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Slow-fast effect and generation mechanism of Brusselator based on coordinate transformation

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Abstract: The Brusselator with different time scales, which behaves in the classical slow-fast effect, is investigated, and is characterized by the coupling of the quiescent and spiking states. In order to reveal the generation mechanism by using the slow-fast analysis method, the coordinate transformation is introduced into the classical Brusselator, so that the transformed system can be divided into the fast and slow subsystems. Furthermore, the stability condition and bifurcation phenomenon of the fast subsystem are analyzed, and the attraction domains of different equilibria are presented by theoretical analysis and numerical simulation respectively. Based on the transformed system, it could be found that the generation mechanism between the quiescent and spiking states is Fold bifurcation and change of the attraction domain of the fast subsystem. The results may also be helpful to the similar system with multiple time scales.

Keywords: Brusselator; slow-fast effect; generation mechanism; coordinate transformation

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1 Introduction

The Brusselator is a theoretical model for a type of autocatalytic reaction, and its nonlinear dynamical behaviors have attracted many scholars. A lot of works about this oscillator, such as stability, analytical and numerical solution, bifurcation, and control, etc, have been studied. For example, the Hopf bifurcation and stability condition of periodic solution for the Brusselator were investigated by using Hopf bifurcation theorem, normal form theory, and center manifold theorem in [1]. Yu [2] had studied the associated Hopf bifurcation and double-Hopf bifurcations for the coupling of double Brusselators. The steady-state bifurcation from the unique positive constant equilibrium point, was investigated in detail [3]. The approximate analytical solution of the steady-state non-linear boundary value problem, was derived by using the Homotopy perturbation method in [4]. Islam [5] studied a meshfree technique for the numerical solution of the two-dimensional reaction-diffusion Brusselator along with Dirichlet and Neumann boundary conditions. Mittal [6] had given a differential quadrature method for numerical study of a two-dimensional reaction-diffusion Brusselator. Bashkirtseva [7] investigated the sensitivity analysis of the Brusselator, subject to small stochastic and periodic disturbances. Guruparan [8] characterized periodic orbits, quasiperiodic orbits, chaotic orbits, hysteresis, and vibrational resonance of the Brusselator. Vaidyanathan [9] used Lyapunov stability theory to discuss the adaptive control of the Brusselator so as to regulate its states to desired steady-state values.

Here, it should be pointed out that the Brusselator is a catalytic reaction with typical coupling of different time scales, because the large gap between different reaction steps will result in the difference in time scale. Some works about the slow-fast effect could be referred to the famous Belousov-Zhabotinsky reaction, CO oxidation on the platinum group metals, metal electrochemical system in sulfuric acid solution, reaction-diffusion model, and so on [10–15].

The dynamical systems with different time scales coupled had attracted the attention of many scholars in different fields, such as electronic circuits, neuronal, chemical kinetics, and population dynamics [16–21]. The slow-fast effect, known as bursting, would happen in these systems, which could be characterized by a combination of quiescent state (QS) and spiking state (SP) during each evolution process. The early works were mainly involved in the approximate analytical solution and numerical simulation of the slow-fast phenomenon, but they could not accurately explain the mechanism of interaction between

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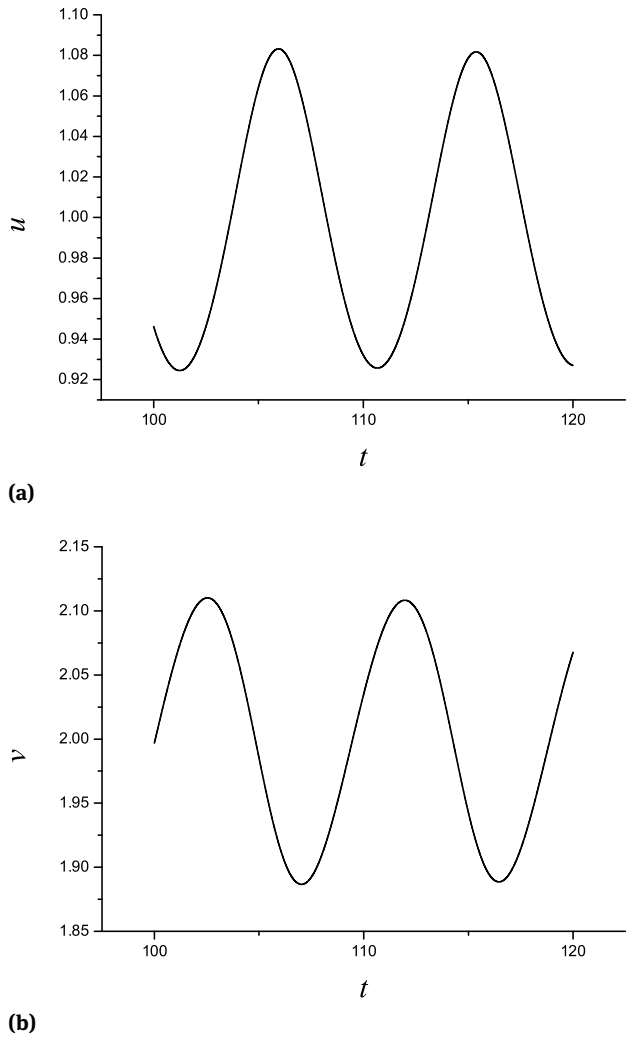


