Open Phys. 2016; 14:200–211 **DE GRUYTER** OPEN

Research Article **Open Access**

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On Negacyclic Codes Over the Ring $\mathbb{Z}_p + u\mathbb{Z}_p + \cdots + u^{k-1}\mathbb{Z}_p$

DOI 10.1515/phys-2016-0027
Received Nov 30, 2015; accepted Apr 19, 2016

Abstract: We investigate the structure of negacyclic codes over the chain ring $\mathbb{Z}_p[u]/\langle u^k\rangle$, which plays an important role in data transmission technologies. We study the set of generators for these codes. We also discuss the rank and hamming distance for these codes. We give some examples of negacyclic codes which are near to optimal.

Keywords: Cyclic codes, Negacyclic codes, Hamming distance.

PACS: 02.10.De, 02.10.Hh, 02.10.Hd

1 Introduction

In the middle last of the century, Shannon, Golay and Hamming discussed codes over finite field. Recently, codes over chain ring have been considered as the finite ring can be related to codes over finite fields via the Gray map. Coding theory, at its origin, is concerned with effective communication of information. Specifically, to detect and correct transmission errors techniques were developed. Coding theory was primarily studied for its application to telephone communication, elementary computing machines and computer to computer communication.

Negacyclic code played a very important role in the area of error correcting codes due to their rich algebraic structure. Negacyclic code has a lot of applications in consumer electronics, data transmission technologies, broadcast systems and computer applications as it has efficient encoding and decoding algorithms. In 1968, Berklamp investigated negacyclic codes over odd prime fields $GF(p)$ and designed a decoding algorithm that corrects up to $t \le \lfloor (p-1)/2 \rfloor$ in [1]. In [2], Wolfmann discussed some interesting results about negacyclic codes of odd length over $\mathbb{Z}_4$, and proposed questions about such codes when the length is even. In [3], Blackford classified all negacyclic codes of even length using a transform approach. This has motivated authors to study the negacyclic codes over the ring $\mathbb{Z}_p[u]/\langle u^k\rangle$. In particular, cyclic and negacyclic codes over finite rings have received much attention in series of paper [4–15].

Dinh and Lopez-Permouth in [16] have considered the ring $\mathbb{Z}_{2^m} (m \ge 2)$ and studied the negacyclic codes of length $2^t (t \ge 1)$ over that. They have provided complete list of negacyclic codes of length 2^t Over $\mathbb{Z}_{2m}$. Dinh in [17], determined the structure of negacyclic codes of length 2^s over the Galois ring $GR(2^a, m)$, as well as that of their duals, and studied the Hamming distance of negacyclic codes of length 2^s over the Galois ring $GR(2^a, m)$.

Let $R_{k,p} = \mathbb{Z}_p + u\mathbb{Z}_p + \cdots + u^{k-1}\mathbb{Z}_p$, where p is a prime number and $u^k = 0$. Note that the ring $R_{k,p}$ can also be viewed as the quotient ring $\mathbb{Z}_p[u]/\langle u^k\rangle$. Let C be a negacyclic code over the ring $R_{k,p} = \mathbb{Z}_p + u\mathbb{Z}_p + \cdots + u^{k-1}\mathbb{Z}_p$, $u^k = 0$. The line of arguments we have used to find a set of generators and a minimal spanning set of a code C are somewhat similar to those discussed in [4]. The idea to find a set of generators is as follows. We view the negacyclic code C as an ideal in $R_{k,p,p^s} = R_{k,p}/\langle x^{p^s} + 1\rangle$. Then we define the isomorphism map from $\mu : R_{k,p}[x]/\langle x^{p^s} - 1\rangle \longrightarrow R_{k,p}[x]/\langle x^{p^s} + 1\rangle$ and discuss that I is a cyclic code of length p^s over $R_{k,p}$ if and only if $\mu(I)$ is a negacyclic code of length p^s over $R_{k,p}$.

This paper is organized as follows. In second section, we present some basic background. In Section 3, we determine a set of generators for the negacyclic codes C over the ring $R_{k,p} = \mathbb{Z}_p + u\mathbb{Z}_p + \cdots + u^{k-1}\mathbb{Z}_p$, $u^k = 0$. We discuss a minimal spanning set for these codes and find the rank in section 4. In Section 5, we study the minimum distance of these codes. We discuss some examples of differ-

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ent lengths of these codes in Section 6. Section 7 concludes the paper.

2 Preliminaries

Let R be a ring. A linear code of length n over R is a R submodule of R^n. A linear code C of length n over R is λ-constacyclic if for some unit $\lambda \in R$, the code is invariant under the automorphism ν. That is $\nu(c_{n-1}, c_1, \dots, c_0) = (\lambda c_0, c_1, \dots, c_{n-1})$. We can consider a negacyclic codes $C(\lambda = -1)$ of length n over R as an ideal in $R[x]/\langle x^n + 1\rangle$ via the following correspondence

$$R^n \longrightarrow R[x]/\langle x^n+1\rangle, \quad (c_0, c_1, \dots, c_{n-1}) \mapsto c_0 + c_1x + \dots + c_{n-1}x^{n-1}.$$

A ring R is called local ring if R has a unique maximal right (left) ideal. A ring is called a chain ring if the set of all right (left) ideals of R is a chain under set-theoretic inclusion. If R is both a right and a left chain ring , we simply call R a chain ring.

For the class of finite commutative rings, We have the following equivalent conditions.

Proposition 2.1. *The following conditions are equivalent for a finite commutative ring R.*

(i) R is a local ring and the maximal ideal M of R is principal;
(ii) R is a local principal ideal ring;
(iii) R is a chain ring.

Let R be a finite commutative local ring with maximal ideal M. We define the residue field $\overline{R} = R/M$. Let $\mu : R[x] \to \overline{R}[x]$ denote the natural ring homomorphism that maps $r \mapsto r + M$ and the variable x to x.We define the degree of the polynomial $f(x) \in R[x]$ as the degree of the polynomial $\mu(f(x))$ in $\overline{R}[x]$, *i.e.*, $deg(f(x)) = deg(\mu(f(x))$. A polynomial $f(x) \in R[x]$ is called regular if it is not a zero divisor.

Proposition 2.2. *Let R be a finite commutative local ring. Let* $f(x) = a_0 + a_1x + \dots + a_nx^n$ *be in* $R[x]$*, then the followings are equivalent.*

(i) $\mu(f(x)) \neq 0$;
(ii) a_i *is a unit for some* i, $0 \le i \le n$;
(iii) $\langle a_0, a_1, \dots, a_n\rangle = R$.

Proof: $(i) \Rightarrow (ii)$ Let $\mu(f(x)) \neq 0 \Rightarrow \exists$ an a_i for some i, $0 \le i \le n$ such that $a_i \notin M$. $\Rightarrow a_i$ is a unit element (Since all element out side of the M are unit).
$(ii) \Rightarrow (iii)$ Let a_i is a unit for some i, $0 \le i \le n \Rightarrow \langle a_i\rangle = R$ $\Rightarrow \langle a_0, a_1, \dots, a_n\rangle = R$.
$(iii) \Rightarrow (i)$ Let $\langle a_0, a_1, \dots, a_n\rangle = R$. We claim that at least one of a_i is a unit for some i, $0 \le i \le n$. If not then all $a_i \in M \Rightarrow \langle a_0, a_1, \dots, a_n\rangle \subseteq M \subset R$. A contradiction. So $\exists$ an $a_i \notin M$ for some i, $0 \le i \le n \Rightarrow \mu(f(x)) \neq 0$.

The following version of the division algorithm holds true for polynomials over finite commutative local rings.

Proposition 2.3. *Let R be a finite commutative local ring. Let f(x) and g(x) be non zero polynomials in R[x]. If g(x) is regular, then there exist polynomials q(x) and r(x) in R[x] such that* $f(x) = g(x)q(x) + r(x)$ *and* $deg(r(x))\langle deg(g(x))$.

