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Perturbation solutions for a micropolar fluid flow in a semi-infinite expanding or contracting pipe with large injection or suction through porous wall

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Abstract: We investigate an unsteady incompressible laminar micropolar flow in a semi-infinite porous pipe with large injection or suction through a deforming pipe wall. Using suitable similarity transformations, the governing partial differential are transformed into a coupled nonlinear singular boundary value problem. For large injection, the asymptotic solutions are constructed using the Lighthill method, which eliminates singularity of solution in the high order derivative. For large suction, a series expansion matching method is used. Analytical solutions are validated against the numerical solutions obtained by Bvp4c.

Keywords: expansion ratio; micropolar fluid; Lighthill method; singular perturbation method; Bvp4c

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1 Introduction

Since Eringen [1, 2] proposed a mathematical model to describe the non-Newtonian behaviour of liquids such as polymers, colloidal suspensions, animal blood and liquid crystals, there has been interest in micropolar fluids. In particular, micropolar fluids flowing in porous channels or pipes have received more attention due to their relevance to a number of practical biological problems. For example, Mekheimer and Elkot [3] presented a micropolar model for axisymmetric blood flow through an axially nonsymmetric but radially symmetric mild-stenosis-tapered artery. Mekheimer et al. [4] investigated the effects of an induced magnetic field on peristaltic transport of an incompressible micropolar fluid in a symmetric channel. Furthermore, Subhardra Ramachandran et al. [5] used Van Dyke's singular perturbation technique to study the heat transfer of a micropolar fluid past a curved surface with suction and injection. Anwar Kamal and Hussain [6] examined the steady, incompressible and laminar flow of micropolar fluids inside an infinite channel where the flow was driven due to a surface velocity proportional to the streamwise coordinates. Joneidi et al. [7] obtained similarity equations for a micropolar fluid in a porous channel and used the homotopy analysis method (HAM) to discuss the velocity distribution. In addition, Ariman et al. [8] and Lukaszewicz [9] gave reviews of micropolar fluid mechanics and its applications.

The purpose of this paper is to extend previous investigations by presenting analytical solutions for the flow inside a deforming porous pipe with large injection or suction. Equations describing the unsteady flow of an incompressible Newtonian fluid in a porous expanding channel are presented by White [10] as one of the new exact Navier-Stokes solutions attributed to Dauenhauer and Majdalani [11]. In their work [11], they numerically discussed the influence of the expansion ratio and Reynolds number on the velocity and pressure distribution. Furthermore, Majdalani, Zhou and Dawson [12] also obtained an asymptotic solution for the flow in a porous channel, with slowly expanding or contracting walls, by considering the permeation Reynolds number and expansion ratio as two small parameters. Boutros et al. [13, 14] also discussed the flow through an expanding porous channel or pipe using the Lie group method and obtained the analytical solution with the perturbation method. Recently Si et al. [15] also investigated the micropolar fluid in a porous deforming channel and discussed the effects of the micropolar parameter and the expansion ratio on the velocity and microrotation distribution. As further research, Li, Lin and Si [16] numerically analyzed the flow of a micropolar fluid through a porous pipe with an expanding or contracting wall.

In this paper, asymptotic solutions are constructed for the flow of a micropolar fluid through an expanding or con-

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tracting porous pipe. For large injection, analytical solutions are constructed using the Lighthill method, which eliminates singularity of the solution in the high order derivative [17–19];a series expansion matching method is used for large suction. The accuracy of the analytical solutions for each case is compared with its numerical results.

2 Preliminaries

Consider a micropolar fluid flowing through a pipe with a vertical moving porous wall. Here we assume that one end of the pipe is closed by a complicated solid membrane and the wall of the pipe moves in the radial direction and expands or contracts uniformly at a time-dependent rate $\dot{a}(t)$. In order to neglect the influence of the opening at the end, the length of the pipe is assumed to be semi-infinite [20]. Under the porous wall stipulation, the fluid is injected or aspirated uniformly and vertically through the pipe wall with an absolute velocity v_w, which is proportional to the moving velocity at the wall surface. $\mathbf{u} = (u, v, 0)$ and $\overline{\omega} = (0, 0, N)$ are the velocity vector and microrotation vector, respectively. N is the component of microrotation in the direction vertical to (r, z) plane, and u, v are the components of velocity in the direction of z and r, respectively. The flow configuration and the coordinate system are shown in Fig. 1.

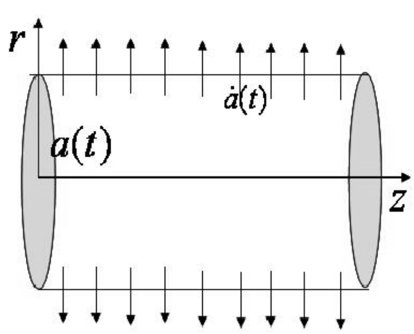

Fig. 1. A model of a micropolar fluid through a porous expanding pipe.

Under these assumptions, the governing equations of the incompressible and homogeneous micropolar fluid flowing with no body force are expressed as follows:

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + (\mu+\kappa)\Delta\mathbf{u} + \kappa\nabla\times\overline{\omega}, \tag{2}$$

$$\rho j\left(\frac{\partial \bar{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\bar{\omega}\right) = -2\kappa\bar{\omega} + y\triangle\bar{\omega} + \kappa\nabla\times\mathbf{u}, \tag{3}$$

where ρ and μ are the density and the dynamic viscosity, and j, y and κ are the micro-inertial coefficient, spin gradient viscosity and vortex viscosity, respectively. Here y is assumed to be

$$y = \left(\mu + \frac{\kappa}{2}\right) j. \tag{4}$$

The corresponding boundary conditions are [11, 12, 15]

$$\begin{aligned} &v = -v_w = -A\dot{a}, \quad u = 0, \quad N = 0, \quad \text{at} \quad r = a(t), \\ &\frac{\partial u}{\partial r} = 0, \quad v = 0, \quad N = 0, \quad \text{at} \quad r = 0, \end{aligned} \tag{5}$$

where A is the measure of the permeability. Here we also assume that there is a strong concentration of microelements, and the microelements close to the wall are unable to rotate [21, 22].