Figure 1: Periodic oscillation of Eq. (1) when $B = 2$ and $A = 1$. (a) time history of u, (b) time history of v.

the fast and slow variables. The generation mechanism of transition between the QS and SP had not been clearly revealed until the slow-fast dynamical analysis method was proposed by Rinzel [22]. Based on the method, various bifurcation phenomena had been found to be related with the generation mechanism in the systems with two time scales. For example, Izhikevich [23] had given the classification of bursting by using bifurcation on the fast subsystem. Shimizu [24] discussed the mixed-mode oscillations of the BVP system with weak periodic perturbation. Chumakov [25] established a kinetic model of catalytic hydrogen oxidation and investigated the generated mechanism of bursting. Simpson [26] analyzed bursting phenomenon in a stochastic piecewise system, and discussed the influence of noise on the bursting oscillation. Shilnikov [27] had summarized the qualitative methods on Hindmarsh-Rose model, and presented the different burstings under Hopf bifurcation. Lu [28] proposed a double-parameter analysis method to reveal the complex behaviors of Chay models. Bi [29, 30] investigated the periodic excited systems and non-smooth systems, and discussed the connection of bursting with bifurcations such as Fold, Hopf, and non-smooth bifurcations. Li and Bi [31] proposed the enveloping slow-fast analysis that could be used to explain the bursting phenomenon in the system with three time scales.

Work about the Brusselator with different time scales, has been found in [8, 32]. However, there is little analytical study on the slow-fast phenomenon or its generation mechanism of a Brusselator due to the coupling of the fast and slow subsystems. In this paper, we focus on the slow-fast effect and generation mechanism of a Brusselator with two time scales. The paper is organized as follows. In Section 2, the classical Brusselator is given and the slow-fast phenomenon is numerically found under certain parameters. The transformed Brusselator and the slow-fast phenomenon are presented in Section 3. In Section 4, the bifurcation phenomenon and attraction domains of the fast subsystem are investigated. Then the slow-fast phenomenon and corresponding generation mechanism are studied by using slow-fast analysis in Section 5. Finally, the main conclusions of this paper are made.

2 The classical Brusselator and slow-fast phenomenon

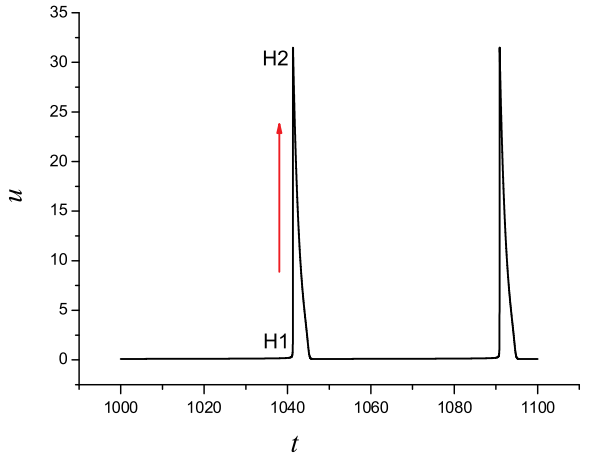
The Brusselator is a coupled differential equation written as

