L$ is said to be a* nullnorm *if it is an associative, commutative and monotone operation with an absorbing element $\mathbf{a} \in L$ such that*

$$\begin{aligned} (\forall x \le \mathbf{a}) \quad & N(x, \mathbf{0}) = x, \\ (\forall x \ge \mathbf{a}) \quad & N(x, \mathbf{1}) = x. \end{aligned}$$

If $\mathbf{a} \in L \setminus \{\mathbf{0}, \mathbf{1}\}$, we say that N is a proper nullnorm.

In the paper by Karaçal et al. [10] it was shown that on every bounded lattice it is possible to construct proper nullnorms.

Remark 2. Properties of proper uninorms on a bounded lattice may substantially differ from those of proper uninorms as operations on $[0,1]$. Particularly, Deschrijver in [7] showed that for interval-valued fuzzy sets, i.e., if $L_{\mathrm{IV}} = \{(x,y) \in [0,1]^2; x \le y\}$, then there exist proper uninorms which are neither conjunctive nor disjunctive. More precisely, he showed that for arbitrary element $\mathbf{e} = (x,x)$ for $x \in]0,1[$ and arbitrary element $\mathbf{a} \in L_{\mathrm{IV}}$ that is incomparable with $\mathbf{e}$ (except for elements from intervals $](0,0),(0,1)[$ and $](0,1),(1,1)[$) there exists a proper uninorm $\mathbf{U}$ whose neutral element is $\mathbf{e}$ and absorbing element is $\mathbf{a}$.

Example 1. Let $U_1 \colon [0,1]^2 \to [0,1]$ and $U_2 \colon [0,1]^2 \to [0,1]$ be uninorms with the same neutral element $e = \frac{1}{2}$. Let U_1 be conjunctive and U_2 be disjunctive and such that

$$U_1(x,y) \le U_2(x,y)$$

for all $(x,y) \in [0,1]^2$. Denote by L_{IV} the lattice of interval-valued fuzzy sets. Then $\mathbf{U} \colon L_{\mathrm{IV}}^2 \to L_{\mathrm{IV}}$ defined by

$$\mathbf{U}((x_1,x_2),(y_1,y_2)) = (U_1(x_1,y_1), U_2(x_2,y_2))$$

is a proper uninorm with neutral element $\mathbf{e} = (\frac{1}{2}, \frac{1}{2})$ which is neither conjunctive nor disjunctive, since the absorbing element of $\mathbf{U}$ is $\mathbf{a} = (0,1)$.

2 Uninorms as operations on $[0,1]^2$

First, we consider *representable* uninorms on $\mathcal{U} \colon [0,1]^4 \to [0,1]^2$ where representability is meant in the sense of Deschrijver [6], i.e., there exist uninorms $U_1 \colon [0,1]^2 \to [0,1]$ and $U_2 \colon [0,1]^2 \to [0,1]$ such that

$$\mathcal{U}((x_1,x_2),(y_1,y_2)) = \big(U_1(x_1,y_1), U_2(x_2,y_2)\big)\,.$$

Lemma 4. **(a)** *Let $(e_1,e_2) \in]0,1]^2$ be arbitrarily chosen. Assume that $U_1 \colon [0,1]^2 \to [0,1]$ and $U_2 \colon [0,1]^2 \to [0,1]$ are conjunctive uninorms with neutral elements e_1 and e_2, respectively. Then*

$$\mathcal{U}_1((x_1,x_2),(y_1,y_2)) = \big(U_1(x_1,y_1), U_2(x_2,y_2)\big)$$

is a conjunctive uninorm on $[0,1]^2$ whose neutral element is (e_1,e_2).
(b) *Let $e_3 \in [0,1[$ and $U_3 \colon [0,1]^2 \to [0,1]$ be a disjunctive uninorm with neutral element e_3. Then*

$$\mathcal{U}_2((x_1,x_2),(y_1,y_2)) = \big(U_1(x_1,y_1), U_3(x_2,y_2)\big)$$

is a uninorm on $[0,1]^2$ whose neutral element is (e_1,e_3) and annihilator is $(0,1)$.

