

## Research Article

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# Numerical solutions of multi-order fractional differential equations by Boubaker Polynomials

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**Abstract:** In this paper, we have applied a numerical method based on Boubaker polynomials to obtain approximate numerical solutions of multi-order fractional differential equations. We obtain an operational matrix of fractional integration based on Boubaker polynomials. Using this operational matrix, the given problem is converted into a set of algebraic equations. Illustrative examples are given to demonstrate the efficiency and simplicity of this technique.

**Keywords:** Boubaker polynomials, Multi-order fractional differential equation, Operational matrix.

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## 1 Introduction

During the last few decades, fractional differential equations (FDEs) have been applied to describe mathematical phenomenon in physics, chemistry, damping laws, rheology, control theory, signal processing, viscoelastic materials, polymers and so on [4, 10, 13, 17, 19]. Since most FDEs do not have exact analytic solutions, many researchers have tried to find solutions of FDEs using approximate and numerical techniques. For example see [3, 12, 16, 20–24]. Our aim in this work is the following type of multi-order FDE:

$$D^\alpha y(x) = \sum_{i=1}^k y_i D_x^{\beta_i} y(x) + y_{k+1} y(x) + g(x), \quad (1)$$

$$y^{(p)}(0) = d_p, \quad p = 0, 1, \dots, n-1, \quad (2)$$

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where  $n-1 < \alpha \leq n$ , the coefficients  $y_i$  ( $i = 1, \dots, k+1$ ) are constant,  $0 < \beta_1 < \beta_2 < \dots < \beta_k < \alpha$  and  $g$  is a known function. Moreover,  $D^\alpha y(x)$  denotes the Caputo fractional derivative of order  $\alpha$ . It is defined as [4, 7, 17]:

$$D_x^\alpha y(x) = \begin{cases} I^{(k-\alpha)} y^{(k)}(x), & k-1 < \alpha < k, \quad k \in \mathbf{N}, \\ \frac{d^k}{dt^k} y(x), & \alpha = k. \end{cases} \quad (3)$$

In (3)  $I^{(k-\alpha)}$  denotes the Riemann-Liouville fractional. It is generally defined as follows:

$$I^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{y(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0. \quad (4)$$

Here we list the few properties of these two operators as follow:

$$(a) \quad D_x^\alpha x^\beta = \begin{cases} 0, & \beta \in \mathbf{N}_0, \beta < [\alpha], \\ \frac{\Gamma(\beta+1)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, & \beta \in \mathbf{N}_0, \beta \geq [\alpha], \\ \text{or } \beta \in \mathbf{N}_0, \beta > [\alpha], \end{cases}$$

$$(b) \quad I^\alpha D_x^\alpha y(x) = y(x) - \sum_{k=0}^{n-1} y^{(k)}(0^+) \frac{x^k}{k!}, \quad (5)$$

$$(c) \quad I^\alpha D_x^\beta y(x) = I^{\alpha-\beta} y(x) - \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{\Gamma(\alpha-\beta+k+1)} (x-a)^{\alpha-\beta+k}, \quad (6)$$

$$m-1 < \beta \leq m.$$

Many approximation and numerical techniques are utilized to determine the numerical solution of multi-order FDE [8, 11].

The Boubaker polynomials were established for the first time by Boubaker [1, 5, 6] as a guide for solving a one-dimensional heat transfer equation and second order differential equations. Kumar used these polynomials to solve Love's equation in a particular physical system [14]. We will generalize the operational matrix for fractional integration using Boubaker polynomials [1, 5, 6, 14].

In this study, we want to solve the multi-order FDE using the operational matrix for fractional integration based on Boubaker Polynomials. The aim of this approach is converting the multi-order FDE into a set of algebraic

equations by expanding the unspecified function within Boubaker polynomials.

This work is organized into six sections. Section 2 deals with some properties of Boubaker polynomials. The operational matrix is computed for fractional order integration in Section 3. We convert the multi-order FDE to system of algebraic equations in Section 4. Following this we solve a selection of numerical examples in section 5 by using the proposed technique. A brief conclusion is presented in Section 6.

## 2 Boubaker polynomials

The Boubaker polynomials monomial definition is given by [5, 6, 15]:

$$B_n(x) = \sum_{p=0}^{\xi(n)} \left[ \frac{(n-4p)}{(n-p)} C_{n-p}^p \right] \cdot (-1)^p x^{n-2p}, \tag{7}$$

where

$$\xi(n) = \left\lfloor \frac{n}{2} \right\rfloor = \frac{2n + ((-1)^n - 1)}{4}, \tag{8}$$

$$C_{n-p}^p = \frac{(n-p)!}{p!(n-2p)!}.$$

The symbol  $\lfloor \cdot \rfloor$  denotes the floor function.