$$u' = A - (B + 1)\,u + u^2 v, \tag{1a}$$

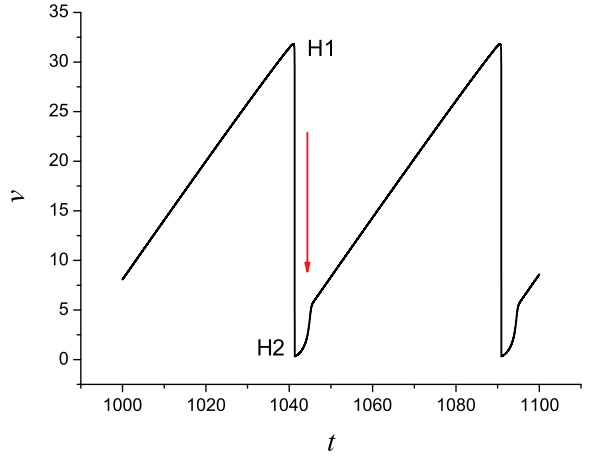
$$v' = Bu - u^2 v, \tag{1b}$$

where $u(t)$ and $v(t)$ are activator and inhibitor variables respectively. A and B are external system parameters, which will determine the system dynamics.

The variation of the parameters may alter the time scales of the system and the shape of the cycle. For example, for $B = 2$ and $A = 1$, the whole system is almost based on single time scale, and the limit cycle is similar to a simple harmonic vibration shown in Fig. 1. If the parameter A is fixed and parameter B increases, the system will gradually exhibit dynamical behavior with two time scales, which may lead the system to interact with the slow and fast process. For example, the fast-slow phenomenon may appear in the system for $B = 10$. As shown in Fig. 2,



(a)



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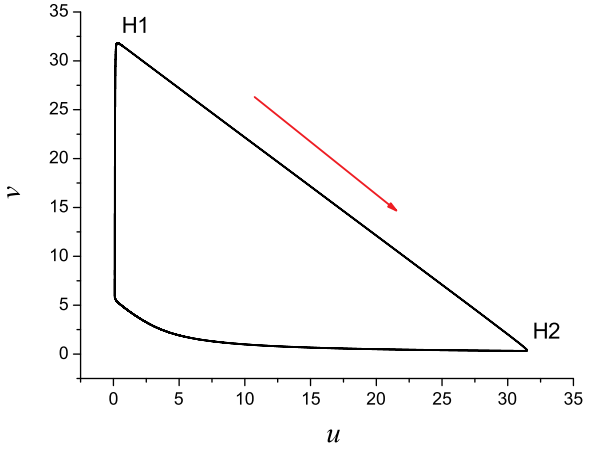


(c)

Figure 2: Periodic oscillation of Eq. (1) when $B = 10$ and $A = 1$. (a) time history of u, (b) time history of v, (c) phase diagram.

the instantaneous jumping from H1 to H2 forms the fast procedure, and the other movement is the slow procedure.

Therefore, under the condition of $B \gg A$, Eq. (1) is coupled with two time scales, and the fast-slow phenomenon will appear in the reaction. Up to now, the classical method to reveal the transition mechanism between the fast and slow process is the slow-fast analysis method. The necessary condition of this method is that the whole system should be divided into two subsystems, *i.e.* the fast subsystem (FS) and slow subsystem (SS). The slow variables are generally treated as the bifurcation parameters of FS, and the bifurcation behaviors of FS can decide the transition mechanism between the fast and slow process, associated with the whole dynamical behaviour.

However, Eq. (1) coupled with two time scales cannot be directly analyzed by the slow-fast analysis method, because the parameter lies both in Eq. (1a) and (1b). Therefore, both of the two variables $u(t)$ and $v(t)$ behave in the fast and slow process simultaneously. It can be seen from Fig. 2, the jumping phenomenon from H1 to H2 is composed of the rapid increase of variable $u(t)$ and instantaneous decrease of the variable $v(t)$. In other words, the classical slow-fast analysis method to explain the jumping phenomenon, could not be used here directly, because one could not separate the fast and slow variables from $u(t)$ and $v(t)$.

3 The transformed Brusselator and slow-fast phenomenon

Based on the above-mentioned analysis, the parameter condition $B \gg A$ will cause the slow-fast phenomenon. In order to facilitate the whole system into separated FS and SS, we will introduce the coordinate transformation into the original system.

Letting $x = v$ and $y = u + v$, Eq. (1) becomes

$$x' = B(y - x) - (y - x)^2 x \tag{2a}$$

$$y' = A - (y - x) \tag{2b}$$

Because the transformation is invertible, the transformed Brusselator Eq. (2) is topologically equivalent to the original classical system Eq. (1).