Introduce the stream function $\Psi(r, z, t)$ such that

$$u = \frac{1}{r}\frac{\partial\Psi}{\partial r}, \quad v = -\frac{1}{r}\frac{\partial\Psi}{\partial z}. \tag{6}$$

In this paper, the stream function Ψ and the microrotation velocity N are assumed as follows:

$$\Psi = \nu z F(\eta, t), \quad N = \nu a^{-3} z \eta^{\frac{1}{2}} G(\eta, t), \tag{7}$$

where $\eta = (\frac{r}{a})^2$ and $\nu = \frac{\mu}{\rho}$.

Similar to Dauenhauer and Majdalani [11], Uchida and Aoki [20], and Boutros et al. [13, 14], we substitute Eqs. (6) and (7) into governing equations and consider the similarity solutions with respect to space and time, then the following ordinary equations can be obtained:

$$\begin{aligned} &(1+K)(\eta f'''' + 2f''') + \frac{\alpha}{2}(\eta f''' + 2f'') + \frac{Re}{2}(ff''' - f'f'') \\ &+ \frac{K}{4}(\eta g'' + 2g') = 0, \end{aligned} \tag{8}$$

$$\begin{aligned} &(1+\frac{K}{2})(\eta^2 g'' + 2\eta g') - \frac{K\zeta}{2}(\eta g + 2\eta f'') + \frac{\alpha}{2}(\eta^2 g' + 2\eta g) \\ &+ \frac{Re}{4}(fg + 2\eta f g' - 2\eta f' g) = 0, \end{aligned} \tag{9}$$

where $(f, g) = (\frac{F}{Re}, \frac{G}{Re})$, $Re = \frac{a v_w}{\nu}$ is the permeation Reynolds number, $\alpha = \frac{a\dot{a}}{\nu}$ is the expansion ratio, and $K = \frac{\kappa}{\mu}$ and $\zeta = \frac{a^2}{j}$ are the micropolar parameters. In physical meaning, α is positive for expansion and negative for contraction.

The corresponding boundary conditions can be written as

$$\begin{aligned} &f(1) = 1, \quad f'(1) = 0, \quad g(1) = 0, \quad f(0) = 0, \\ &\lim_{\eta\to 0} \eta^{\frac{1}{2}} f'' = 0, \\ &\lim_{\eta\to 0} \eta^{\frac{1}{2}} g = 0. \end{aligned} \tag{10}$$

3 Perturbation analysis for this problem

3.1 Solution for the large injection Reynolds number

For large injection Reynolds numbers, the asymptotic solution of (8) and (9), subject to the boundary conditions (10), is obtained by the Lighthill method. One treats $\varepsilon = \frac{2}{Re}$ as the perturbation parameter, the equations (8) and (9) then become

$$\begin{aligned}&\varepsilon(1+K)(\eta f''' + f'') + \frac{\varepsilon\alpha}{2}(\eta f'' + f') + \frac{\varepsilon K}{4}(\eta g' + g)\\&-f'^2 + ff'' = \lambda,\end{aligned} \tag{11}$$

$$\begin{aligned}&\varepsilon(1+\frac{K}{2})(\eta^2 g'' + 2\eta g') - \frac{\varepsilon K\zeta}{2}(\eta g + 2\eta f'')\\&+\frac{\varepsilon\alpha}{2}(\eta^2 g' + 2\eta g) + \frac{1}{2}(fg + 2\eta fg' - 2\eta f'g) = 0,\end{aligned} \tag{12}$$

where λ is an integral constant. Firstly, we introduce a variable transformation of η

$$\eta = \xi + \varepsilon X_1(\xi) + \varepsilon^2 X_2(\xi) + O(\varepsilon^3), \tag{13}$$

where the functions X_1, X_2 are unknown and will be determined in the following process. One assumes that the functions f, g and the constant λ are expanded as

$$f(\eta) = \sum_{i=0}^{\infty}\varepsilon^i f_i(\xi),\quad g(\eta) = \sum_{i=0}^{\infty}\varepsilon^i g_i(\xi),\quad \lambda = \sum_{i=0}^{\infty}\varepsilon^i\lambda_i. \tag{14}$$

Substituting (13)-(14) into (11)-(12) and collecting the same powers of ε, one can obtain the leading solution

$$f_0\ddot{f}_0 - \dot{f}_0^{\,2} = \lambda_0,\quad \xi\dot{f}_0 g_0 - \xi f_0\dot{g}_0 - \frac{1}{2}f_0 g_0 = 0, \tag{15}$$

and the first order solution

$$\begin{aligned}f_0\ddot{f}_1 - 2\dot{f}_0\dot{f}_1 + \ddot{f}_0 f_1 =& -(1+K)(\xi\dddot{f}_0 + \ddot{f}_0) - \frac{\alpha}{2}(\xi\ddot{f}_0 + \dot{f}_0)\\&-\frac{K}{4}(\xi\dot{g}_0 + g_0) + \lambda_1 + 2\lambda_0\dot{X}_1,\end{aligned} \tag{16}$$

$$\begin{aligned}2\xi f_0\dot{g}_1 - 2\xi\dot{f}_0 g_1 + f_0 g_1 =& -(2+K)(\xi^2\ddot{g}_0 + 2\xi\dot{g}_0)\\&+K\zeta(\xi g_0 + 2\xi\ddot{f}_0)\\&-\alpha(\xi^2\dot{g}_0 + 2\xi g_0)\\&-f_1 g_0 - f_0 g_0\dot{X}_1 - 2X_1 f_0\dot{g}_0\\&-2\xi f_1\dot{g}_0 + 2X_1\dot{f}_0 g_0 + 2\xi\dot{f}_1 g_0.\end{aligned} \tag{17}$$

Here $\dot{}$ denotes the derivative with respect to ξ.