The Boubaker polynomials could be calculated by following recursive formula:

$$\begin{cases} B_m(x) = xB_{m-1}(x) - B_{m-2}(x), & \text{for } m \geq 2, \\ B_0(x) = 1, \quad B_1(x) = x. \end{cases} \tag{9}$$

### 2.1 Approximation of function

The function  $f$  is approximated by Boubaker polynomials as following:

$$f(x) \simeq \sum_{i=0}^N c_i B_i = C^T \mathbf{B}(x), \tag{10}$$

where  $\mathbf{B}(x)^T = [B_0, B_1, \dots, B_N]$ ,  $B_i(x)$ ,  $i = 0, 1, 2, \dots, N$  denote the Boubaker polynomials,  $C^T = [c_0, c_1, \dots, c_N]$  are unknown Boubaker coefficients and  $N$  is chosen as any positive integer.

Then  $C^T$  can be obtained by

$$C^T \langle \mathbf{B}(x), \mathbf{B}(x) \rangle = \langle f, \mathbf{B}(x) \rangle, \tag{11}$$

where

$$\langle f, \mathbf{B}(x) \rangle = \int_0^1 f(x) \mathbf{B}(x)^T dx = [\langle f, B_0 \rangle, \langle f, B_1 \rangle, \dots, \langle f, B_m \rangle], \tag{12}$$

and  $\langle \mathbf{B}(x), \mathbf{B}(x) \rangle$  is called the dual matrix of  $\phi$  denoted by  $Q$ , and  $Q$  is obtained as:

$$Q = \langle \mathbf{B}(x), \mathbf{B}(x) \rangle = \int_0^1 \mathbf{B}(x) \mathbf{B}(x)^T dx, \tag{13}$$

and then

$$C^T = \left( \int_0^1 f(x) \mathbf{B}(x)^T dx \right) Q^{-1}. \tag{14}$$

By using the expression (7) and taking  $n = 0, \dots, N$ , we can express Boubaker polynomials in terms of power basis functions

$$\mathbf{B}(x) = ZX(x), \tag{15}$$

where

$$X(x) = [1 \ x \ \dots \ x^N]^T, \tag{16}$$

and if  $N$  is odd,

$$Z = \begin{bmatrix} \varphi_{0,0} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \varphi_{1,0} & 0 & 0 & \dots & 0 & 0 \\ \varphi_{2,1} & 0 & \varphi_{2,0} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{N-1, \frac{N-1}{2}} & 0 & \varphi_{N-1, \frac{N-3}{2}} & 0 & \dots & \varphi_{N-1,0} & 0 \\ 0 & \varphi_{N, \frac{N-1}{2}} & 0 & \varphi_{N, \frac{N-3}{2}} & \dots & 0 & \varphi_{N,0} \end{bmatrix}$$

(default) and if  $N$  is even,

$$Z = \begin{bmatrix} \varphi_{0,0} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \varphi_{1,0} & 0 & 0 & \dots & 0 & 0 \\ \varphi_{2,1} & 0 & \varphi_{2,0} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \varphi_{N-1, \frac{N-1}{2}} & 0 & \varphi_{N-1, \frac{N-3}{2}} & \dots & \varphi_{N-1,0} & 0 \\ \varphi_{N, \frac{N}{2}} & 0 & \varphi_{N, \frac{N-2}{2}} & 0 & \dots & 0 & \varphi_{N,0} \end{bmatrix}$$

where

$$B_n(x) = \sum_{p=0}^{\xi(n)} \varphi_{n,p} x^{n-2p},$$

$$n = 0, 1, \dots, N, \quad p = 0, 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor. \tag{17}$$

$$\varphi_{n,p} = \left[ \frac{(n-4p)}{(n-p)} C_{n-p}^p \right] (-1)^p. \tag{18}$$

### 3 Operational matrix for fractional order integration

For a vector  $\mathbf{B}(x)$ , we can approximate the operational matrices of fractional order integration as:

$${}_0I_x^\alpha \mathbf{B}(x) \simeq P^\alpha \mathbf{B}(x), \tag{19}$$

where  $P^\alpha$  is the  $(N+1) \times (N+1)$  Riemann-Liouville fractional operational matrix of integration for Boubaker polynomials. We compute  $P^\alpha$  as follows:

$${}_0I_x^\alpha \mathbf{B}(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} \mathbf{B}(\tau) d\tau. \tag{20}$$