This assertion is straightforward and therefore we have skipped the proof.

Dually to Lemma 4 we have the following lemma.

Lemma 5. **(a)** *Let $(e_1,e_2) \in [0,1[^2$ be arbitrarily chosen. Assume that $U_1 \colon [0,1]^2 \to [0,1]$ and $U_2 \colon [0,1]^2 \to [0,1]$ are disjunctive uninorms with neutral elements e_1 and e_2, respectively. Then*

$$\mathcal{U}_3((x_1,x_2),(y_1,y_2)) = \big(U_1(x_1,y_1), U_2(x_2,y_2)\big)$$

is a disjunctive uninorm on $[0,1]^2$ whose neutral element is (e_1,e_2).
(b) *Let $e_3 \in]0,1]$ and $U_3 \colon [0,1]^2 \to [0,1]$ be a conjunctive uninorm with neutral element e_3. Then*

$$\mathcal{U}_4((x_1,x_2),(y_1,y_2)) = \big(U_1(x_1,y_1), U_3(x_2,y_2)\big)$$

is a uninorm on $[0,1]^2$ whose neutral element is (e_1,e_3) and annihilator is $(1,0)$.

We have seen in Lemma 4 that there exists a conjunctive representable uninorm with a neutral element equal to (e_1,e_2), where $e_1 > 0$ and $e_2 > 0$. Now we show that there exist conjunctive uninorms with arbitrary neutral element $\mathbf{e} \ne (0,0)$.

Definition 5. *Let $\star \colon [0,1]^2 \to [0,1]$ be a binary operation. We say that an element $\mathbf{x} : [0,1]^2$, $\mathbf{x} \ne (0,0)$, is a* zero-divisor of $\star$ *if there exists an element $\mathbf{y} : [0,1]^2$, $\mathbf{y} \ne (0,0)$, such that $\mathbf{x} \star \mathbf{y} = (0,0)$.*

Proposition 1. *Let $\mathbf{e} = (e_1,e_2) \in [0,1]^2$ be such that $\mathbf{e} \ne (0,0)$. Further, let $U_1 \colon [0,1]^2 \to [0,1]$ and $U_2 \colon [0,1]^2 \to [0,1]$ be uninorms without zero-divisors and with neutral elements equal to e_1 and e_2, respectively. Assume that U_1 as well as U_2 are conjunctive uninorms if $e_1 > 0$ and $e_2 > 0$, respectively. Otherwise, if $e_1 = 0$ or $e_2 = 0$, the respective uninorm is a t-conorm. Denote*

$$\mathbf{U}(\mathbf{x},\mathbf{y}) = (U_1(x_1,y_1), U_2(x_2,y_2))$$

and $\mathbf{0} = (0,0)$. Then

$$\mathcal{U}_5((x_1,x_2),(y_1,y_2)) = \begin{cases} \mathbf{0} & \text{if } (x_1,x_2) = \mathbf{0} \\ & \text{ or } (y_1,y_2) = \mathbf{0}, \\ \mathbf{U}(\mathbf{x},\mathbf{y}) & \text{otherwise,} \end{cases} \tag{1}$$

is a conjunctive uninorm with the neutral element equal to $\mathbf{e} = (e_1,e_2)$.

Proof. We distinguish two cases. First, if $(e_1,e_2) \in]0,1]^2$, then $\mathcal{U}_5$ is just a representable conjunctive uninorm whose definition coincides with that of $\mathcal{U}_1$ from Lemma 4.