It should be stressed that the coordinate transformation successfully separates parameter B and A. Therefore, the variable x and y denote FS and SS subsystems respectively if $B \gg A$ and $B \gg 1$. For example, when the same parameters are selected as those in Fig. 2, the phase diagram and time history of Eq. (2) are plotted in Fig. 3. It could be found that the variable x can quickly decrease

from 31.8 to 0.32, while the variable y is nearly unchanged in the procedure. That means the instantaneous jumping behaviour appears only in the fast variable x, not in the slow variable y. Therefore, the generation mechanism of the slow-fast phenomenon in Eq. (2) can be analyzed by the slow-fast analysis method.

In the following part, we will study the slow-fast behaviour in Eq. (2) whose parameters are consistent with Eq. (1).

4 Stability and bifurcation analysis of the fast subsystem in the transformed Brusselator

In order to reveal the generation mechanism of the slow-fast phenomenon, we will discuss the bifurcation and stability of the fast subsystem, in which the slow variable y is considered only as the bifurcation parameter.

The equilibriums of the subsystem Eq. (2a) should meet

$$B(y-x)-(y-x)^2 x=0, \tag{3}$$

i.e.

$$x=y, x+\frac{B}{x}=y. \tag{4}$$

The two equilibrium lines for $B = 10$ are shown in Fig. 4, denoted by L1 and L2 respectively. By calculating the eigenvalues, it could be concluded L1 is unconditionally stable, while L2 is stable for $0 < x < \sqrt{B}$ and unstable when $x > \sqrt{B}$. Accordingly, there exists Fold bifurcation at the critical point LP $\left(\sqrt{B}, 2\sqrt{B}\right)$ in subsystem (2a). Furthermore, there are two stable equilibriums for $y > 2\sqrt{B}$, and one stable equilibrium for $y < 2\sqrt{B}$.

Now we discuss the attraction domain of the two stable attractors of the subsystem (2a) under the variation of parameter $y > 2\sqrt{B}$. Firstly, the analytical solution is presented. Letting $y - x = s$ and $ds = -dx$, Eq. (2a) could be transformed into

$$\begin{aligned}
&\int \frac{ds}{Bs+s^2(s-y)} \\
&=\int\left[\frac{1}{Bs}+\frac{\frac{1}{B}(-s+y)}{s^2-sy+B}\right]ds \\
&=\frac{1}{B}ln|s|-\frac{1}{2B}\int\frac{2s-y}{s^2-sy+B}ds+\frac{y}{2B}\int\frac{1}{s^2-sy+B}ds \\
&=\frac{1}{B}ln|s|-\frac{1}{2B}ln\left|s^2-sy+B\right| \\
&+\frac{y}{2B\sqrt{y^2-4B}}ln\left|\frac{2s-y-\sqrt{y^2-4B}}{2s-y+\sqrt{y^2-4B}}\right| \\
&=\frac{1}{B}ln|y-x|-\frac{1}{2B}ln\left|x^2-xy+B\right| \\
&+\frac{y}{2B\sqrt{y^2-4B}}ln\left|\frac{y-2x-\sqrt{y^2-4B}}{y-2x+\sqrt{y^2-4B}}\right|=-t+C
\end{aligned} \tag{5}$$