By substituting  $\mathbf{B}(x) = ZX(x)$ , we get:

$$\begin{aligned} {}_0I_x^\alpha \mathbf{B}(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} ZX(\tau) d\tau \\ &= Z \left[ I^\alpha 1, I^\alpha x, \dots, I^\alpha x^N \right]^T \\ &= Z \left[ \frac{0!}{\Gamma(\alpha+1)} x^\alpha, \dots, \frac{N!}{\Gamma(\alpha+N+1)} x^{\alpha+N} \right]^T \\ &= ZD\bar{X}(x), \end{aligned} \tag{21}$$

where the matrix  $D_{(N+1) \times (N+1)}$  is given by

$$D = \begin{bmatrix} \frac{0!}{\Gamma(\alpha+1)} & 0 & \dots & 0 \\ 0 & \frac{1!}{\Gamma(\alpha+2)} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \frac{N!}{\Gamma(\alpha+N+1)} \end{bmatrix},$$

and

$$\bar{X}(x) = \begin{bmatrix} x^\alpha \\ x^{\alpha+1} \\ \vdots \\ x^{\alpha+N} \end{bmatrix}.$$

Now we approximate  $x^{k+\alpha}$  by  $N+1$  terms of the Boubaker basis

$$x^{\alpha+i} \simeq E_i^T \mathbf{B}(x), \tag{22}$$

where  $E_i = [E_{i,0}, E_{i,1}, \dots, E_{i,N}]$  and

$$\begin{aligned} E_{i,j} &= Q^{-1} \int_0^1 x^{\alpha+i} B_{i,j}(x) dx = \frac{N! \Gamma(i+j+\alpha+1)}{j! \Gamma(i+N+\alpha+2)}, \\ &i, j = 0, 1, \dots, N, \end{aligned} \tag{23}$$

where  $E$  is an  $(N+1) \times (N+1)$  matrix with  $E_i$  as its column. Therefore, we can write

$$\begin{aligned} I^\alpha \phi(x) &= ZD[E_0^T \mathbf{B}(x), E_1^T \mathbf{B}(x), \dots, E_m^T \mathbf{B}(x)]^T \\ &= ZDE^T \mathbf{B}(x). \end{aligned} \tag{24}$$

Finally, we obtain

$${}_0I_x^\alpha \phi(x) \simeq P^\alpha \mathbf{B}(x), \tag{25}$$

where

$$P^\alpha = ZDE, \tag{26}$$

is called the operational matrix of fractional integration Boubaker polynomials.

### 4 Operational matrix for multi-order FDE

In this section, we employ the Boubaker polynomials for solving the multi-order FED (1). First we apply  $I^\alpha$  on both sides of (1). It gives the following fractional order integral equation

$$\begin{aligned} y(x) &- \sum_{k=0}^{n-1} y^{(k)}(0^+) \frac{x^k}{k!} = \\ &\sum_{i=1}^k y_i (I^{\alpha-\beta_i} y(x)) - \sum_{j=0}^{n_i-1} \frac{y^{(j)}(0)}{\Gamma(\alpha-\beta_i+j+1)} x^{\alpha-\beta_i+k} \\ &+ y_{k+1} I^\alpha y(x) + I^\alpha g(x), \\ y^{(i)}(0) &= d_i, \quad i = 0, 1, \dots, n-1, \end{aligned} \tag{27}$$

where  $n_i - 1 < \beta_i \leq n_i$ ,  $n_i \in N$ . This implies that

$$y(x) = \sum_{i=1}^k y_i (I^{\alpha-\beta_i} y(x)) + y_{k+1} I^\alpha y(x) + h(x), \tag{28}$$

$$y^{(i)}(0) = d_i, \quad i = 0, 1, \dots, n-1, \tag{29}$$

where

$$h(x) = I_x^\alpha f(x) + \sum_{k=0}^{n-1} y^{(k)}(0^+) \frac{x^k}{k!} - \sum_{j=0}^{n_i-1} \frac{y^{(j)}(0)}{\Gamma(\alpha-\beta_i+j+1)} x^{\alpha-\beta_i+k}. \tag{30}$$

Now, we approximate  $y$  and  $g$  by Boubaker polynomials  $\mathbf{B}(x)$  as follows:

$$y(x) \simeq \sum_{i=0}^N c_j B_i(x) = C^T \mathbf{B}(x), \tag{31}$$

$$h(x) \simeq \sum_{i=0}^N g_j B_i(x) = G^T \mathbf{B}(x), \tag{32}$$

such that

$$C^T = [c_0, \dots, c_N]^T, \quad G^T = [g_0, \dots, g_N]^T, \quad (33)$$

where  $G^T$  and  $C^T$  are known and unknown vectors, respectively.