Second, assume that $e_1 = 0$ (we could treat the case when $e_2 = 0$ similarly). Directly from formula (1) we have that $(0, 0)$ is the annihilator of $\mathcal{U}_5$. Commutativity and monotonicity of U_1 and U_2 (as well as formula (1)) ensure commutativity and monotonicity of $\mathcal{U}_5$. Now we prove that $\mathcal{U}_5$ is associative. Recall that U_1, U_2 are uninorms without zero-divisors and, since we assume $e_1 = 0$, U_1 is a t-conorm, i.e.,

$$(U_1(x_1, x_2), U_2(y_1, y_2)) = \mathbf{0} \Leftrightarrow (x_1, x_2) = \mathbf{0} \text{ or } (y_1, y_2) = \mathbf{0}.$$

Hence, for $\mathbf{x} = (x_1, x_2) \in [0, 1]^2$, $\mathbf{y} = (y_1, y_2) \in [0, 1]^2$, $\mathbf{z} = (z_1, z_2) \in [0, 1]^2$ we have two possibilities – either none of the elements $\mathbf{x}, \mathbf{y}, \mathbf{z}$ is equal to $\mathbf{0}$, or at least one of them is equal to $\mathbf{0}$.
In the former case we get

$$\mathcal{U}_5(\mathcal{U}_5(\mathbf{x}, \mathbf{y}), \mathbf{z}) = (U_1(U_1(x_1, y_1), z_1), U_2(U_2(x_2, y_2), z_2)) = (U_1(x_1, U_1(y_1, z_1)), U_2(x_2, U_2(y_2, z_2))) = \mathcal{U}_5(\mathbf{x}, \mathcal{U}_5(\mathbf{y}, \mathbf{z}))$$

using associativity of U_1 and U_2.
In the latter case (assume $\mathbf{x} = \mathbf{0}$) we have

$$\mathcal{U}_5(\mathcal{U}_5(\mathbf{0}, \mathbf{y}), \mathbf{z}) = \mathcal{U}_5(\mathbf{0}, \mathbf{z}) = \mathbf{0} = \mathcal{U}_5(\mathbf{0}, \mathcal{U}_5(\mathbf{y}, \mathbf{z}))$$

directly from (1) and this finishes the proof of associativity of $\mathcal{U}_5$.
The fact that $\mathbf{e} = (e_1, e_2)$ is the neutral element of $\mathcal{U}_5$ follows directly from the fact that e_1 and e_2 are the neutral elements of U_1 and U_2, respectively. □

Next we show that we can choose arbitrary incomparable elements $\mathbf{a} \in [0, 1]^2$ and $\mathbf{e} \in [0, 1]^2$ and there exists a uninorm on $[0, 1]^2$ with the annihilator $\mathbf{a}$ and the neutral element $\mathbf{e}$.

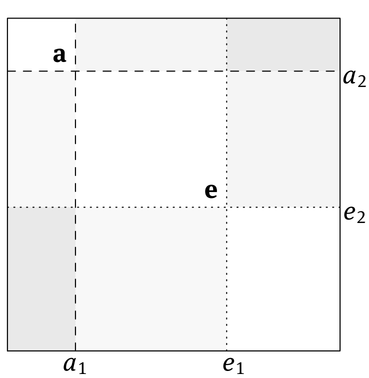

Figure 1: A uninorm on $[0, 1]^2$ whose annihilator, **a**, and neutral element, **e**, are incomparable

Proposition 2. *Let* $\mathbf{a}$ *and* $\mathbf{e}$ *be incomparable elements such that* $a_1 < e_1$ *and* $e_2 < a_2$ *(see Fig. 1). Let* $T_1 \colon [0, 1]^2 \to [0, 1]$, $T_2 \colon [0, 1]^2 \to [0, 1]$, $T_3 \colon [0, 1]^2 \to [0, 1]$ *be arbitrary t-norms and* $S_1 \colon [0, 1]^2 \to [0, 1]$, $S_2 \colon [0, 1]^2 \to [0, 1]$, $S_3 \colon [0, 1]^2 \to [0, 1]$ *be arbitrary t-conorms. Let* $\varphi_1 : [0, a_1] \to [0, 1]$, $\varphi_2 : [a_1, e_1] \to [0, 1]$, $\varphi_3 : [e_1, 1] \to [0, 1]$, $\psi_1 : [a_2, 1] \to [0, 1]$, $\psi_2 : [e_2, a_2] \to [0, 1]$ *and* $\psi_3 : [0, e_2] \to [0, 1]$ *denote the corresponding increasing affine transformations. We further denote the endomorphic transformations*