3.1.1 A. the transformed boundary conditions at the wall of the pipe

We assume $\tilde{\xi}$ is the root of (13) at $\eta = 1$, then

$$\begin{aligned}\tilde{\xi} &= 1 - \varepsilon X_1(\tilde{\xi}) - \varepsilon^2 X_2(\tilde{\xi}) + O(\varepsilon^3)\\&= 1 - \varepsilon\{X_1(1) + \dot{X}_1(1)[-\varepsilon X_1(\tilde{\xi}) - \varepsilon^2 X_2(\tilde{\xi})] + \cdots\}\\&\quad - \varepsilon^2\{X_2(1) + \dot{X}_1(1)[-\varepsilon X_1(\tilde{\xi}) - \varepsilon^2 X_2(\tilde{\xi})] + \cdots\} + \cdots\\&= 1 - \varepsilon X_1(1) - \varepsilon^2[X_2(1) - \dot{X}_1(1)X_1(1)] + O(\varepsilon^3),\end{aligned} \tag{18}$$

thus the conditions at the wall can be obtained

$$\begin{aligned}f|_{\eta=1} = 1 \Rightarrow 1 = f|_{\xi=\tilde{\xi}} &= f|_{\xi=1} + \dot{f}|_{\xi=1}\{-\varepsilon X_1(1)\\&\quad - \varepsilon^2[X_2(1) - \dot{X}_1(1)X_1(1)] + \cdots\} + \cdots\\&= f_0|_{\xi=1} + \varepsilon(f_1 - X_1\dot{f}_0)|_{\xi=1} + O(\varepsilon^2),\end{aligned} \tag{19}$$

$$\begin{aligned}f'|_{\eta=1} = 0 \Rightarrow 0 = \dot{f}|_{\xi=\tilde{\xi}} &= \dot{f}|_{\xi=1} + \ddot{f}|_{\xi=1}\{-\varepsilon X_1(1)\\&\quad - \varepsilon^2[X_2(1) - \dot{X}_1(1)X_1(1)] + \cdots\} + \cdots\\&= \dot{f}_0|_{\xi=1} + \varepsilon(\dot{f}_1 - X_1\ddot{f}_0)|_{\xi=1} + O(\varepsilon^2),\end{aligned} \tag{20}$$

$$\begin{aligned}g|_{\eta=1} = 0 \Rightarrow 0 = g|_{\xi=\tilde{\xi}} &= g|_{\xi=1} + \dot{g}|_{\xi=1}\{-\varepsilon X_1(1)\\&\quad - \varepsilon^2[X_2(1) - \dot{X}_1(1)X_1(1)] + \cdots\} + \cdots\\&= g_0|_{\xi=1} + \varepsilon(g_1 - X_1\dot{g}_0)|_{\xi=1} + O(\varepsilon^2).\end{aligned} \tag{21}$$

Hence, the boundary conditions of f_i and g_i at $\eta = 1$ are

$$f_0|_{\xi=1} = 1,\quad f_1 - X_1\dot{f}_0|_{\xi=1} = 0,\quad \cdots, \tag{22}$$

$$\dot{f}_0|_{\xi=1} = 0,\quad \dot{f}_1 - X_1\ddot{f}_0|_{\xi=1} = 0,\quad \cdots, \tag{23}$$

$$g_0|_{\xi=1} = 0,\quad g_1 - X_1\dot{g}_0|_{\xi=1} = 0,\quad \cdots. \tag{24}$$

3.1.2 B. the transformed boundary conditions at the center of the pipe

One supposes that $\hat{\xi}$ is the root of (13) at $\eta = 0$, then

$$\begin{aligned}\hat{\xi} &= -\varepsilon X_1(\hat{\xi}) - \varepsilon^2 X_2(\hat{\xi}) + O(\varepsilon^3)\\&= -\varepsilon X_1(0) - \varepsilon^2[X_2(0) - \dot{X}_1(0)X_1(0)] + O(\varepsilon^3),\end{aligned} \tag{25}$$

thus we can induce

$$\begin{aligned}f|_{\eta=0} = 0 \Rightarrow 0 = G|_{\xi=\hat{\xi}} &= f|_{\xi=0} + \dot{G}|_{\xi=0}\{-\varepsilon X_1(0)\\&\quad - \varepsilon^2[X_2(0) - \dot{X}_1(0)X_1(0)] + \cdots\} + \cdots\\&= f_0|_{\xi=0} + \varepsilon(f_1 - X_1\dot{f}_0)|_{\xi=0} + O(\varepsilon^2).\end{aligned} \tag{26}$$

Hence, the boundary conditions of f_i at $\eta = 0$ are

$$f_0|_{\xi=0} = 0,\quad (f_1 - X_1\dot{f}_0)|_{\xi=0} = 0,\quad \cdots. \tag{27}$$