Let $R_{k,p} = \mathbb{Z}_p + u\mathbb{Z}_p + \dots + u^{k-1}\mathbb{Z}_p$, $u^k = 0$. It is easy to see that the ring $R_{k,p}$ is a finite chain ring with unique maximal ideal $\langle u\rangle$. Let $g(x)$ be a non zero polynomial in $\mathbb{Z}_p[x]$. It is also easy to see that the polynomial $g(x)+up_1(x)+u^2p_2(x)+\dots+u^{k-1}p_{k-1}(x) \in R_{k,p}[x]$ is regular. Throughout the paper, we repeatedly make use for the polynomial $g(x) + up_1(x) + u^2p_2(x) + \dots + u^{k-1}p_{k-1}(x) \in R_{k,p}[x]$. Note that $\deg(g(x) + up_1(x) + u^2p_2(x) + \dots + u^{k-1}p_{k-1}(x)) = \deg(g(x))$.

3 A generator for negacyclic codes of length p^s over the ring $R_{k,p}$

Let p be a odd prime number. Let $R_{k,p} = \mathbb{Z}_p + u\mathbb{Z}_p + \dots + u^{k-1}\mathbb{Z}_p$, $u^k = 0$.

Proposition 3.1. *Let* μ *be the map* $\mu : R_{k,p}[x]/\langle x^{p^s} - 1\rangle \longrightarrow R_{k,p}[x]/\langle x^{p^s} + 1\rangle$ *defined by* $\mu(f(x)) = f(-x)$ *. Then* μ *is a ring isomorphism.*

Proof: For the polynomials $f(x), g(x) \in R_{k,p}[x]$,

$$f(x) \equiv g(x) \bmod (x^n - 1);$$

if and only if there exists a polynomial $h(x) \in R_{k,p}[x]$, such that

$$f(x) - g(x) = h(x)(x^{p^s} - 1);$$

if and only if p^s is odd and

$$f(-x) - g(-x) = h(-x)((-x)^{p^s} - 1); \text{if and only if}$$
$$f(-x) - g(-x) = h(-x)((-1)^{p^s}x^{p^s} - 1) = -h(-x)(x^{p^s} + 1));$$

if and only if

$$f(-x) \equiv g(-x) \bmod (x^{p^s} + 1);$$

This implies that $f(x), g(x) \in R_{k,p}[x]/\langle x^{p^s}-1\rangle$, $\mu(f(x)) \equiv \mu(g(x)) \bmod (x^{p^s}+1)$ if and only if $f(x)=g(x)$. Hence, μ is well-defined and one-to-one. Since the ring $R_{k,p}[x]/\langle x^{p^s}-1\rangle$ and $R_{k,p}[x]/\langle x^{p^s}+1\rangle$ are finite and of same order, μ is an onto map. It is easy to see that μ is a ring homomorphism. So μ is a ring isomorphism.

Remark 3.1. *We restrict the isomorphism μ to an isomorphism $\mu : \mathbb{Z}_p[x]/\langle x^{p^s}-1\rangle \longrightarrow \mathbb{Z}_p[x]/\langle x^{p^s}+1\rangle$*

Throughout this paper we use the isomorphism μ and restriction of μ defined in Proposition [3.1] and Remark [3.1].

Corollary 3.2. *If I is a subset of $R_{k,p}[x]/\langle x^{p^s}-1\rangle$ and $\mu(I)$ is a subset of $R_{k,p}[x]/\langle x^{p^s}+1\rangle$, then I is an ideal of $R_{k,p}[x]/\langle x^{p^s}-1\rangle$ if and only if $\mu(I)$ is an ideal of $R_{k,p}[x]/\langle x^{p^s}+1\rangle$. Equivalently I is a cyclic code of length p^s over $R_{k,p}$ if and only if $\mu(I)$ is a negacyclic code of length p^s over $R_{k,p}$.*

Proof: The proof is obvious since the map μ is a ring isomorphism.

Let p be a odd prime number. Let $R_{k,p}=\mathbb{Z}_p+u\mathbb{Z}_p+\cdots+u^{k-1}\mathbb{Z}_p$, $u^k=0$. and $R_{k,p,p^s}=R_{k,p}[x]/\langle x^{p^s}+1\rangle$. Let C_k be a negacyclic code of length p^s over $R_{k,p}$. From Corollary [3.1], we know that for the negacyclic code C_k there exist a cyclic code, say A over the same ring and of same length. We know the structure of a cyclic code over the ring $R_{k,p}$ from [4]. Since p^s is not relatively prime to p, then we know that cyclic code of length p^s over $R_{k,p}$ is $A=\langle g(x)+up_1(x)+u^2p_2(x)+\cdots+u^{k-1}p_{k-1}(x), ua_1(x)+u^2q_1(x)+\cdots+u^{k-1}q_{k-2}(x), u^2a_2(x)+u^3l_1(x)+\cdots+u^{k-1}l_{k-3}(x), \cdots, u^{k-2}a_{k-2}(x)+u^{k-1}t_1(x), u^{k-1}a_{k-1}(x)\rangle$ with $a_{k-1}(x)|a_{k-2}(x)|\cdots|a_2(x)|a_1(x)|g(x)|(x^{p^s}-1) \bmod p$, $a_{k-2}(x)|p_1(x)\left(\frac{x^{p^s}-1}{g(x)}\right), \cdots, a_{k-1}|t_1(x)\left(\frac{x^{p^s}-1}{a_{k-2}(x)}\right), \cdots, a_{k-1}|p_{k-1}\times\left(\frac{x^{p^s}-1}{g(x)}\right)\cdots\left(\frac{x^{p^s}-1}{a_{k-2}(x)}\right)$. Moreover, $\deg p_{k-1}(x)\langle \deg a_{k-1}(x), \cdots,$ $\deg t_1(x)\langle a_{k-1}(x), \cdots,$ and $\deg p_1(x)\langle \deg a_{k-2}(x)$. Now the polynomials $g(x)+up_1(x)+u^2p_2(x)+\cdots+u^{k-1}p_{k-1}(x), ua_1(x)+u^2q_1(x)+\cdots+u^{k-1}q_{k-2}(x), \cdots, u^{k-2}a_{k-2}(x)+u^{k-1}t_1(x)$, $u^{k-1}a_{k-1}(x)\in R_{k,p}[x]/\langle x^{p^s}-1\rangle$ Therefore from the definition of μ from Proposition [3.1], we get $\mu(g(x)+up_1(x)+u^2p_2(x)+\cdots+u^{k-1}p_{k-1}(x)) = g(-x)+up_1(-x)+u^2p_2(-x)+\cdots+u^{k-1}p_{k-1}(-x) = g_1(x)+up_{11}(x)+u^2p_{12}(x)+\cdots+u^{k-1}p_{1(k-1)}(x)$, $\mu(ua_1(x)+u^2q_1(x)+\cdots+u^{k-1}q_{k-2}(x)) = ua_1(-x)+u^2q_1(-x)+\cdots+u^{k-1}q_{k-2}(-x) = ua_{11}(x)+u^2q_{11}(x)+\cdots+u^{k-1}q_{1(k-2)}(x), \cdots$, $\mu(u^{k-2}a_{k-2}(x)+u^{k-1}t_1(x)) = u^{k-2}a_{k-2}(-x)+u^{k-1}t_1(-x) = u^{k-2}a_{1(k-2)}(x)+u^{k-1}t_{11}(x)$, $\mu(u^{k-1}a_{k-1}(x)) = u^{k-1}a_{k-1}(-x) = u^{k-1}a_{1(k-1)}(x)$ and $\mu(x^{p^s}-1) = -(x^{p^s}+1)$, where $g(-x) = g_1(x)$, $p_1(-x)=p_{11}(x)$, $p_2(-x)=p_{12}(x), \cdots$, $a_{k-2}(-x)=a_{1(k-2)}(x)$, $t_1(-x)=t_{11}(x)$, $a_{k-1}(-x)=a_{1(k-1)}(x)$. Again $\mu(A)=C_k$. Therefore the code C can be written as $C_k=\langle g_1(x)+up_{11}(x)+u^2p_{12}(x)+\cdots+u^{k-1}p_{1(k-1)}(x), ua_{11}(x)+u^2q_{11}(x)+\cdots+u^{k-1}q_{1(k-2)}(x), u^2a_{12}(x)+u^3l_{11}(x)+\cdots+u^{k-1}l_{1(k-3)}(x), \ldots, u^{k-2}a_{1(k-2)}(x)+u^{k-1}t_{11}(x), u^{k-1}a_{1(k-1)}(x)\rangle$ with $a_{1(k-1)}(x)$ $|a_{1(k-2)}(x)|\cdots|a_{12}(x)|a_{11}(x)|g_1(x)|(x^{p^s}+1) \bmod p$, $a_{1(k-2)}(x)|p_{11}(x)\left(\frac{x^{p^s}+1}{g(x)}\right), \cdots, a_{1(k-1)}|t_{11}(x)\left(\frac{x^{p^s}+1}{a_{1(k-2)}(x)}\right), \cdots, a_{1(k-1)}|p_{1(k-1)}\times\left(\frac{x^{p^s}+1}{g_1(x)}\right)\cdots\left(\frac{x^{p^s}+1}{a_{1(k-2)}(x)}\right)$.