$$\begin{aligned} \tilde{T}_1 &= \varphi_1^{-1} \circ T_1 \circ \varphi_1, & \tilde{S}_1 &= \psi_1^{-1} \circ S_1 \circ \psi_1, \\ \tilde{T}_2 &= \varphi_2^{-1} \circ T_2 \circ \varphi_2, & \tilde{S}_2 &= \psi_2^{-1} \circ S_2 \circ \psi_2, \\ \tilde{T}_3 &= \varphi_3^{-1} \circ T_3 \circ \varphi_3, & \tilde{S}_3 &= \psi_3^{-1} \circ S_3 \circ \psi_3. \end{aligned}$$

Define functions $\mathbf{U}_1 : [0, 1]^4 \to [0, 1]^2$ *and* $\mathbf{U}_2 : [0, 1]^4 \to [0, 1]^2$ *by*

$$\mathbf{U}_1(\mathbf{x}, \mathbf{y}) = \begin{cases} \tilde{T}_1(x_1, y_1) & \text{if } x_1 < a_1,\ y_1 < a_1, \\ \tilde{T}_2(x_1, y_1) & \text{if } x_1 \in [a_1, e_1[\,,\ y_1 \in [a_1, e_1[\,, \\ \tilde{S}_3(x_1, y_1) & \text{if } x_1 \geq e_1,\ y_1 \geq e_1, \\ \min\{x_1, y_1\} & \text{if } x_2 \leq e_2, x_1 \geq a_1, y_1 < a_1, \\ & \text{ or } y_2 \leq e_2, y_1 \geq a_1, x_1 < a_1, \\ & \text{ or } x_1 \geq e_1, y_1 \in [a_1, e_1[\,, \\ & \text{ or } y_1 \geq e_1, x_1 \in [a_1, e_1[\,, \\ a_1 & \text{if } x_1 < a_1, y_1 \geq a_1, y_2 > e_2 \\ & \text{ or } y_1 < a_1, x_1 \geq a_1, x_2 > e_2, \end{cases} \tag{2}$$

$$\mathbf{U}_2(\mathbf{x}, \mathbf{y}) = \begin{cases} \tilde{T}_3(x_2, y_2) & \text{if } x_2 \leq e_2,\ y_2 \leq e_2, \\ \tilde{S}_2(x_2, y_2) & \text{if } x_2 \in]e_2, a_2],\ y_3 \in]e_2, a_2], \\ \tilde{S}_1(x_2, y_2) & \text{if } x_2 > a_2,\ y_2 > a_2, \\ \max\{x_2, y_2\} & \text{if } x_2 \leq e_2, y_2 \in]e_2, a_2], \\ & \text{ or } y_2 \leq e_2, x_2 \in]e_2, a_2], \\ & \text{ or } x_1 \geq e_1, x_2 \leq a_2\ y_2 > a_2, \\ & \text{ or } y_1 \geq e_1, y_2 \leq a_2\ x_2 > a_2, \\ a_2 & \text{if } x_2 > a_2, y_1 < e_1, y_2 \leq a_2 \\ & \text{ or } y_2 > a_2, x_1 < e_1, x_2 \leq a_2. \end{cases} \tag{3}$$

Then $\mathcal{U}_6 : [0, 1]^4 \to [0, 1]^2$, *given by*

$$\mathcal{U}_6(\mathbf{x}, \mathbf{y}) = \left(\mathbf{U}_1(\mathbf{x}, \mathbf{y}), \mathbf{U}_2(\mathbf{x}, \mathbf{y})\right), \tag{4}$$

is a uninorm whose annihilator is $\mathbf{a}$ *and neutral element is* $\mathbf{e}$.