Using (22) – (24) and (27), the solution for (15) can be obtained

$$f_0 = \sin(\frac{\pi}{2}\xi), \ \ g_0 = 0, \tag{28}$$

and then $\lambda_0 = -\frac{\pi^2}{4}$ can be obtained. Substituting above results into (16), (17) yields the equations for f_1, g_1

$$\begin{aligned}&\sin\left(\frac{\pi}{2}\xi\right)\ddot{f}_1 - \pi\cos\left(\frac{\pi}{2}\xi\right)\dot{f}_1 - \frac{\pi^2}{4}\sin\left(\frac{\pi}{2}\xi\right)f_1\\ &= (1+K)\left[\frac{\pi^3}{8}\xi\cos\left(\frac{\pi}{2}\xi\right) + \frac{\pi^2}{4}\sin\left(\frac{\pi}{2}\xi\right)\right]\\ &-\frac{\alpha}{2}\left[-\frac{\pi^2}{4}\xi\sin\left(\frac{\pi}{2}\xi\right) + \frac{\pi}{2}\cos\left(\frac{\pi}{2}\xi\right)\right] - \frac{\pi^2}{2}\dot{X}_1 + \lambda_1,\end{aligned} \tag{29}$$

and

$$\begin{aligned}&\xi\sin\left(\frac{\pi}{2}\xi\right)\dot{g}_1 - \frac{\pi}{2}\xi\cos\left(\frac{\pi}{2}\xi\right)g_1 + \frac{1}{2}\sin\left(\frac{\pi}{2}\xi\right)g_1\\ &+\frac{K\zeta\pi^2}{4}\xi\sin\left(\frac{\pi}{2}\xi\right) = 0.\end{aligned} \tag{30}$$

Here it should be noted that direct use of the method of variation of parameters will cause a singularity in the third-order derivative of f_1 [17–19]. In order to eliminate the singularity and to simplify the equation of f_1, we can set

$$\begin{aligned}&(1+K)\left[\frac{\pi^3}{8}\xi\cos\left(\frac{\pi}{2}\xi\right) + \frac{\pi^2}{4}\sin\left(\frac{\pi}{2}\xi\right)\right]\\ &-\frac{\alpha}{2}\left[-\frac{\pi^2}{4}\xi\sin\left(\frac{\pi}{2}\xi\right) + \frac{\pi}{2}\cos\left(\frac{\pi}{2}\xi\right)\right] - \frac{\pi^2}{2}\dot{X}_1 + \lambda_1 = 0.\end{aligned} \tag{31}$$

Then we have

$$X_1(\xi) = \frac{1+K}{2}\xi\sin\left(\frac{\pi}{2}\xi\right) - \frac{\alpha}{2\pi}\xi\cos\left(\frac{\pi}{2}\xi\right) + \frac{2\lambda_1}{\pi^2}\xi + \hat{C}_1, \tag{32}$$

where $\hat{C}_1$ is a constant. Thus, the equation for f_1 becomes

$$\sin\left(\frac{\pi}{2}\xi\right)\ddot{f}_1 - \pi\cos\left(\frac{\pi}{2}\xi\right)\dot{f}_1 - \frac{\pi^2}{4}\sin\left(\frac{\pi}{2}\xi\right)f_1 = 0. \tag{33}$$

The solution of Eq. (33) is

$$f_1(\xi) = \hat{C}_2\cos\left(\frac{\pi}{2}\xi\right) + \hat{C}_3\left[\frac{2}{\pi}\sin\left(\frac{\pi}{2}\xi\right) - \xi\cos\left(\frac{\pi}{2}\xi\right)\right], \tag{34}$$

where $\hat{C}_2$, $\hat{C}_3$ are still integral constants. According to the boundary conditions (22) – (24) and (27), $\hat{C}_1 = 0$, $\hat{C}_2 = 0$, $\hat{C}_3 = 0$, $\lambda_1 = -\frac{(1+K)\pi^2}{4}$ can be determined. Thus, f_1, g_1 can be achieved as follows:

$$f_1 = 0, \ \ g_1 = \frac{K\zeta\pi^2}{4}\xi^{-\frac{1}{2}}\sin\left(\frac{\pi}{2}\xi\right)\int_\xi^1 t^{\frac{1}{2}}\csc\left(\frac{\pi}{2}t\right)dt. \tag{35}$$

Finally, one obtains the asymptotic solutions of f, g in terms of ξ

$$\begin{aligned}f(\eta) &= \sin\left(\frac{\pi}{2}\xi\right),\\ g(\eta) &= \frac{\varepsilon K\zeta\pi^2}{4}\xi^{-\frac{1}{2}}\sin\left(\frac{\pi}{2}\xi\right)\int_\xi^1 t^{\frac{1}{2}}\csc\left(\frac{\pi}{2}t\right)dt,\end{aligned} \tag{36}$$

where

$$\eta = \xi + \varepsilon[\frac{1+K}{2}\xi\sin(\frac{\pi}{2}\xi) - \frac{\alpha}{2\pi}\xi\cos(\frac{\pi}{2}\xi) - \frac{1+K}{2}\xi]. \tag{37}$$

Fig. **??** shows the profiles of $f'(\eta)$ and $g(\eta)$ against η for the asymptotic and numerical results. Tables **??** and **??** give asymptotic and numerical values of $f''(1)$ and $g'(1)$ for some values of large injection Reynolds number and expansion ratio α, respectively. The results agree well.

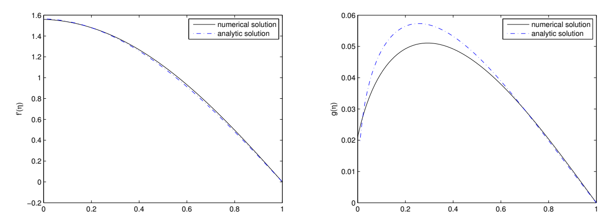

Fig. 2. Variation of $f'(\eta)$ and $g(\eta)$ as $Re = 100$, $\alpha = -5$, $K = 0.2$, $\zeta = 10$.