Lemma 3.3. *Let C_2 be a negacyclic code over $R_2=\mathbb{Z}_p+u\mathbb{Z}_p, u^2=0$. If $C_2=\langle g_1(x)+up_{11}(x), ua_{11}(x)\rangle$, and $g_(x)=a_{11}(x)$ with* $\deg g(x)=r$, *then*

$$C_2=\langle g_1(x)+up_{11}(x)\rangle \text{ and } (g_1(x)+up_{11}(x))|(x^{p^s}+1) \text{ in } R_2.$$

Proof: We have $u(g_1(x)+up_{11}(x)) = ug_1(x)$ and $g(x)=a_{11}(x)$. It is clear that $C_2 \subset \langle g_1(x)+up_{11}(x)\rangle$. Hence, $C_2=\langle g_1(x)+up_{11}(x)\rangle$. By the division algorithm, we have $x^{p^s}-1 = (g_1(x)+up_{11}(x))q(x)+r(x)$, where $r(x)=0$ or $\deg r(x)\langle r$. This implies that $r(x)=(x^{p^s}-1)-(g_1(x)+up_{11}(x))q(x)$. This gives, $r(x)\in C_2$. Thus, we have $r(x)=0$ and hence $(g_1(x)+up_{11}(x))|(x^{p^s}-1)$ in R_2.

Lemma 3.4. *Let C_3 be a negacyclic code over $R_3=\mathbb{Z}_p+u\mathbb{Z}_p+u^2\mathbb{Z}_p, u^3=0$. If $C_3=\langle g_1+up_{11}(x)+u^2p_{12}(x), ua_{11}(x)+u^2q_{11}(x), u^2a_{12}(x)\rangle$, and $a_{12}(x)=g_1(x)$, then $C_3=\langle g_1(x)+up_{11}(x)+u^2p_{12}(x)\rangle$, $(g_1(x)+up_{11}(x)|(x^{p^s}+1)$ in R_2 and $(g_1+up_{11}(x)+u^2p_{12}(x))|(x^{p^s}+1)$ in R_3.*

Theorem 3.5. *Let C_k be a negacyclic code over $R_{k,p}=\mathbb{Z}_p+u\mathbb{Z}_p+u^2\mathbb{Z}_p+\cdots++u^{k-1}\mathbb{Z}_p, u^k=0$. Since p^s is not relatively prime to p, then,*

(1) $C_k=\langle g_1(x)+up_{11}(x)+u^2p_{12}(x)+\cdots+u^{k-1}p_{1(k-1)}(x)\rangle$ where $g_1(x)$ and $p_{1i}(x)$ are polynomials in $\mathbb{Z}_p[x]$ with $g_1(x)|(x^{p^s}+1) \bmod p$, $(g_1(x)+up_{11}(x)+u^2p_{12}(x)+\cdots+u^{k-1}p_{1(k-1)}(x))|(x^{p^s}+1)$ in R_i and $\deg p_{1i}\le \deg p_{1(i-1)}$ for all $1\le i\le k$.

(2) $C_k = \langle g_1(x)+up_{11}(x)+u^2p_{12}(x)+\cdots+u^{k-1}p_{1(k-1)}(x), ua_{11}(x)+u^2q_{11}(x)+\cdots+u^{k-1}q_{1(k-2)}(x)$, $u^2a_{12}(x)+u^3l_{11}(x)+\cdots+u^{k-1}l_{1(k-3)}(x), \ldots, u^{k-2}a_{1(k-2)}(x)+u^{k-1}t_{11}(x), u^{k-1}a_{1(k-1)}(x)\rangle$ with $a_{1(k-1)}(x)|a_{1(k-2)}(x)|\cdots|a_{12}(x)|a_{11}(x)|g_1(x)|(x^{p^s}+1) \bmod p$, $a_{1(k-2)}(x)|p_{11}(x)\left(\frac{x^{p^s}+1}{g(x)}\right)$, $\cdots, a_{1(k-1)}|t_{11}(x)\left(\frac{x^{p^s}+1}{a_{1(k-2)}(x)}\right), \cdots, a_{1(k-1)}|p_{1(k-1)}\times\left(\frac{x^{p^s}+1}{g_1(x)}\right)\cdots\left(\frac{x^{p^s}+1}{a_{1(k-2)}(x)}\right)$. Moreover, $\deg p_{1(k-1)}(x) < \deg a_{1(k-1)}(x), \ldots, \deg t_{11}(x) < a_{1(k-1)}(x), \ldots$, and $\deg p_{11}(x) < \deg a_{1(k-2)}(x)$.

4 Ranks and minimal spanning sets

If negacyclic code C over the finite chain ring R are free R-submodules, then the basis of C over R is called the minimal generating set of C, and the number of element include in the minimal generating set is called the rank of the code C, denoted the $rank(C)$.

Theorem 4.1. *Let $C = \langle g_0, ag_1, \cdots, a^{t-1}g_{t-1}\rangle$ be any negacyclic code over the finite chain ring R, where $g_{t-1} \mid g_{t-2} \mid \cdots \mid g_1 \mid g_0 \mid (x^{p^s}+1)$, $deg(g_i) = r_i$, $i = 0, 1, \cdots t-1$, $r_{t-1} < r_{t-2} < \cdots < r_0$, then C is a free R-submodule with $rank(C) = p^s - r_{t-1}$ and its minimal generating set is $\beta = \{g_0, xg_0, \ldots, x^{n-r_0-1}g_0; ag_1, xag_1, \cdots, x^{r_0-r-1}ag_1; \ldots ; a^{t-2}g_{t-2}, xa^{t-2}g_{t-2}, \cdots, x^{r_{t-3}-r_{t-2}-1} a^{t-2}g_{t-2}; a^{t-1}g_{t-1}, xa^{t-1}g_{t-1}, \cdots, x^{r_{t-2}-r_{t-1}-1}a^{t-1}g_{t-1}\}$*

Theorem 4.2. *Let p^s is not relatively prime to p. Let C_2 be a negacyclic code of length p^s over $R_2 = \mathbb{Z}_p + u\mathbb{Z}_p$, $u^2 = 0$.*

(1) If $C_2 = \langle g_1(x) + up_{11}(x)\rangle$ with deg $g_1(x) = r$ and $(g_1(x)+up_{11}(x)|(x^{p^s}-1)$, then C_2 is a free module with rank $n-r$ and a basis $B_1 = \{g_1(x)+up_{11}(x), x(g_1(x)+up_{11}(x)), \cdots, x^{n-r-1}(g_1(x)+up_{11}(x))\}$, and $|C_2| = p^{2p^s-2r}$.