Proof. Formulae (2) and (3) imply immediately that $\mathcal{U}_6$ is commutative and increasing. We show that $\mathbf{a}$ is the annihilator. In the first coordinate, the unit interval is split into

$$I_1 = [0, a_1[\,,\ I_2 = [a_1, e_1[\,,\ I_3 = [e_1, 1],$$

and in the second coordinate, the unit interval is split into

$$J_1 = [0, e_2],\ J_2 =]e_2, a_2],\ J_3 =]a_2, 1].$$

Let $\mathbf{x} = \mathbf{a}$. Then we get $\mathbf{U}_1(\mathbf{x}, \mathbf{y}) = \mathbf{U}_1(\mathbf{a}, \mathbf{y}) = a_1$ from the last, the second and the last but two item of (2), respectively, if $y_1 \in I_1$, $y_1 \in I_2$, or $y_1 \in I_3$. Similarly, we get $\mathbf{U}_2(\mathbf{x}, \mathbf{y}) = \mathbf{U}_1(\mathbf{a}, \mathbf{y}) = a_2$ from the fifth, the second and the last item of (3), respectively, if $y_2 \in J_1$, $y_2 \in J_2$, or $y_2 \in J_3$. This implies that $\mathbf{a}$ is the annihilator of $\mathcal{U}_6$.
Let $\mathbf{x} = \mathbf{e}$. Then we get $\mathbf{U}_1(\mathbf{x}, \mathbf{y}) = \mathbf{U}_1(\mathbf{e}, \mathbf{y}) = y_1$ from the fourth, the sixth and the third item of (2), respectively, if $y_1 \in I_1$, $y_1 \in I_2$, or $y_1 \in I_3$. We get $\mathbf{U}_2(\mathbf{x}, \mathbf{y}) = \mathbf{U}_1(\mathbf{e}, \mathbf{y}) = y_2$ from the first, the fourth and the sixth item of (3), respectively, if $y_2 \in J_1$, $y_2 \in J_2$, or $y_2 \in J_3$. This implies that $\mathbf{e}$ is the neutral element of $\mathcal{U}_6$.
We show that $\mathcal{U}_6$ is associative. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in [0, 1]^2$ are arbitrary elements. First we are going to prove the following

$$\mathbf{U}_1(\mathbf{U}_1(\mathbf{x}, \mathbf{y}), \mathbf{z}) = \mathbf{U}_1(\mathbf{x}, \mathbf{U}_1(\mathbf{y}, \mathbf{z})). \tag{5}$$

We may assume that not all of the first coordinates, x_1, y_1, z_1 are from he same interval I_1, I_2 or I_3, since otherwise formula (5) is obvious. Let us distinguish several cases.

- Assume $x_1 \in I_1$ and $y_1 \notin I_1$, $z_1 \notin I_1$. Then if $y_2 \le e_2$ and $z_2 \le e_2$, we get immediately

  $$\mathbf{U}_1(\mathbf{U}_1(\mathbf{x}, \mathbf{y}), \mathbf{z}) = \mathbf{U}_1(\mathbf{x}, \mathbf{U}_1(\mathbf{y}, \mathbf{z})) = x_1.$$

  If $y_2 > e_2$ or $z_2 > e_2$, we have

  $$\mathbf{U}_1(\mathbf{U}_1(\mathbf{x}, \mathbf{y}), \mathbf{z}) = \mathbf{U}_1(\mathbf{x}, \mathbf{U}_1(\mathbf{y}, \mathbf{z})) = a_1.$$

  In both of these cases (5) holds.
- Assume $x_1 \in I_1$ and $y_1 \in I_1$, $z_1 \notin I_1$. Then if $z_2 \le e_2$ we get

  $$\mathbf{U}_1(\mathbf{U}_1(\mathbf{x}, \mathbf{y}), \mathbf{z}) = \mathbf{U}_1(\mathbf{x}, \mathbf{U}_1(\mathbf{y}, \mathbf{z})) = \tilde{T}_1(x_1, y_1),$$

  otherwise

  $$\mathbf{U}_1(\mathbf{U}_1(\mathbf{x}, \mathbf{y}), \mathbf{z}) = \mathbf{U}_1(\mathbf{x}, \mathbf{U}_1(\mathbf{y}, \mathbf{z})) = a_1.$$