3.2 Solution for large suction Reynolds number

The boundary layer happens near the wall not only for the velocity but also for the microrotation when there is large suction. The solutions of (8) and (9), subject to the boundary conditions (10), can be obtained for large suction by using the method of matched asymptotic expansion. One treats $\varepsilon = -\frac{2}{Re}$ as the perturbation parameter, the equations (8) and (9) then become

$$\begin{aligned}&\varepsilon(1+K)(\eta f''' + f'') + \frac{\varepsilon\alpha}{2}(\eta f'' + f') + \frac{\varepsilon K}{4}(\eta g' + g)\\ &+f'^2 - ff'' = k,\end{aligned} \tag{38}$$

$$\begin{aligned}&\varepsilon(2+K)(\eta^2 g'' + 2\eta g') - \varepsilon K\zeta(\eta g + 2\eta f'') + \varepsilon\alpha(\eta^2 g' + 2\eta g)\\ &-fg - 2\eta fg' + 2\eta f'g = 0,\end{aligned} \tag{39}$$

where k is a constant of integration,

$$k = \varepsilon(1+K)\sigma + \frac{\varepsilon\alpha\delta}{2} + \frac{\varepsilon K\omega}{4} + \delta^2, \tag{40}$$

Table 1. Values of $f''(1)$ and $g'(1)$ for large injection Reynolds number ($\alpha = -5$, $K = 0.2$, $\zeta = 10$).

	$f''(1)$		$g'(1)$	
Re	**Numerical**	**Asymptotic**	**Numerical**	**Asymptotic**
50	**−2.6641619556581**	**−2.7339624379749**	**−0.2128340193109**	**−0.2077811452861**
100	**−2.5700001017056**	**−2.5955567129755**	**−0.1036184958045**	**−0.1012267118060**
150	**−2.5366158007420**	**−2.5517506351567**	**−0.0681380895087**	**−0.0669125722108**
200	**−2.5195961015861**	**−2.5302623043972**	**−0.0507066428440**	**−0.0499726805118**
500	**−2.4884675811385**	**−2.4922614078153**	**−0.0199682862609**	**−0.0198384008062**

Table 2. A comparison of $f''(1)$ and $g'(1)$ for different α ($Re = 100$, $K = 0.3$, $\zeta = 10$).

	$f''(1)$		$g'(1)$	
α	**Numerical**	**Asymptotic**	**Numerical**	**Asymptotic**
−5	**−2.569310686399770**	**−2.595556712975505**	**−0.155025023543228**	**−0.151840067709067**
−2	**−2.507901592906879**	**−2.517499337080236**	**−0.147379010820407**	**−0.149539460622566**
2	**−2.430977747070986**	**−2.418783550899264**	**−0.138027490934619**	**−0.146578283184495**
5	**−2.376702021744101**	**−2.348507888421025**	**−0.131584147099143**	**−0.144433235137893**

and

$$\begin{aligned}\delta &= f'(0) = \delta_0 + \varepsilon\delta_1 + \varepsilon^2\delta_2 + O(\varepsilon^3),\\ \sigma &= f''(0) = \sigma_0 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + O(\varepsilon^3),\\ \omega &= g(0) = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + O(\varepsilon^3),\end{aligned} \tag{41}$$

where the coefficients δ_i, σ_i and ω_i $(i = 0, 1, 2, \cdots)$ are constants determined by matching with the inner solution.

We assume that the forms of the outer solutions are

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + O(\varepsilon^3), \;\; g = g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + O(\varepsilon^3). \tag{42}$$

Substituting (41) – (42) into (38) – (39), and equating the same power of the coefficient ε, one obtains

$$\varepsilon^0 : f_0'^2 - f_0 f_0'' = \delta_0^2, \tag{43}$$

$$f_0 g_0 + 2\eta f_0 g_0' - 2\eta f_0' g_0 = 0. \tag{44}$$

$$\begin{aligned}\varepsilon^1 :\; & 2f_0'f_1' - f_0''f_1 - f_0 f_1'' = -(1+K)(\eta f_0''' + f_0'')\\ & - \frac{\alpha}{2}(\eta f_0'' + f_0') - \frac{K}{4}(\eta g_0' + g_0)\\ & + 2\delta_0\delta_1 + (1+K)\sigma_0 + \frac{\alpha\delta_0}{2} + \frac{K\omega_0}{4},\end{aligned} \tag{45}$$

$$\begin{aligned}& 2\eta f_0' g_1 - 2\eta f_0 g_1' - f_0 g_1 = -(2+K)(\eta^2 g_0'' + 2\eta g_0')\\ & + K\zeta(\eta g_0 + 2\eta f_0'') - 2\eta f_1' g_0 - \alpha(\eta^2 g_0' + 2\eta g_0)\\ & + 2\eta f_1 g_0' + f_1 g_0.\end{aligned} \tag{46}$$

$$\begin{aligned}\varepsilon^2 :\; & 2f_0'f_2' - f_0''f_2 - f_0 f_2'' = -(1+K)(\eta f_1''' + f_1'') - \frac{\alpha}{2}(\eta f_1'' + f_1')\\ & - \frac{K}{4}(\eta g_1' + g_1) + f_1 f_1'' - f_1'^2 + 2\delta_0\delta_2 + \delta_1^2 + (1+K)\sigma_1\\ & + \frac{\alpha\delta_1}{2} + \frac{K\omega_1}{4},\end{aligned} \tag{47}$$

$$\begin{aligned}& 2\eta f_0' g_2 - 2\eta f_0 g_2' - f_0 g_2 = -(2+K)(\eta^2 g_1'' + 2\eta g_1')\\ & + K\zeta(\eta g_1 + 2\eta f_1'') + f_1 g_1 - \alpha(\eta^2 g_1' + 2\eta g_1) + f_2 g_0\\ & + 2\eta f_1 g_1' - 2\eta f_1' g_1 - 2\eta f_2' g_0 + 2\eta f_2 g_0'.\end{aligned} \tag{48}$$