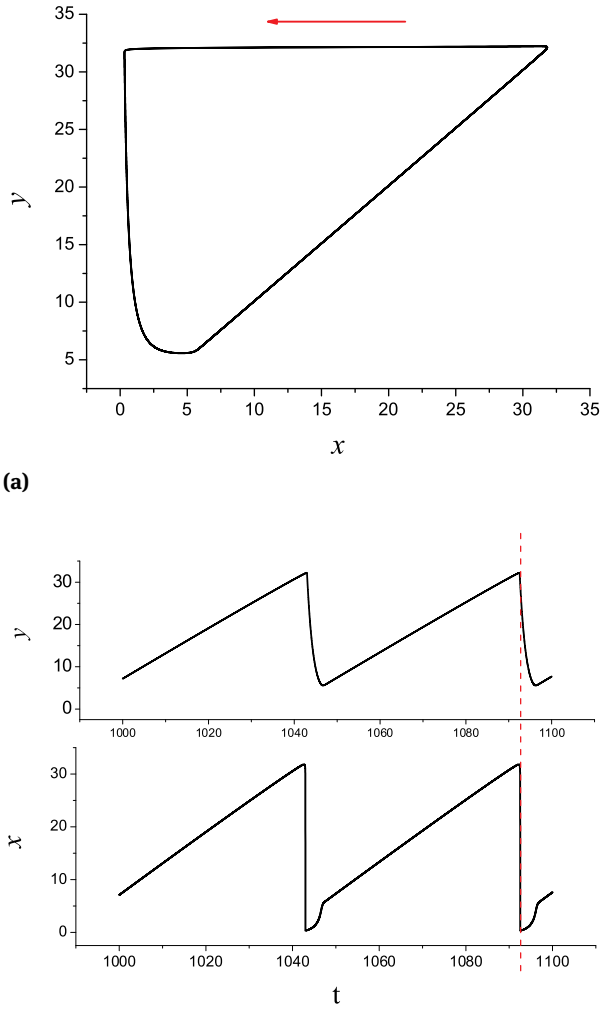


Figure 3: Periodic oscillation of Eq. (2) when $B = 10$ and $A = 1$. (a) phase diagram, (b) time histories of x and y.

where C is an integration constant. When $y > 2\sqrt{B}$ there are two stable attractors in subsystem (2a), *i.e.* $x = y$ and $x = \frac{y-\sqrt{y^2-4B}}{2}$. The region for $y > 2\sqrt{B}$ can be divided into three parts in the plane xoy, denoted by R1, R2 and R3 respectively, which are shown in Fig. 4. The ranges for the three parts are defined as

$$\begin{aligned}
&R1: x>\sqrt{B}, \quad 2\sqrt{B}<y<x+\frac{B}{x}, \\
&R2: y>x+\frac{B}{x}, \\
&R3: 0<x<\sqrt{B}, 2\sqrt{B}<y<x+\frac{B}{x}.
\end{aligned} \tag{6}$$

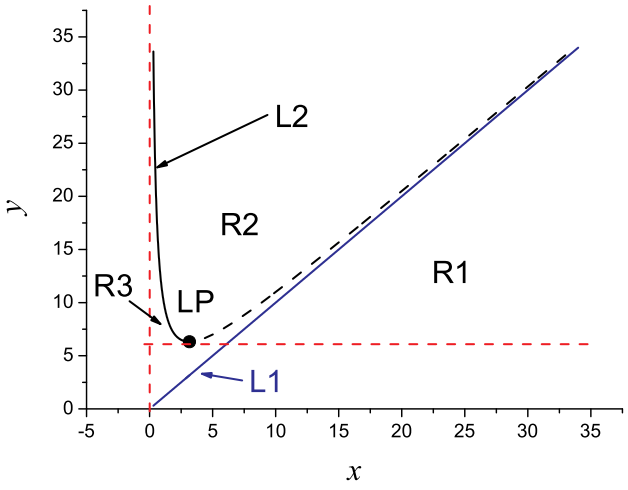


Figure 4: The equilibrium lines and bifurcation diagram of the fast subsystem (2a). (L1 and L2 are equilibrium lines, LP is the critical point of Fold bifurcation, R1, R2 and R3 are three region divided by equilibrium lines.).

In region R1, the following condition is met

$$y - 2x - \sqrt{y^2 - 4B} < \frac{B}{x} - x - \sqrt{y^2 - 4B} < 0.$$

The analytical solution of Eq. (2a) can be expressed as

$$\frac{1}{B} ln\,|y - x| - \left(\frac{1}{2B} + \frac{y}{2B\sqrt{y^2 - 4B}}\right) ln\left|x^2 - xy + B\right| + \frac{y}{B\sqrt{y^2 - 4B}} ln\left|\frac{y - 2x - \sqrt{y^2 - 4B}}{2}\right| = -t + C. \tag{7}$$

It is obvious that, in region R1 the following condition holds unconditionally

$$\left|\frac{y - 2x - \sqrt{y^2 - 4B}}{2}\right| \neq 0.$$

If $t \to \infty$, it will lead to $x = y$, that means the initial points starting from region R1 will be convergent into the stable attractor $x = y$.