(2) If $C_2 = \langle g_1(x) + up_{11}(x), ua_{11}(x)\rangle$ with deg $g_1(x) = r$ and deg $a_{11}(x) = t$, then C_2 has rank $n-t$ and a minimal spanning set $B_2 = \{g_1(x) + up_{11}(x), x(g_1(x) + up_{11}(x)), \cdots, x^{n-r-1}(g_1(x) + up_{11}(x)), ua_{11}(x), xua_{11}(x), \ldots, x^{r-t-1}ua_{11}(x)\}$, and $|C_2| = p^{2p^s-r-t}$.

Proof: (1) Suppose $x^{p^s} + 1 = (g_1(x) + up_{11}(x))(h_1(x) + uh_{11}(x))$ over R_2. Let $c(x) \in C_2 = \langle g_1(x) + up_{11}(x)\rangle$, then $c(x) = (g_1(x) + up_{11}(x))f(x)$ for some polynomial $f(x)$. If deg $f(x) \le p^s - r - 1$, then $c(x)$ can be written as linear combinations of elements of B_1. Otherwise by the division algorithm there exist polynomials $q(x)$ and $r(x)$ such that $f(x) = \left(\frac{x^{p^s}+1}{g_1(x)+up_{11}(x)}\right) q(x) + r(x)$ where $r(x) = 0$ or deg $r(x) \le p^s - r - 1$. This gives,

$$\begin{aligned}(g_1(x) + up_{11}(x))f(x) &= (g_1(x) + up_{11}(x))\\ &\times \left(\left(\frac{x^{p^s}+1}{g_1(x)+up_{11}(x)}\right) q(x) + r(x)\right)\\ &= (g_1(x) + up_{11}(x))r(x).\end{aligned}$$

Since deg $r(x) \le p^s-r-1$, this shows that B_1 spans C_2. Now we only need to show that B_1 is linearly independent. Let $g_1(x) = g_{10} + g_{11}x + \cdots + g_{1r}x^r$ and $p_{11}(x) = p_{110} + p_{111}x + \cdots + p_{11l}x^l$, $g_0 \in \mathbb{Z}_p^{\times}, g_{1i}, p_{11(i-1)} \in \mathbb{Z}_p, i \ge 1$. Suppose $(g_1(x)+up_{11}(x))c_0+x(g_1(x)+up_{11}(x))c_1+\cdots+x^{p^s-r-1}(g_1(x)+up_{11}(x))c_{p^s-r-1} = 0$. By comparing the coefficients in the above equation, we get

$$(g_{10} + up_{110})c_0 = 0. \text{ (constant coefficient)}$$

Since $(g_{10} + up_{110})$ is unit, we get $c_0 = 0$. Thus,

$$\begin{aligned}&x(g_1(x) + up_{11}(x))c_1 + \cdots\\ &\quad + x^{p^s-r-1}(g_1(x) + up_{11}(x))c_{p^s-r-1} = 0.\end{aligned}$$

Again comparing the coefficients, we get

$$(g_{10} + up_{110})c_1 = 0. \text{ (coefficient of } x).$$

As above, this gives $c_1 = 0$. Continuing in this way we get that $c_i = 0$ for all $i = 0, 1. \ldots, n-r-1$. Therefore, the set B_1 is linearly independent and hence a basis for C_2.
(2) If $C_2 = \langle g_1(x) + up_{11}(x), ua_{11}(x)\rangle$ with deg $g_1(x) = r$ and deg $a_{11}(x) = t$. The lowest degree polynomial in C_2 is $ua_{11}(x)$. It is suffices to show that B_2 spans $B = \{g_1(x) + up_{11}(x), x(g_1(x) + up_{11}(x)), \cdots, x^{p^s-r-1}(g_1(x) + up_{11}(x)), ua_{11}(x), xua_{11}(x), \cdots, x^{p^s-t-1}ua_{11}(x)\}$. We first show that $ux^{r-t}a_{11}(x) \in$ span(B_2). Let the leading coefficients of $x^{r-t}a_{11}(x)$ be a_0 and of $g_1(x) + up_{11}(x)$ be g_{10}. There exists a constant $c_0 \in \mathbb{Z}_p$ such that $a_{10} = c_0g_{10}$. Then we have

$$ux^{r-t}a_{11}(x) = uc_0(g_1(x) + up_{11}(x)) + um(x),$$

where $um(x)$ is a polynomial in C_2 of degree less than r. Since $C_2 =< g_1(x) + up_{11}(x), ua_{11}(x) >$, any polynomial in C_2 must have degree greater or equal to deg $a_{11}(x) = t$. Hence, $t \le$ deg $m(x) < r$ and

$$um(x) = \alpha_0ua_{11} + \alpha_1xua_{11} + \cdots + \alpha_{r-t-1}x^{r-t-1}ua_{11}.$$

Thus, $ux^{r-t}a_{11} \in$ span(B_2). Inductively, we can show that $ux^{r-t+1}a(x), \ldots,$
$ux^{p^s-t-1}a(x) \in$ span(B_2). Hence, B_2 is a generating set. As in (1), by comparing the coefficients we can see that B_2 is linearly independent. Therefore, B_2 is a minimal spanning set and $|C_2| = p^{2p^s-r-t}$.
Following the same process as in the above theorem, we can find the rank and the minimal spanning set of any cyclic code over the ring $R_{k,p}$, $k \ge 1$.

Theorem 4.3. *Let p^s is not relatively prime to p. Let C_k be a negacyclic code of length p^s over $R_{k,p} = \mathbb{Z}_p + u\mathbb{Z}_p + \cdots + u^{k-1}\mathbb{Z}_p$, $u^k = 0$. We assume the constraints on the generator polynomials of C_k as in Theorem 3.5.*

(1) If $C_k = \langle g_1(x)+up_{11}(x)+u^2p_{12}(x)+\cdots+u^{k-1}p_{1(k-1)}(x)\rangle$ with deg $g_1(x) = r$, then C_k is a free module with rank $p^s - r$ and a basis $B_1 = \{g_1(x) + up_{11}(x) + u^2p_{12}(x) + \cdots + u^{k-1}p_{1(k-1)}(x), x(g_1(x) + up_{11}(x) + u^2p_{12}(x) + \cdots + u^{k-1}p_{1(k-1)}(x)), \cdots, x^{n-r-1}(g_1(x) + up_{11}(x) + u^2p_{12}(x) + \cdots + u^{k-1}p_{1(k-1)}(x))\}$.

Table 1: Non free module negacyclic codes of length 5 over $\mathbb{Z}_5 + u\mathbb{Z}_5 + u^2\mathbb{Z}_5, u^3 = 0$.