$$\begin{aligned}\varepsilon^3 :\; & 2f_0'f_3' - f_0''f_3 - f_0 f_3'' = -(1+K)(\eta f_2''' + f_2'') - \frac{\alpha}{2}(\eta f_2'' + f_2')\\ & - \frac{K}{4}(\eta g_2' + g_2) + f_1'' f_2 + f_1 f_2'' - 2f_1' f_2' + 2\delta_0\delta_3 + 2\delta_1\delta_2\\ & + (1+K)\sigma_2 + \frac{\alpha\delta_2}{2} + \frac{K\omega_2}{4},\end{aligned} \tag{49}$$

$$\begin{aligned}& 2\eta f_0' g_3 - 2\eta f_0 g_3' - f_0 g_3 = -(2+K)(\eta^2 g_2'' + 2\eta g_2')\\ & + K\zeta(\eta g_2 + 2\eta f_2'') - \alpha(\eta^2 g_2' + 2\eta g_2) + f_1 g_2 + f_2 g_1 + f_3 g_0\\ & + 2\eta f_1 g_2' + 2\eta f_2 g_1' - 2\eta f_1' g_2 - 2\eta f_2' g_1 - 2\eta f_3' g_0 + 2\eta f_3 g_0'.\end{aligned} \tag{50}$$

$$\cdots\cdots.$$

The corresponding boundary conditions for the outer solutions are

$$f_i(0) = 0, \;\; \lim_{\eta\to 0} \eta^{\frac{1}{2}} f_i'' = 0, \;\; \lim_{\eta\to 0} \eta^{\frac{1}{2}} g_i = 0, \;\; i = 0, 1, 2, \cdots. \tag{51}$$

According to the boundary conditions (51), the solutions for (43) and (44) can be obtained

$$f_0 = \delta_0\eta, \;\; g_0 = \tilde{C}_0 \eta^{\frac{1}{2}}, \tag{52}$$

where $\tilde{C}_0$ is an integral constant. Because g_0 means the microrotation velocity of particles, its first-order derivative should be bounded, thus $g_0'(0) \to \infty$ leads to $\tilde{C}_0 = 0$. Then $\omega_0 = g_0(0) = 0$, $\sigma_0 = f_0''(0) = 0$. Substituting (52) into (45), one obtains

$$2f_1' - \eta f_1'' = 2\delta_1. \tag{53}$$

Then the solution of (53) that satisfies the boundary conditions (51) is

$$f_1 = \delta_1\eta + \tilde{C_1}\eta^3, \tag{54}$$

where $\tilde{C_1}$ is an integral constant, which will be determined next. Substituting (52), (54) into (46), one obtains

$$g_1 - 2\eta g_1^{'} = 0, \tag{55}$$

whose solution is

$$g_1 = \tilde{C_2}\eta^{\frac{1}{2}}, \tag{56}$$

where $\tilde{C_2}$ is an integration constant. Similarly, $g_1^{'}(0) \to \infty$ leads to $\tilde{C_2} = 0$. Then

$$f_1 = \delta_1\eta + \tilde{C_1}\eta^3, \;\; g_1 = 0. \tag{57}$$

Thus one obtains $\sigma_1 = f_1^{''}(0) = 0, \omega_1 = g_1(0) = 0$. Substituting Eqs. (51) and (57) into (47), one obtains

$$f_2 = -\frac{6(1+K)\tilde{C_1}}{\delta_0}\eta^2 + \frac{\alpha\tilde{C_1}}{2\delta_0}(3\eta^3\ln\eta - \eta^3) + \delta_2\eta + \frac{3\tilde{C_1}^2}{10\delta_0}\eta^5 + \tilde{C_3}\eta^3, \tag{58}$$

where $\tilde{C_3}$ is an integral constant. Similarly, $f_2^{'''}(0) \to \infty$ leads to $\tilde{C_1} = 0$, thus

$$f_1 = \delta_1\eta, \;\; f_2 = \delta_2\eta + \tilde{C_3}\eta^3. \tag{59}$$

Then one obtains $\sigma_2 = f_2^{''}(0) = 0$. Substituting (51), (57) and (59) into (48), one obtains

$$g_2 = \tilde{C_4}\eta^{\frac{1}{2}}. \tag{60}$$

Similarly, $g_2^{'}(0) \to \infty$ leads to $\tilde{C_4} = 0$. Then one obtains $\omega_2 = g_2(0) = 0$. Substituting (51), (57), (59) and (60) into (49), one obtains

$$f_3 = -\frac{6(1+K)\tilde{C_3}}{\delta_0}\eta^2 + \frac{\alpha\tilde{C_3}}{2\delta_0}\eta^3(3\ln\eta - 1) + \delta_3\eta + \tilde{C_5}\eta^3, \tag{61}$$

where $\tilde{C_5}$ is an integration constant. Similarly, $f_3^{'''}(0) \to \infty$ leads to $\tilde{C_3} = 0$, thus

$$f_2 = \delta_2\eta, \;\; g_2 = 0. \tag{62}$$

In order to obtain the inner solution in the viscous layer, we introduce a stretching transformation $\tau = (1-\eta)/\varepsilon$. Substituting into Eqs. (11) and (12) yields

$$\begin{aligned}&(1+K)[-\dddot{f} + \varepsilon(\tau\dddot{f} + \ddot{f})] + \frac{\alpha}{2}[\varepsilon\ddot{f} - \varepsilon^2(\tau\ddot{f} + \dot{f})] \\ &+ \frac{K}{4}[-\varepsilon^2\dot{g} + \varepsilon^3(\tau\dot{g} + g)] + \dot{f}^2 - f\ddot{f} \\ &= \varepsilon^2\delta^2 + \varepsilon^3[(1+K)\sigma + \frac{\alpha\delta}{2} + \frac{K\omega}{4}],\end{aligned} \tag{63}$$

$$\begin{aligned}&(2+K)[\varepsilon^2(\tau^2\ddot{g} + 2\tau\dot{g}) - \varepsilon(2\tau\ddot{g} + 2\dot{g}) + \ddot{g}] \\ &- K\zeta[-\varepsilon^3\tau g + \varepsilon^2 g - 2\varepsilon\tau\ddot{f} + 2\ddot{f}] \\ &+ \alpha[-\varepsilon^3(\tau^2\dot{g} + 2\tau g) + \varepsilon^2(2\tau\dot{g} + 2g) - \varepsilon\dot{g}] + 2f\dot{g} \\ &- 2\dot{f}g + \varepsilon(2\tau\dot{f}g - fg - 2\tau f\dot{g}) = 0.\end{aligned} \tag{64}$$