In region R2 the inequality

$$y - 2x + \sqrt{y^2 - 4B} > \frac{B}{x} - x + \sqrt{\left(x + \frac{b}{x}\right)^2 - 4B} \geq 0$$

is met. The analytical solution of Eq. (2a) can become

$$\frac{1}{B} ln\,(y - x) + \left(\frac{y}{2B\sqrt{y^2 - 4B}} - \frac{1}{2B}\right) ln\left|x^2 - xy + B\right| + \frac{y}{B\sqrt{y^2 - 4B}} ln\left|\frac{2}{y - 2x + \sqrt{y^2 - 4B}}\right| = -t + C, \tag{8}$$

where

$$\frac{2}{y - 2x + \sqrt{y^2 - 4B}} \neq 0.$$

When $t \to \infty$, the left side of Eq. (8) will approach minus infinity, which results in $x = y$ or $x^2 - xy + B = 0$. Because the attractor $x = y$ is not in R2, the points starting from region R2 will be attracted into the stable equilibrium $x = \frac{y - \sqrt{y^2 - 4B}}{2}$, which is the left boundary of region R2.

In region R3 the inequality

$$y - 2x + \sqrt{y^2 - 4B} > \frac{B}{x} - x + \sqrt{y^2 - 4B} > 0$$

is satisfied, and the subsystem (2a) will be convergent into the stable attractor $x = \frac{-y - \sqrt{y^2 - 4B}}{2}$. The analysis procedure is similar to that for region R2.

From the above analysis, it could be concluded that if $y > 2\sqrt{B}$ the attraction domain of the equilibrium $x = \frac{y - \sqrt{y^2 - 4B}}{2}$ is $x < \frac{y + \sqrt{y^2 - 4B}}{2}$, and the attraction domain of another equilibrium $x = y$ is $x > \frac{y + \sqrt{y^2 - 4B}}{2}$. Furthermore, the subsystem (2a) possesses only one stable attractor $x = y$ when $y < 2\sqrt{B}$, and the attraction domain for this single attractor is $x \in R^+$, where R^+ is positive real number set.

The numerical simulation for the attraction domain is plotted in Fig. 5, where the boundary of the attraction domain for different attractors is the right branch of curve $y = x + \frac{B}{x}$ $(x > 0)$, *i.e.*, the unstable equilibrium line located on L2. It is obvious that the numerical result agrees well with the above-mentioned theoretical one. The numerical method is selected as a variable order method for a

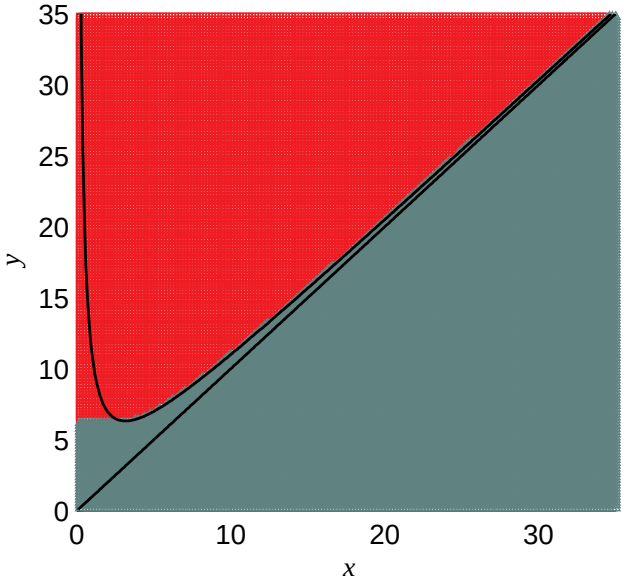


Figure 5: The numerical attraction domain of the subsystem (2a).

stiff differential equation, *i.e.*, the routine ODE15s in MATLAB. Here the total computation time is 200 s, and then one could determine the attraction property of every initial point. In the *xoy* plane, we restrict the initial point and parameter as $x \in (0.2,\ 35)$ and $y \in (0.2,\ 35)$. The sample steps of x and y are all selected as 0.2.

5 Generation mechanism of the slow-fast effect

Based on the transformed Brusselator Eq. (2), we will reveal the generation mechanism of the slow-fast phenomenon by use of the slow-fast analysis method. Overlapping the bifurcation diagram Fig. 4 and the phase diagram Fig. 3(a), one could obtain Fig. 6. Now we describe one revolution of the system in detail. The trajectory starting at point E moves almost along the stable equilibrium manifold L2, resulting in QS. When the trajectory reaches point F near the critical point LP of Fold bifurcation, it will move to point G due to the attraction of stable equilibrium manifold L1. The system will keep QS along L1 for a long time. At point H, the trajectory will go into the attraction domain of stable equilibrium on L2, so that it quickly jumps to point E and forms the instantaneous SP. The whole periodic procedure forms the slow-fast effect of the system.