Non-zero generator polynomials	$d(c)$	Ranks
$< u, u^2 >, < g^i, u, u^2 >, 1 \le i \le 4.$	1	5
$< g^2 + uc_0 + u^2c_1, ug + u^2c_2, u^2g >, c_0, c_1, c_2 \in \mathbb{Z}_5$	2	4
$< g^2 + uc_0, ug, u^2 >, c_0 \in \mathbb{Z}_5$	1	5
$< g^3 + u(c_0 + c_1) + u^2(c_2 + c_3), ug^2$ $+u^2(c_4 + c_5), u^2g^2 >, c_i \in \mathbb{Z}_5, 0 \le i \le 5$	3	3
$< g^3 + u(c_0 + c_1) + u^2c_2, ug^2 + u^2c_3), u^2g >, c_i \in \mathbb{Z}_5, 0 \le i \le 3$	2	4
$< g^3 + u(c_0 + c_1), ug^2, u^2 >, c_0, c_1 \in \mathbb{Z}_5.$	1	5
$< g^3 + uc_0 + u^2c_1, ug + u^2c_2, u^2g >, c_0, c_1, c_2 \in \mathbb{Z}_5.$	2	4
$< g^3 + uc_0, ug, u^2 >, c_0, \in \mathbb{Z}_5.$	1	5
$< g^4 + uc_0g^2 + u^2(c_1 + c_2x + c_3x^2), ug^3 + u^2(c_4 + c_5x)g, u^2g^3 > c_i \in \mathbb{Z}_5, 0 \le i \le 5.$	4	2
$< g^4 + uc_0g^2 + u^2(c_1 + c_2x), ug^3 + u^2(c_3 + c_4x), u^2g^2 >, c_i \in \mathbb{Z}_5, 0 \le i \le 4.$	3	3
$< g^4 + uc_0g^2 + u^2c_1, ug^3 + u^2c_2, u^2g >, c_0, c_1, c_2 \in \mathbb{Z}_5.$	2	4
$< g^4 + uc_0g^2, ug^3, u^2 >, c_0 \in \mathbb{Z}_5.$	1	5
$< g^4 + uc_0g + u^2(c_1 + c_2x), ug^2 + u^2(c_3 + c_4x), u^2g^2 >, c_i \in \mathbb{Z}_5, 0 \le i \le 4.$	3	3
$< g^4 + uc_0g + u^2c_1, ug^2 + u^2c_2, u^2g >, c_0, c_1, c_2 \in \mathbb{Z}_5.$	2	4
$< g^4 + uc_0g, ug^2, u^2 >, c_0 \in \mathbb{Z}_5.$	1	5
$< g^4 + uc_0g + u^2c_1, ug + u^2c_2, u^2g >, c_0, c_1, c_2 \in \mathbb{Z}_5.$	2	4
$< g^4 + uc_0g, ug, u^2 >, c_0 \in \mathbb{Z}_5.$	1	5
$< ug^4 + u^2c_0g^3, u^2g^4 >, c_0 \in \mathbb{Z}_5.$	5	1
$< ug^4 + u^2c_0g^2, u^2g^3 >, c_0 \in \mathbb{Z}_5.$	4	2
$< ug^4 + u^2c_0g, u^2g^2 >, c_0 \in \mathbb{Z}_5.$	3	3

(2) *If* $C_k = \langle g_1(x) + up_{11}(x) + u^2p_{12}(x) + \cdots + u^{k-1}p_{1(k-1)}(x), ua_{11}(x) + u^2q_{11}(x) + \cdots + u^{k-1}q_{1(k-2)}(x), u^2a_{12}(x) + u^3l_{11}(x) + \cdots + u^{k-1}l_{1(k-3)}(x), \cdots, u^{k-2}a_{1(k-2)}(x) + u^{k-1}t_{11}(x), u^{k-1}a_{1(k-1)}(x)\rangle$ *with deg* $g_1(x) = r_1$, *deg* $a_{11}(x) = r_2$, *deg* $a_{12}(x) = r_3, \ldots,$ *deg* $a_{1(k-1)}(x) = r_k$, *then* C_k *has rank* $p^s - r_k$ *and a minimal spanning set* $B_2 = \{g_1(x) + up_{11}(x) + u^2p_{12}(x) + \cdots + u^{k-1}p_{1(k-1)}(x), x(g_1(x) + up_{11}(x) + u^2p_{12}(x) + \cdots + u^{k-1}p_{1(k-1)}(x)), \cdots, x^{p^s-r_1-1}(g_1(x) + up_{11}(x) + u^2p_{12}(x) + \cdots + u^{k-1}p_{1(k-1)}(x)), ua_{11}(x) + u^2q_{11}(x) + \cdots + u^{k-1}q_{1(k-2)}(x), x(ua_{11}(x) + u^2q_{11}(x) + \cdots + u^{k-1}q_{1(k-2)}(x)), \cdots, x^{r_1-r_2-1}(ua_{11}(x) + u^2q_{11}(x) + \cdots + u^{k-1}q_{1(k-2)}(x)), u^2a_{12}(x) + u^3l_{11}(x) + \cdots + u^{k-1}l_{1(k-3)}(x), x(u^2a_{12}(x) + u^3l_{11}(x) + \cdots + u^{k-1}l_{1(k-3)}(x)), \cdots, x^{r_2-r_3-1}(u^2a_{12}(x) + u^3l_{11}(x) + \cdots + u^{k-1}l_{1(k-3)}(x)), \cdots, u^{k-1}a_{1(k-1)}(x), xu^{k-1}a_{1(k-1)}(x), \cdots, x^{r_{k-1}-r_k-1}u^{k-1}a_{1(k-1)}(x)\}$.

5 Minimum Distance

Since p^s is not relatively prime to p. Let $C_2 = \langle g_1(x) + up_{11}(x), ua_{11}(x)\rangle$ be a negacyclic code of length p^s over $R_2 = \mathbb{Z}_p + u\mathbb{Z}_p, u^2 = 0$. We define $C_{2,u} = \{k(x) \in R_{2,n} : uk(x) \in C_2\}$. It is easy to see that $C_{2,u}$ is a negacyclic code over $\mathbb{Z}_p$. Let C_k be a negacyclic code of length p^s over $R_{k,p} = \mathbb{Z}_p + u\mathbb{Z}_p + \cdots + u^{k-1}\mathbb{Z}_p, u^k = 0$. We define $C_{k,u^{k-1}} = \{k(x) \in R_{k,n} : u^{k-1}k(x) \in C_k\}$. Again it is easy to see that $C_{k,u^{k-1}}$ is a negacyclic code over $\mathbb{Z}_p$.

Theorem 5.1. *Let* p^s *is not relatively prime to* p. *If* $C_k = \langle g_1(x) + up_{11}(x) + u^2p_{12}(x) + \cdots + u^{k-1}p_{1(k-1)}(x), ua_{11}(x) + u^2q_{11}(x) + \cdots + u^{k-1}q_{1(k-2)}(x), u^2a_{12}(x) + u^3l_{11}(x) + \cdots + u^{k-1}l_{1(k-3)}(x), \ldots, u^{k-2}a_{1(k-2)}(x) + u^{k-1}t_{11}(x), u^{k-1}a_{1(k-1)}(x)\rangle$ *is a negacyclic code of length* p^s *over* $R_{k,p} = \mathbb{Z}_p + u\mathbb{Z}_p + \cdots + u^{k-1}\mathbb{Z}_p, u^k = 0$. *Then* $C_{k,u^{k-1}} =< a_{k-1}(x) >$ *and* $w_H(C_k) = w_H(C_{k,u^{k-1}})$.

Proof: We have $u^{k-1}a_{1(k-1)}(x) \in C_k$, thus $\langle a_{1(k-1)}(x)\rangle \subseteq C_{k,u^{k-1}}$. If $b(x) \in C_{k,u^{k-1}}$, then $u^{k-1}b(x) \in C_k$ and hence there exist polynomials $b_1(x), \cdots, b_k(x) \in \mathbb{Z}_p[X]$ such that $u^{k-1}b(x) = b_1(x)u^{k-1}g_1(x) + b_2(x)u^{k-1}a_{11}(x) + b_2(x)u^{k-1}a_{12}(x) + \cdots + b_k(x)u^{k-1}a_{1(k-1)}(x)$. Since $a_{1(k-1)}(x) | a_{1(k-2)}(x)| \ldots |a_{12}(x)|a_{11}(x)|g_1(x)$, we have $u^{k-1}b(x) = m(x)u^{k-1}a_{1(k-1)}(x)$ for some polynomial $m(x) \in \mathbb{Z}_p[x]$. So, $C_{k,u^{k-1}} \in \langle a_{1(k-1)}(x)\rangle$, and hence $C_{k,u^{k-1}} = \langle a_{1(k-1)}(x)\rangle$. Let $m(x) = m_0(x) + um_1(x) + \cdots + u^{k-1}m_{k-1}(x) \in C_k$, where $m_0(x), m_1(x), \ldots, m_{k-1}(x) \in \mathbb{Z}_p[x]$. We have $u^{k-1}m(x) =$

Table 2: Non free module negacyclic codes of length 5 over $\mathbb{Z}_5 + u\mathbb{Z}_5 + u^2\mathbb{Z}_5, u^3 = 0$.