Here $\dot{}$ denotes the derivative with respect to τ. According to the boundary conditions (10), we assume the inner solutions near the wall to be

$$f(\tau) = 1 + \sum_{i=1}^{\infty}\varepsilon^i\phi_i(\tau), \;\; g(\tau) = \sum_{i=1}^{\infty}\varepsilon^i\psi_i(\tau). \tag{65}$$

Substituting (65) into (63) and (64) yields the following equations

$$\varepsilon^1 : (1+K)\dddot{\phi}_1 + \ddot{\phi}_1 = 0, \tag{66}$$

$$(2+K)\ddot{\psi}_1 - 2K\zeta\ddot{\phi}_1 + 2\dot{\psi}_1 = 0. \tag{67}$$

$$\varepsilon^2 : (1+K)(\dddot{\phi}_2 - \tau\dddot{\phi}_1 - \ddot{\phi}_1) - \frac{\alpha}{2}\ddot{\phi}_1 - \dot{\phi}_1^{\,2} + \ddot{\phi}_2 + \phi_1\ddot{\phi}_1 = -\delta_0^2, \tag{68}$$

$$\begin{aligned}&(2+K)(\ddot{\psi}_2 - \tau\ddot{\psi}_1 - 2\dot{\psi}_1) - 2K\zeta\ddot{\phi}_2 - \alpha\dot{\psi}_1 + 2\dot{\psi}_2 \\ &+ 2\phi_1\dot{\psi}_1 - 2\dot{\phi}_1\psi_1 - \psi_1 = 0.\end{aligned} \tag{69}$$

$$\cdots\cdots.$$

The boundary conditions corresponding to the inner solution are

$$\phi_i(0) = 0, \;\; \dot{\phi}_i(0) = 0, \;\; \psi_i(0) = 0, \;\; i = 1, 2, \cdots. \tag{70}$$

The solution of (66) satisfying the boundary conditions (70) is

$$\phi_1 = D_1(e^{-A_0\tau} + A_0\tau - 1), \tag{71}$$

where $A_0 = \frac{1}{1+K}$, and D_1 is an integral constant. Then the first two terms of the inner solution of f can be expressed as

$$f(\tau) = 1 + D_1(e^{-A_0\tau} + A_0\tau - 1)\varepsilon. \tag{72}$$

The outer solution of f, expressed in terms of the inner variable τ, is

$$f(\eta) = \delta_0 + (\delta_1 - \delta_0\tau)\varepsilon + (\delta_2 - \delta_1\tau)\varepsilon^2 + \cdots. \tag{73}$$

As $\tau \to \infty$, matching the inner solution (72) with (73) gives

$$\delta_0 = 1, \;\; \delta_1 = \frac{1}{A_0}, \;\; D_1 = -\frac{1}{A_0}. \tag{74}$$

The solution of (67) satisfying the boundary conditions (70) is

$$\psi_1 = \frac{KB_0\zeta}{B_0 - A_0}(e^{-A_0\tau} - 1) + D_2(e^{-B_0\tau} - 1), \tag{75}$$

where $B_0 = \frac{2}{2+K}$, and D_2 is an integral constant. Similarly, as $\tau \to \infty$ in (75), D_2 can be determined:

$$D_2 = -\frac{KB_0\zeta}{B_0 - A_0}. \tag{76}$$

The solution of (68) satisfying the boundary conditions (70) is

$$\begin{aligned}\phi_2 = &-\left(\frac{\alpha}{2} + \frac{2}{A_0}\right)\tau e^{-A_0\tau} - \left(\frac{\alpha}{A_0} + \frac{4}{A_0^2} - \frac{D_3}{A_0}\right)e^{-A_0\tau} \\ &+ \left(D_3 - \frac{\alpha}{2} - \frac{2}{A_0}\right)\tau + \frac{4}{A_0^2} + \frac{\alpha}{A_0} - \frac{D_3}{A_0},\end{aligned} \tag{77}$$

where D_3 is an integral constant. Then the inner solution of f can be expressed as

$$\begin{aligned}f = 1 &- \frac{\varepsilon}{A_0}(e^{-A_0\tau} + A_0\tau - 1) + \varepsilon^2\left[-\left(\frac{\alpha}{2} + \frac{2}{A_0}\right)\tau e^{-A_0\tau}\right. \\ &- \left(\frac{\alpha}{A_0} + \frac{4}{A_0^2} - \frac{D_3}{A_0}\right)e^{-A_0\tau} \\ &\left.+ \left(D_3 - \frac{\alpha}{2} - \frac{2}{A_0}\right)\tau + \frac{4}{A_0^2} + \frac{\alpha}{A_0} - \frac{D_3}{A_0}\right] + \cdots.\end{aligned} \tag{78}$$