It is obvious that there are two transitions between different attractors in the system. The one from F to G is caused by Fold bifurcation, and the other transition from H to E is due to the change of attraction domains. Here it is important to point out that the trajectory from G to H is located between L1 and L2. The distance between L1 and L2 is

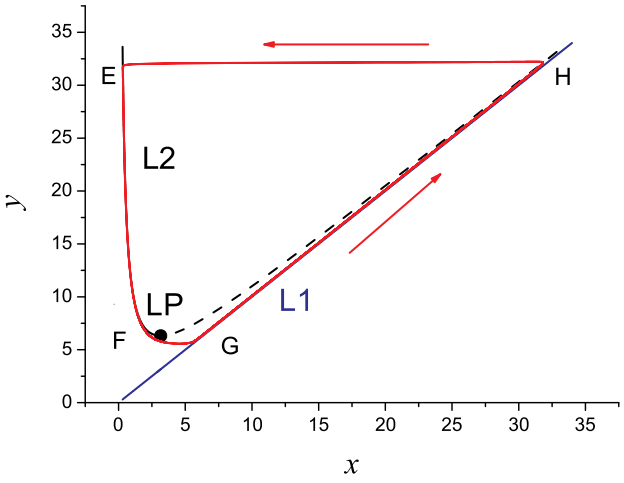
$$x + \frac{B}{x} - x = \frac{B}{x}.$$

With the increase of x, the distance between L1 and L2 will become shorter and shorter, so that the attraction of L1 will become smaller and smaller. This may make the trajectory go across the unstable part of L2 and be attracted by the stable part of L2. Therefore, the change of attraction domains results in the transition from H to E.

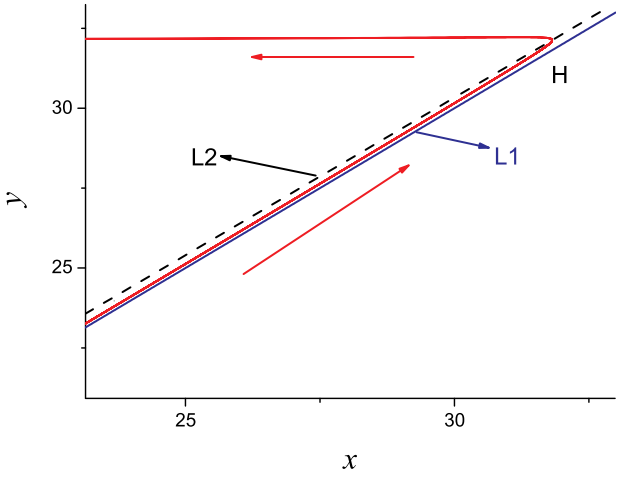
6 Conclusions

The Brusselator is a typical catalytic reaction, where the reaction procedure may behave in different time scales. Under certain conditions of parameters, the classical slow-fast effect appears in the system, which is characterized by coupling of QS and SP. Because the original system is not convenient to explain the transitions between QS and SP, the coordinate transformation is introduced to establish an equivalent model of the original Brusselator. The advantage of the transformed system is that the whole system can be divided into the fast and slow subsystems.

The bifurcation and attraction domain of the fast subsystem in the transformed system are analyzed in detail to reveal the generation mechanism of the slow-fast phenomenon. The slow-fast analysis method is used to reveal the bifurcation mechanism of transitions between QS and SP, where the slow variable is considered as the parame-



(a)



(b)

Figure 6: Mechanism analysis of slow-fast effect. (a) overlapping of portrait with bifurcation diagram, (b) local enlargement near point H. (L1 and L2 are equilibrium lines, LP is the critical point of Fold bifurcation, E, F, G and H are special points used to describe the movement of system.).

ter of the fast subsystem. It has been found that the movement along the stable equilibrium line forms QS, and the jumping behavior represents SP. Fold bifurcation and the change of attraction domain result in the transition between QS and SP. The theoretical analysis of bifurcation and the attraction domain of the fast subsystem coincide well with the numerical simulation. This method of the coordinate transformation may also be helpful to the similar system with multiple time scales.

Conflict of Interests: The authors declare that there is no conflict of interests regarding the publication of this article.

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