$< ug^4 + u^2c_0, u^2g >, c_0 \in \mathbb{Z}_5$.	2	4
$< ug^4, u^2 >$.	1	5
$< ug^3 + u^2(c_0 + c_1x)g, u^2g^3 >, c_0, c_1 \in \mathbb{Z}_5$.	4	2
$< ug^3 + u^2(c_0 + c_1x), u^2g^2 >, c_0, c_1 \in \mathbb{Z}_5$.	3	3
$< ug^3 + u^2c_0, u^2g >, c_0 \in \mathbb{Z}_5$.	2	4
$< ug^3, u^2 >$.	1	5
$< ug^2 + u^2(c_0 + c_1x), u^2g^2 >, c_0, c_1 \in \mathbb{Z}_5$.	3	3
$< ug^2 + u^2c_0, u^2g >, c_0 \in \mathbb{Z}_5$.	2	4
$< ug^2, u^2 >$.	1	5
$< ug + u^2c_0, u^2g >, c_0 \in \mathbb{Z}_5$.	2	4
$< ug, u^2 >$.	1	5

Table 3: Non-zero free module negacyclic codes of length 5 over $\mathbb{Z}_5 + u\mathbb{Z}_5 + u^2\mathbb{Z}_5, u^3 = 0$.

Non-zero generator polynomials	$d(c)$	Ranks
$< 1 >$	1	5
$< g + uc_0 >, c_0 \in \mathbb{Z}_5$.	2	4
$< g^2 + u(c_0 + c_1x) + u^2c_2 >, c_0, c_1, c_2 \in \mathbb{Z}_5$.	3	3
$< g^3 + u(c_0 + c_1x - (c_0 - c_1)x^2) + u^2(c_2 + c_3x) >$ $, c_0, c_1, c_2, c_3 \in \mathbb{Z}_5$.	4	2
$< g^4 + uc_0(1 - 2x + 3x^2 - 4x^3) + u^2c_1(1 + 2x + x^2) >, c_0, c_1 \in \mathbb{Z}_5$.	5	1

$u^{k-1}m_0(x)$, $w_H(u^{k-1}m(x)) \le w_H(m(x))$ and $u^{k-1}C_k$ is subcode of C_k with $w_H(u^{k-1}C_k) \le w_H(C_k)$. Therefore, it is sufficient to focus on the subcode $u^{k-1}C_k$ in order to prove the theorem. Since $u^{k-1}C_k = \langle u^{k-1}a_{1(k-1)}(x)\rangle$, we get $w_H(C_k) = w_H(C_{k,u^{k-1}})$.

Definition 5.2. *Let $m = b_{s-1}p^{s-1} + b_{s-2}p^{s-2} + \cdots + b_1p + b_0$, $b_i \in \mathbb{Z}_p, 0 \le i \le s-1$, be the p-adic expansion of m.*

(1) *If $b_{s-i} \neq 0$ for all $1 \le i \le q, q < s$, and $b_{s-i} = 0$ for all $i, q+1 \le i \le s$, then m is said to have a p-adic length q zero expansion.*
(2) *If $b_{s-i} \neq 0$ for all $1 \le i \le q, q < s$, $b_{s-q-1} = 0$ and $b_{s-i} \neq 0$ for some $i, q+2 \le i \le s$, then m is said to have p-adic length q non-zero expansion.*
(3) *If $b_{s-i} \neq 0$ for $1 \le i \le s$, then m is said to have a p-adic length s expansion or p-adic full expansion.*

Lemma 5.3. *Let C be a negacyclic code over $R_{k,p}$ of length p^s where s is a positive integer. Let $C = \langle a(x)\rangle$ where $a(x) = (x^{p^{s-1}} - 1)^b h(x)$, $1 \le b < p$. If $h(x)$ generates a negacyclic code of length p^{s-1} and minimum distance d then the minimum distance $d(C)$ of C is $(b+1)d$.*

Proof: For $c \in C$, we have $c = (x^{p^{s-1}} - 1)^b h(x)m(x)$ for some $m(x) \in \frac{R_{k,p}[x]}{(x^{p^s}-1)}$. Since $h(x)$ generates a negacyclic code of length p^{s-1}, we have $w(c) = w((x^{p^{s-1}} - 1)^b h(x)m(x)) = w(x^{p^{s-1}b}h(x)m(x)) + w({}^bC_1x^{p^{s-1}(b-1)}h(x)m(x)) + \cdots + w({}^bC_{b-1}x^{p^{s-1}}h(x)m(x)) + w(h(x)m(x))$. Thus, $d(C) = (b+1)d$.

Theorem 5.4. *Let C_k be a negacyclic code over $R_{k,p}$ of length p^s where s is a positive integer. Then, $C_k = \langle g_1(x) + up_{11}(x) + u^2p_{12}(x) + \cdots + u^{k-1}p_{1(k-1)}(x), ua_{11}(x) + u^2q_{11}(x) + \cdots + u^{k-1}q_{1(k-2)}(x), u^2a_{12}(x) + u^3l_{11}(x) + \cdots + u^{k-1}l_{1(k-3)}(x), \ldots, u^{k-2}a_{1(k-2)}(x) + u^{k-1}t_{11}(x), u^{k-1}a_{1(k-1)}(x)\rangle$ where $g_1(x) = (x+1)^{t_1}, a_{11}(x) = (x+1)^{t_2}, \cdots, a_{1(k-1)}(x) = (x+1)^{t_k}$. for some $t_1 > t_2 > \cdots > t_k > 0$.*

(1) *If $t_k \le p^{s-1}$, then $d(C) = 2$.*
(2) *If $t_k > p^{s-1}$, let $t_k = b_{s-1}p^{s-1} + b_{s-2}p^{s-2} + \cdots + b_1p + b_0$ be the p-adic expansion of t_k and $a_{k-1}(-x) = (x+1)^{t_k} = (x^{p^{s-1}} - 1)^{b_{s-1}}(x^{p^{s-2}} - 1)^{b_{s-2}} \cdots (x^{p^1} - 1)^{b_1}(x^{p^0} - 1)^{b_0}$.*
    (a) *If t_k has a p-adic length q zero expansion or full expansion $(s = q)$. Then, $d(C_k) = (b_{s-1}+1)(b_{s-2}+1)\cdots(b_{s-q}+1)$.*
    (b) *If t_k has a p-adic length q non-zero expansion. Then, $d(C_k) = 2(b_{s-1}+1)(b_{s-2}+1)\cdots(b_{s-q}+1)$*

6 Examples

In this section, we give some examples of negacyclic codes of different lengths over the ring $R_{k,p}$.

Example 6.1. *Negacyclic codes of length 5 over $R_{3,5} = \mathbb{Z}_5 + u\mathbb{Z}_5 + u^2\mathbb{Z}_5, u^3 = 0$: We have $C = \langle g_1(x) + up_{11}(x) + u^2p_{12}(x), ua_{11}(x) + u^2q_{11}(x), u^2a_{12}(x)\rangle$ where $a_{12}(x)|a_{11}(x)|g_1(x)|(x^{p^s}+1) \bmod p$ and $a_{11}(x) \mid \left(\frac{x^{p^s}+1}{g_1(x)}\right)$, $a_{12}(x) \mid \left(\frac{x^{p^s}+1}{a_{11}(x)}\right)$ and $a_2(x) \mid \left(\frac{x^{p^s}+1}{a_{11}(x)}\right)\left(\frac{x^{p^s}+1}{g_1(x)}\right)$, $deg p_{12}(x) < deg a_{12}(x)$, $deg q_{11}(x) < deg a_{12}(x)$ and $deg p_{11}(x) < deg a_{11}(x)$*

$$x^5 + 1 = (x+1)^5 \text{ over } R_{3,5}.$$

Let $g_1(x) = (x+1) = g$. The non-zero negacyclic codes of length 5 over $R_{3,5}$ with generator polynomial are given in Tables 1, 2 and table 3.