  Also in these two cases (5) holds.
- Assume $x_1 \in I_2$, $y_1 \in I_3$, $z_1 \in I_3$. In this case we have

  $$\mathbf{U}_1(\mathbf{U}_1(\mathbf{x}, \mathbf{y}), \mathbf{z}) = \mathbf{U}_1(\mathbf{x}, \mathbf{U}_1(\mathbf{y}, \mathbf{z})) = x_1.$$

- Assume $x_1 \in I_2$, $y_1 \in I_2$, $z_1 \in I_3$. we get

  $$\mathbf{U}_1(\mathbf{U}_1(\mathbf{x}, \mathbf{y}), \mathbf{z}) = \mathbf{U}_1(\mathbf{x}, \mathbf{U}_1(\mathbf{y}, \mathbf{z})) = \tilde{T}_2(x_1, y_1).$$

Any permutation of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ gives the same result because of the commutativity of $\mathbf{U}_1$. This finishes the proof of associativity of $\mathcal{U}_6$ in the first coordinate. The proof of associativity of $\mathcal{U}_6$ in the second coordinate follows the same idea and therefore we have skipped this part of the proof. □

Remark 3. In Proposition 2 we have constructed the uninorm $\mathcal{U}_6$ whose neutral element, $\mathbf{e}$, and annihilator, $\mathbf{a}$, are incomparable and such that $a_1 < e_1$ and $e_2 < a_2$. Similarly it is possible to construct a uninorm with incomarable annihilator $\hat{\mathbf{a}}$ and neutral element $\hat{\mathbf{e}}$ such that $\hat{e}_1 < \hat{a}_1$ and $\hat{a}_2 < \hat{e}_2$. It is enough to exchange the coordinates in (4). We skip this construction.

Remark 4. If we restrict the uninorm $\mathcal{U}_6$ form Proposition 2 just to the rectangle $[a_1, e_1] \times [e_2, a_2]$, we get the following operation $\tilde{\mathcal{U}}_6 : \left([a_1, e_1] \times [e_2, a_2]\right)^2 \to [a_1, e_1] \times [e_2, a_2]$ which is defined by the formula

$$\tilde{\mathcal{U}}_6((x_1, x_2), (y_1, y_2)) = (\tilde{T}_2(x_1, y_1), \tilde{S}_2(x_2, y_2)).$$

Using the transformations φ_2 and ψ_2 we can transform $\tilde{\mathcal{U}}_6$ to an operation $\mathcal{U}_7 : [0, 1]^4 \to [0, 1]$ which is of the form

$$\mathcal{U}_7((x_1, x_2), (y_1, y_2)) = (T_2(x_1, y_1), S_2(x_2, y_2)).$$

$\mathcal{U}_7$ is a uninorm on $[0, 1]^2$ with the neutral element $(1, 0)$ and the annihilator $(0, 1)$. But this operation is also a nullnorm with the same annihilator $(0, 1)$. This means that we have constructed a proper nullnorm with a neutral element. As we have already mentioned in Lemma 3, this type of operation cannot be constructed on $[0, 1]$. In the rest of this paper we will deal with nullnorms with a neutral element.

3 Nullnorms with neutral element

In the rest of this paper we will assume that $L = L_1 \times L_2$ is a bounded lattice wich is a direct product of bounded lattices L_1 and L_2, i.e., we will assume that there exist incomparable elements $\mathbf{b}_1, \mathbf{b}_2 \in L$ such that

$$\mathbf{b}_1 \wedge \mathbf{b}_2 = \mathbf{0}, \quad \mathbf{b}_1 \vee \mathbf{b}_2 = \mathbf{1}, \tag{6}$$
$$(\forall x \in L)(x = (x \wedge \mathbf{b}_1) \vee (x \wedge \mathbf{b}_2)). \tag{7}$$

Remark 5. The direct product of lattices $L = L_1 \times L_2$ is a lattice whose set of elements L can be expressed as the set of all pairs $L = \{(x_1, x_2); x_1 \in L_1,\ x_2 \in L2\}$ and all lattice-theoretical operations are defined coordinate-wise.