As $\tau \to \infty$, matching the inner solution with outer solution, one obtains

$$D_3 = \frac{1}{A_0} + \frac{\alpha}{2}, \quad \delta_2 = \frac{3}{A_0^2} + \frac{\alpha}{2A_0}. \tag{79}$$

similarly, one can obtain

$$\begin{aligned}\psi_2 = &\frac{\zeta}{K(2+K)(4+3K)}[(48K + 108K^2 + 78K^3 + 18K^4 \\ &+ 8K\alpha + 10K^2\alpha + 3K^3\alpha)\tau e^{-A_0\tau} - (24K + 46K^2 + 25K^3 \\ &+ 3K^4 + 8K\alpha + 14K^2\alpha + 6K^3\alpha)\tau e^{-B_0\tau} + (8K\alpha + 18K^2\alpha \\ &+ 13K^3\alpha + 3K^4\alpha - 48 - 156K - 186K^2 - 96K^3 \\ &- 18K^4)e^{-A_0\tau} + (48 + 156K + 182K^2 + 84K^3 + 6K^4 \\ &- 4K^5 - 8K\alpha - 18K^2\alpha - 13K^3\alpha - 3K^4\alpha)e^{-B_0\tau} \\ &+ (4K^2 + 12K^3 + 12K^4 + 4K^5)e^{-(A_0+B_0)\tau}].\end{aligned} \tag{80}$$

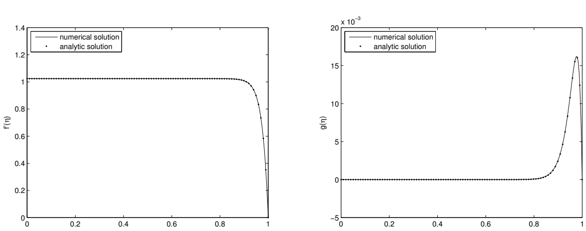

Fig. 3. variation of $f'(\eta)$ and $g(\eta)$ as $Re = -100$, $\alpha - 5$, $K = 0.2$, $\zeta = 10$.

Hence, the complete solutions of (8) and (9) satisfying the boundary conditions (10) for large suction can be obtained as follows:

$$\begin{aligned}f = &\eta + \varepsilon\{(1+K)\eta - e^{\frac{\eta-1}{\varepsilon(1+K)}}[\frac{\alpha}{2} + 3(1+K) - (2 + 2K + \frac{\alpha}{2})\eta]\} \\ &+ \varepsilon^2\{3(1+K)^2 + \frac{\alpha(1+K)}{2} - e^{\frac{\eta-1}{\varepsilon(1+K)}}[\frac{\alpha(1+K)}{2} + 3(1+K)^2]\},\end{aligned} \tag{81}$$

and

$$\begin{aligned}g = &\varepsilon\zeta\{e^{\frac{\eta-1}{\varepsilon(1+K)}}[8(1+K) + \alpha - (\alpha + 6K + 6)\eta] \\ &+ \frac{1+K}{2+K}e^{\frac{2(\eta-1)}{\varepsilon(2+K)}}[-10 - 3K - 2\alpha + (6 + K + 2\alpha)\eta]\} \\ &+ \varepsilon^2\zeta\{e^{\frac{\eta-1}{\varepsilon(1+K)}}\frac{(1+K)(K\alpha - 6 - 6K)}{K} \\ &+ e^{\frac{2(\eta-1)}{\varepsilon(2+K)}}[\frac{6 + 12K + 2K^2}{K} + \frac{4K(7 + 7K - K^3)}{(2+K)(4+3K)} - (1+K)\alpha] \\ &+ \frac{4K(1+K)^3}{(2+K)(4+3K)}e^{\frac{(4+3K)(\eta-1)}{\varepsilon(1+K)(2+K)}}\}.\end{aligned} \tag{82}$$

Fig. 3 shows the profiles of $f'(\eta)$ and $g(\eta)$ against η for the asymptotic and numerical results. Tables 3 and 4 give asymptotic and numerical values of $f''(1)$ and $g'(1)$ for some values of large suction Reynolds number and expansion ratio, respectively.

4 Conclusions

In this paper, we have proposed a model for the flow of micropolar flow through an expanding porous pipe. Using suitable similarity transformations, the governing equations are transformed into a coupled nonlinear singular boundary value problem, and the analytical solutions are compared with the numerical ones, showing good agreement. Some conclusions can be drawn:

i) Analytical solutions can be obtained for large injection or suction using the Lighthill method and series-expansion matching method;

ii) The microrotation velocity also exists at the boundary layer for large suction;

iii) The Lighthill method can also be used to solve similar problems.

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Table 3. Values of $f''(1)$ and $g'(1)$ for large suction Reynolds number ($\alpha = -5$, $K = 0.2$, $\zeta = 10$)

	$f''(1)$		$g'(1)$	
Re	**Numerical**	**Asymptotic**	**Numerical**	**Asymptotic**
−60	**−26.029180587850**	**−26.083333333333**	**−1.9697087541999**	**−1.9620553359684**
−90	**−38.553222227512**	**−38.583333333333**	**−1.9168665563568**	**−1.9140974967062**
−100	**−42.723767379927**	**−42.750000000000**	**−1.9066699865053**	**−1.9045059288538**
−125	**−53.146883680302**	**−53.166666666667**	**−1.8885407560495**	**−1.8872411067194**
−150	**−63.567422760150**	**−63.583333333333**	**−1.8766007832311**	**−1.8757312252964**

Table 4. The comparison of $f''(1)$ and $g'(1)$ for different α ($Re = -100$, $K = 0.2$, $\zeta = 10$)

	$f''(1)$		$g'(1)$	
α	**Numerical**	**Asymptotic**	**Numerical**	**Asymptotic**
−5	**−42.7237673799268**	**−42.7500000000000**	**−1.90666998651**	**−1.90450592885**
−2	**−41.4054335101753**	**−41.5000000000000**	**−1.91280714642**	**−1.90450592885**
2	**−39.6395810279868**	**−39.8333333333333**	**−1.92210393045**	**−1.90450592885**
5	**−38.3082775719552**	**−38.5833333333333**	**−1.93007245250**	**−1.90450592885**

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