Example 6.2. *Negacyclic codes of length 9 over $R_{3,3} = \mathbb{Z}_3 + u\mathbb{Z}_3 + u^2\mathbb{Z}_3, u^3 = 0$:*
We have $x^9 + 1 = (x+1)^9 over R_{3,3}$. Let $g_1(x) = (x+1) = g$. The non-zero negacyclic codes of length 9 over $R_{3,3}$ with generator polynomial are given in Tables 4, 5 and Tables 6–11.

Table 4: Non-zero free module negacyclic codes of length 9 over $\mathbb{Z}_3 + u\mathbb{Z}_3 + u^2\mathbb{Z}_3, u^3 = 0$.

Non-zero generator polynomials	$d(c)$	Ranks
$< 1 >$.	1	9
$< g + uc_0 >, c_0 \in \mathbb{Z}_3$.	2	8
$< g^2 + u(c_0 + c_1x) + u^2c_2 >, c_0, c_1, c_2 \in \mathbb{Z}_3$.	3	7
$< g^3 + u(c_0 + c_1x + c_2x^2) + u^2(c_3 + c_4x) >, c_0, c_1, c_2, c_3, c_4 \in \mathbb{Z}_3$.	4	6
$< g^4 + u(c_0 + c_1x + c_2x^2 + c_3x^3) + u^2(c_4 + c_5x + c_6x^2) >, c_i \in \mathbb{Z}_3, 0 \le i \le 6$.	5	5
$< g^5 + u(c_0 + c_1x + c_2x^2 + c_3x^3 - (c_0 - c_1 + c_2 - c_3)x^4) + u^2(c_4 + c_5x + c_6x^2 + c_7x^3) >, c_i \in \mathbb{Z}_3, 0 \le i \le 7$.	6	4
$< g^6 + u(c_0 + c_1x + c_2x^2 + c_0x^3 + c_1x^4 + c_2x^5 + u^2(c_3 + c_4x, +c_5x^2 + c_6x^3 + c_7x^4) >, c_i \in \mathbb{Z}_3, 0 \le i \le 7$.	7	3

Table 5: Non-zero free module negacyclic codes of length 9 over $\mathbb{Z}_3 + u\mathbb{Z}_3 + u^2\mathbb{Z}_3, u^3 = 0$.

Non-zero generator polynomials	$d(c)$	Ranks
$< g^7 + u(c_0 + c_1x - c_1x^2 + (c_1 - c_0)x^3 + c_1x^4 - c_1x^5 + (c_0 + c_1)x^6) + u^2(c_2 + c_3x + c_4x^2 + c_5x^3 + c_6x^4 + c_7x^5) >, c_i \in \mathbb{Z}_3, 0 \le i \le 7$.	8	2
$< g^8 + uc_0(1 + x - x^3 - x^4 + x^6 + x^7) + u^2c_1(1 - x^3 + x^6) >, c_0, c_1, c_2 \in \mathbb{Z}_3$.	9	1

Table 6: Non free module negacyclic codes of length 9 over $\mathbb{Z}_3 + u\mathbb{Z}_3 + u^2\mathbb{Z}_3, u^3 = 0$.

Non-zero generator polynomials	$d(c)$	Ranks
$< u, u^2 >, < g^i, u, u^2 >, 1 \le i \le 9$.	1	9
$< g^2 + uc_0 + u^2c_1, ug + u^2c_2, u^2g >, c_0, c_1, c_2 \in \mathbb{Z}_3$	2	8
$< g^2 + uc_0, ug, u^2 >, c_0 \in \mathbb{Z}_3$	1	9
$< g^3 + u(c_0 + c_1) + u^2(c_2 + c_3), ug^2 + u^2(c_4 + c_5), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 5$	2	7
$< g^3 + u(c_0 + c_1) + u^2c_2, ug^2 + u^2c_3), u^2g >, c_i \in \mathbb{Z}_3, 0 \le i \le 3$	2	8
$< g^3 + u(c_0 + c_1), ug^2, u^2 >, c_0, c_1 \in \mathbb{Z}_3$.	1	9
$< g^3 + uc_0 + u^2c_1, ug + u^2c_2, u^2g >, c_0, c_1, c_2 \in \mathbb{Z}_3$.	2	8
$< g^3 + uc_0, ug, u^2 >, c_0, \in \mathbb{Z}_3$.	1	9
$< g^4 + u(c_0 + c_1x + c_2x^2) + u^2(c_3 + c_4x + c_5x^2), ug^3 + u^2(c_6 + c_7x + c_8x^2), u^2g^3 >, c_i \in \mathbb{Z}_3, 0 \le i \le 8$.	2	6
$< g^4 + u(c_0 + c_1x + c_2x^2) + u^2(c_3 + c_4x), ug^3 + u^2(c_5 + c_6x), u^2g^2 > c_i \in \mathbb{Z}_3, 0 \le i \le 6$.	2	7
$< g^4 + u(c_0 + c_1x + c_2x^2) + u^2c_3, ug^3 + u^2c_4, u^2g >, c_i \in \mathbb{Z}_3, 0 \le i \le 4$	2	8
$< g^4 + u(c_0 + c_1x + c_2x^2), ug^3, u^2 >, c_0, c_1, c_2, c_3 \in \mathbb{Z}_3$	1	9
$< g^4 + u(c_0 + c_1x) + u^2(c_2 + c_3x), ug^2 + u^2(c_4 + c_5x), u^2g^2 > c_i \in \mathbb{Z}_3, 0 \le i \le 5$.	2	7
$< g^4 + u(c_0 + c_1x) + u^2c_2, ug^2 + u^2c_3, u^2g >, c_0, c_1, c_2, c_3 \in \mathbb{Z}_3$	2	8
$< g^4 + u(c_0 + c_1x), ug^2, u^2 >, c_0, c_1, \in \mathbb{Z}_3$	1	9
$< g^5 + u(c_0 + c_1x + c_2x^2) + u^2(c_3 + c_4x), ug^3 + u^2(c_5 + c_6x), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 6$.	2	7
$< g^5 + u(c_0 + c_1x + c_2x^2) + u^2c_3, ug^3 + u^2c_4, u^2g >, c_i \in \mathbb{Z}_3, 0 \le i \le 5$.	2	8

Table 7: Non free module cyclic codes of length 9 over $\mathbb{Z}_9 + u\mathbb{Z}_9 + u^2\mathbb{Z}_9, u^3 = 0$.