As an example of a lattice fulfilling (6) and (7) we may consider $L_2 = [0, 1]^2$ with lattice-theoretical operations defined coordinate-wise, which we have considered in the foregoing section.

Example 2. Let $L_n = [0,1]^n$. Then $(L_n, \leq_n)$ is the lattice with lattice-theoretical operations defined coordinate-wise. Set

$$\mathbf{b}_1 = (\underbrace{1,\dots,1}_{k\times},0,\dots,0), \quad \mathbf{b}_2 = (0,\dots,0,\underbrace{1,\dots,1}_{(n-k)\times}).$$

Then formulae (6) and (7) are fulfilled.

In the next proposition we learn how it is possible to construct a nullnorm with a neutral element on a lattice $(L, \leq)$ fulfilling formulae (6) and (7). Particularly, we construct a nullnorm whose neutral element is $\mathbf{b}_1$ and annihilator is $\mathbf{b}_2$.

Proposition 3. *Let $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ be the ordinal sum of two bounded lattices. Let $\mathbf{b}_1, \mathbf{b}_2$ be incomparable elements such that formulae* (6) *and* (7) *are valid for $\mathbf{b}_1$ and $\mathbf{b}_2$. For arbitrary $x \in L$ denote*

$$\bar{x} = x \wedge \mathbf{b}_1, \quad \hat{x} = x \wedge \mathbf{b}_2. \tag{8}$$

Further, let $T\colon [0,\mathbf{b}_1]^2 \to [0,\mathbf{b}_1]$ be a t-norm and $S\colon [0,\mathbf{b}_2]^2 \to [0,\mathbf{b}_2]$ be a t-conorm. Then $\star\colon L^2 \to L$, defined by

$$x \star y = T(\bar{x},\bar{y}) \vee S(\hat{x},\hat{y}), \tag{9}$$

is a proper uninorm (which is neither conjunctive nor disjunctive), and at the same time $\star$ is also a proper nullnorm. The operation $\star$ has $\mathbf{b}_1$ as neutral element and $\mathbf{b}_2$ as absorbing element.

Proof. Since all operations involved in formula (9) are commutative and monotone (increasing), also $\star$ is monotone and increasing. We prove the associativity of $\star$. First realize that for arbitrary $x, y \in L$ we have the following equalities implied by (6)

$$\overline{a \star y} = (x \star y) \wedge \mathbf{b}_1 = T(\bar{x},\bar{y}), \tag{10}$$

$$\widehat{x \star y} = (x \star y) \wedge \mathbf{b}_2 = S(\hat{x},\hat{y}), \tag{11}$$

since $T(\bar{x},\bar{y}) \leq \mathbf{b}_1$ and $S(\hat{x},\hat{y}) \leq \mathbf{b}_2$. Then for all $x, y, z \in L$ we get

$$\begin{aligned}
(x \star y) \star z &= T(\overline{(x \star y)}, \bar{z}) \vee S(\widehat{(x \star y)}, \hat{z}) \\
&= T(T(\bar{x},\bar{y}),\bar{z}) \vee S(S(\hat{x},\hat{y}),\hat{z}) \\
&= T(\bar{x}, T(\bar{y},\bar{z})) \vee S(\hat{x}, S(\hat{y},\hat{z})) \\
&= T(\bar{x}, \overline{(y \star z)}) \vee S(\hat{x}, \widehat{(y \star z)}) \\
&= x \star (y \star z),
\end{aligned}$$

which completes the proof of the associativity of $\star$.
Now we prove that $\mathbf{b}_1$ is the neutral element of $\star$ and $\mathbf{b}_2$ is the annihilator of $\star$. Formula (6) implies

$$\begin{aligned}
\bar{\mathbf{b}}_1 &= \mathbf{b}_1, \quad \hat{\mathbf{b}}_1 = \mathbf{0}, \\
\hat{\mathbf{b}}_2 &= \mathbf{b}_2, \quad \bar{\mathbf{b}}_2 = \mathbf{0}.
\end{aligned}$$