We also approximate the fractional order integrals by using (26) as follow:

$$I_x^\alpha y(x) \simeq C^T P^\alpha \mathbf{B}(x), \quad I_x^{\alpha-\beta_i} y(x) \simeq C^T P^{\alpha-\beta_i} \mathbf{B}(x). \quad (34)$$

By substituting (31)-(34) in (28), we obtain:

$$\left( C^T - C^T \sum_{i=1}^k y_i P^{\alpha-\beta_i} - y_{k+1} C^T P^\alpha + G^T \right) \mathbf{B}(x) = 0. \quad (35)$$

Finally, we get:

$$C^T - C^T \sum_{i=1}^k y_i I^{\alpha-\beta_i} - y_{k+1} C^T I^\alpha + G^T = 0. \quad (36)$$

Finally, by solving the above system of algebraic equations we find vector C can be obtained. Consequently  $y(x)$  can be approximated by (31).

## 5 Applications

Here, we use the presented numerical approach to solve several illustrative examples.

**Example 1.** We solve the following FDE [3, 12]

$$D^{0.5} y(x) + y(x) = \sqrt{x} + \frac{\sqrt{\pi}}{2}, \quad 0 < \alpha \leq 1, \quad (37)$$

$$y(0) = 0,$$

with the exact solution  $y(x) = \sqrt{x}$ .

In Fig. 1, we plotted the exact solution and the approximate solutions of  $y$  for  $N = 3$  and  $N = 5$ . Definitely, by increasing the value of  $N$ , the approximate value of  $y(x)$  will close to the exact values.

**Example 2.** Consider the following FDE [12, 16]

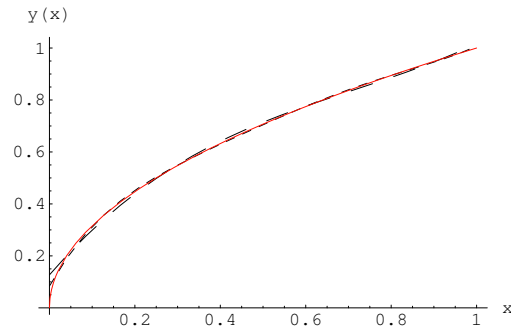
$$D^\alpha y(x) = -y(x) + x^2 + \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)}, \quad 0 < x < 1, \quad (38)$$

$$y(0) = 0,$$

the exact solution in this case:

$$y(x) = x^2. \quad (39)$$

We applied Boubaker polynomials approach to solve (38) with  $N = 3$ . In this case we obtain  $y(x) = x^2$



**Fig. 1.** The exact solution (Red line) and approximation solutions for  $N = 3$  (dotted line) and  $N = 5$  (dashed line).

**Example 3.** Consider the inhomogeneous Bagley-Torvik equation as a multi-order FED [2, 12, 20]

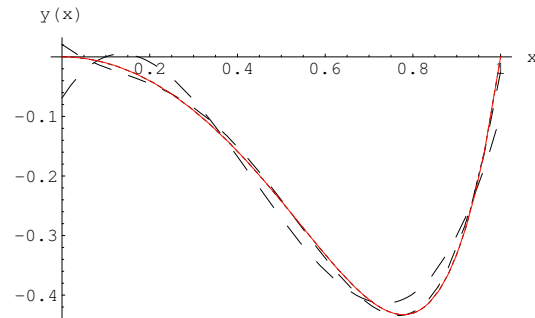
$$D^2 y(x) + D^{\frac{3}{2}} y(x) + y(x) = 1 + x, \quad 0 < x < 1, \quad (40)$$

$$y(0) = 1, \quad y'(0) = 1.$$

The exact solution of (40) is:

$$y(x) = 1 + x. \quad (41)$$

We solved this equation by Boubaker polynomials with  $N = 3$  and obtained the exact solution  $1 + x$ .



**Fig. 2.** The exact solution (red line), approximation solutions for  $N = 6$  (dotted line),  $N = 4$  (dashed line) and  $N = 3$  (long-dashed line).

**Example 4.** The last examined equation [3, 12] is

$$D^2 y(x) - 2Dy(x) + D^{0.5} y(x) = x^7 + \frac{2048}{429\sqrt{\pi}} x^{6.5} - 14x^6 + 42x^5 - x^2 - \frac{8}{3\sqrt{\pi}} x^{1.5} + 4x - 2, \quad (42)$$

$$y(0) = 0, \quad y'(0) = 0, \quad 0 < x < 1,$$

The exact solution  $y(x) = x^7 - x^2$ .

We applied this method for  $N = 3, 4, 6$ , and the result is plotted in Fig. 2. It can be seen that by increasing the value

of  $N$ , the approximate solution will converge to the exact values.

## 6 Conclusion

In this work we applied Boubaker polynomials for solving multi-order FDE. The Boubaker polynomials operational matrices of fractional integration was used. Illustrative examples were presented to show the applicability and validity of the approach.

*Mathematica* was used for computation in this paper.

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