Then for arbitrary $x \in L$ we get

$$\begin{aligned}
x \star \mathbf{b}_1 &= T(\bar{x}, \bar{\mathbf{b}}_1) \vee S(\hat{x}, \hat{\mathbf{b}}_1) \\
&= T(\bar{x}, \mathbf{b}_1) \vee S(\hat{x}, \mathbf{0}) = \bar{x} \vee \hat{x} = x, \\
x \star \mathbf{b}_2 &= T(\bar{x}, \bar{\mathbf{b}}_2) \vee S(\hat{x}, \hat{\mathbf{b}}_2) \\
&= T(\bar{x}, \mathbf{0}) \vee S(\hat{x}, \mathbf{b}_2) = \mathbf{0} \vee \mathbf{b}_2 = \mathbf{b}_2.
\end{aligned}$$

This finishes the proof of the fact that $\star$ is a uninorm on L with the neutral element $\mathbf{b}_1$ and annihilator $\mathbf{b}_2$.
Yet we have to prove that $\star$ is also a nullnorm on L, i.e., we have to prove that

$$\begin{aligned}
&(\forall x \leq \mathbf{b}_2)(x \star \mathbf{0} = x), \\
&(\forall x \geq \mathbf{b}_2)(x \star \mathbf{1} = x).
\end{aligned}$$

From fromulae (8) and (11) we have

$$x \leq \mathbf{b}_2 \Rightarrow x \star \mathbf{0} = S(x, \mathbf{0}) = x.$$

If $x \geq \mathbf{b}_2$ then $\hat{x} = \mathbf{b}_2$ and thence

$$x \star \mathbf{1} = T(\bar{x}, \mathbf{b}_1) \vee S(\hat{x}, \mathbf{b}_2) = \bar{x} \vee \mathbf{b}_2 = x,$$

and the proof is complete. □

Example 3. Let $L = [0,1]^n$, $n \geq 2$. For some $1 \leq j < n$ we choose j t-norms $T_i : [0,1]^2 \to [0,1]$, $i \in \{1,2,\dots,j\}$, and $n-j$ t-conorms $S_k : [0,1]^2 \to [0,1]$, $k \in \{1,2,\dots,n-j\}$. Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in L$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in L$ be arbitrary elements. Then the following operation

$$\begin{aligned}
\mathbf{x} \star \mathbf{y} = \big(&T_1(x_1,y_1), \dots, T_j(x_j,y_j), \\
&S_1(x_{j+1},y_{j+1}), \dots, S_{n-j}(x_n,y_n)\big)
\end{aligned}$$

is a uninorm and a nullnorm and the neutral element and annihilator are

$$\mathbf{e} = (\underbrace{1,\dots,1}_{j\times},0\dots,0), \quad \mathbf{a} = (0,\dots,0,\underbrace{1,\dots,1}_{(n-j)\times}),$$

respectively.

4 Conclusion

In this paper we have discussed possible positions of the neutral element and the annihilator of uninorms on the lattice $([0,1]^2, \leq_2)$. We have shown that arbitrary pair of incomparable elements $(\mathbf{e}, \mathbf{a})$ can be chosen and we are then able to construct a uninorm with the neutral element equal to $\mathbf{e}$ and the annihilator equal to $\mathbf{a}$. As a special case we have $\mathbf{e} = (1, 0)$ and $\mathbf{a} = (0, 1)$. In this case the constructed uninorm is also a nullnorm. This means that on $[0, 1]^2$ there exist nullnorms with neutral element. In the last section we have shown that on the direct product of arbitrary two bounded lattices it is possible to construct nullnorms with neutral element (or in other words nullnorms which are also uninorms).

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