Non-zero generator polynomials	$d(c)$	Ranks
$< g^5 + u(c_0 + c_1x + c_2x^2), ug^3, u^2 >, c_0, c_1, c_2 \in \mathbb{Z}_3.$	1	9
$< g^5 + u(c_0 + c_1x) + u^2(c_2 + c_3x), ug^2 + u^2(c_4 + c_5x), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 6.$	2	7
$< g^5 + u(c_0 + c_1x) + u^2c_2, ug^2 + u^2c_3, u^2g >, c_i \in \mathbb{Z}_3, 0 \le i \le 3.$	2	8
$< g^5 + u(c_0 + c_1x), ug^2, u^2 >, c_0, c_1 \in \mathbb{Z}_3.$	1	9
$< g^5 + uc_0 + u^2c_1, ug + u^2c_2, u^2g >, c_0, c_1 \in \mathbb{Z}_3.$	2	8
$< g^5 + uc_0, ug, u^2 >, c_0 \in \mathbb{Z}_3.$	1	9
$< g^6 + u(c_0 + c_1x + c_2x^2)g^2 + u^2(c_3 + c_4x + c_5x^2 + c_6x^3 + c_7x^4), ug^5 + u^2(c_8 + c_9x + c_{10}x^2 + c_{11}x^3)g, u^2g^5 >, c_i \in \mathbb{Z}_3, 0 \le i \le 11.$	6	4
$< g^6 + u(c_0 + c_1x + c_2x^2)g^2 + u^2(c_3 + c_4x + c_5x^2 + c_6x^3), ug^5 + u^2(c_7 + c_8x + c_9x^2 + c_{10}x^3), u^2g^4 >, c_i \in \mathbb{Z}_3, 0 \le i \le 10.$	4	5
$< g^6 + u(c_0 + c_1x + c_2x^2)g^2 + u^2(c_3 + c_4x + c_5x^2), ug^5 + u^2(c_6 + c_7x + c_8x^2), u^2g^3 >, c_i \in \mathbb{Z}_3, 0 \le i \le 8.$	2	6
$< g^6 + u(c_0 + c_1x + c_2x^2)g^2 + u^2(c_3 + c_4x), ug^5 + u^2(c_5 + c_6x), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 6.$	2	7
$< g^6 + u(c_0 + c_1x + c_2x^2)g^2 + u^2c_3, ug^5 + u^2c_4, u^2g >, c_i \in \mathbb{Z}_3, 0 \le i \le 4$	2	8
$< g^6 + u(c_0 + c_1x + c_2x^2)g^2, ug^5, u^2 >, c_0, c_1, c_2 \in \mathbb{Z}_3.$	1	9
$< g^6 + u(c_0 + c_1x + c_2x^2)g + u^2(c_3 + c_4x + c_5x^2 + c_6x^3), ug^4 + u^2(c_7 + c_8x + c_9x^2 + c_{10}x^3), u^2g^4 >, c_i \in \mathbb{Z}_3, 0 \le i \le 10.$	4	5
$< g^6 + u(c_0 + c_1x + c_2x^2)g + u^2(c_3 + c_4x + c_5x^2), ug^4 + u^2(c_6 + c_7x + c_8x^2), u^2g^3 >, c_i \in \mathbb{Z}_3, 0 \le i \le 8.$	2	6
$< g^6 + u(c_0 + c_1x + c_2x^2)g + u^2(c_3 + c_4x), ug^4 + u^2(c_5 + c_6x), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 6.$	2	7
$< g^6 + u(c_0 + c_1x + c_2x^2)g + u^2c_3, ug^4 + u^2c_4, u^2g >, c_i \in \mathbb{Z}_3, 0 \le i \le 4$	2	8
$< g^6 + u(c_0 + c_1x + c_2x^2)g, ug^4, u^2 >, c_0, c_1, c_2 \in \mathbb{Z}_3.$	1	9
$< g^6 + u(c_0 + c_1x + c_2x^2) + u^2(c_3 + c_4x + c_5x^2), ug^3 + u^2(c_6 + c_7x + c_8x^2), u^2g^3 >, c_i \in \mathbb{Z}_3, 0 \le i \le 8.$	2	6
$< g^6 + u(c_0 + c_1x + c_2x^2) + u^2(c_3 + c_4x), ug^3 + u^2(c_5 + c_6x), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 6.$	2	7
$< g^6 + u(c_0 + c_1x + c_2x^2) + u^2c_3, ug^3 + u^2c_4, u^2g >, c_i \in \mathbb{Z}_3, 0 \le i \le 4$	2	8
$< g^6 + u(c_0 + c_1x + c_2x^2), ug^3, u^2 >, c_0, c_1, c_2 \in \mathbb{Z}_3.$	1	9
$< g^6 + u(c_0 + c_1x) + u^2(c_2 + c_3x), ug^2 + u^2(c_4 + c_5x), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 6.$	2	7

Table 8: Non free module negacyclic codes of length 9 over $\mathbb{Z}_3 + u\mathbb{Z}_3 + u^2\mathbb{Z}_3$, $u^3 = 0$.

Non-zero generator polynomials	$d(c)$	Ranks
$< g^6 + u(c_0 + c_1x) + u^2c_2, ug^2 + u^2c_4, u^2g >, c_0, c_1, c_2, c_3 \in \mathbb{Z}_3.$	2	8
$< g^6 + u(c_0 + c_1x), ug^2, u^2 >, c_0, c_1, \in \mathbb{Z}_3.$	1	9
$< g^6 + uc_0 + u^2c_1, ug + u^2c_2, u^2g >, c_0, c_1, c_2, \in \mathbb{Z}_3.$	2	8
$< g^6 + uc_0, ug, u^2 >, c_0 \in \mathbb{Z}_3.$	1	9
$< g^7 + u(c_0+c_1x)g^4 + u^2(c_2+c_3x+c_4x^2+c_5x^3+c_6x^4)g, ug^6 + u^2(c_7+c_8x+c_9x^2)g^3, u^2g^6 >, c_i \in \mathbb{Z}_3, 0 \le i \le 9.$	8	3
$< g^7 + u(c_0+c_1x)g^4 + u^2(c_2+c_3x+c_4x^2+c_5x^3+c_6x^4), ug^6 + u^2(c_7+c_8x+c_9x^2)g^2, u^2g^5 >, c_i \in \mathbb{Z}_3, 0 \le i \le 9.$	6	4
$< g^7 + u(c_0 + c_1x)g^4 + u^2(c_2 + c_3x + c_4x^2 + c_5x^3), ug^6 + u^2(c_6 + c_7x + c_8x^2)g, u^2g^4 >, c_i \in \mathbb{Z}_3, 0 \le i \le 8.$	4	5
$< g^7 + u(c_0 + c_1x)g^4 + u^2(c_2 + c_3x + c_4x^2), ug^6 + u^2(c_5 + c_6x + c_7x^2), u^2g^3 >, c_i \in \mathbb{Z}_3, 0 \le i \le 7.$	2	6
$< g^7 + u(c_0 + c_1x)g^4 + u^2(c_2 + c_3x), ug^6 + u^2(c_4 + c_5x), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 5.$	2	7
$< g^7 + u(c_0 + c_1x)g^4 + u^2c_2, ug^6 + u^2c_3, u^2g >, c_0, c_1, c_2, c_3 \in \mathbb{Z}_3.$	2	8
$< g^7 + u(c_0 + c_1x)g^4, ug^6, u^2 >, c_0, c_1 \in \mathbb{Z}_3.$	1	9
$< g^7 + u(c_0 + c_1x)g^3 + u^2(c_2 + c_3x + c_4x^2 + c_5x^3 + c_6x^4), ug^5 + u^2(c_7 + c_8x + c_9x^2 + c_10x^3)g, u^2g^5 >, c_i \in \mathbb{Z}_3, 0 \le i \le 10.$	6	4
$< g^7 + u(c_0+c_1x)g^3 + u^2(c_2+c_3x+c_4x^2+c_5x^3), ug^5 + u^2(c_6+c_7x+c_8x^2+c_9x^3), u^2g^4 >, c_i \in \mathbb{Z}_3, 0 \le i \le 9.$	4	5
$< g^7 + u(c_0 + c_1x)g^3 + u^2(c_2 + c_3x + c_4x^2), ug^5 + u^2(c_5 + c_6x + c_7x^2), u^2g^3 >, c_i \in \mathbb{Z}_3, 0 \le i \le 7.$	2	6
$< g^7 + u(c_0 + c_1x)g^3 + u^2(c_2 + c_3x), ug^5 + u^2(c_4 + c_5x), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 5.$	2	7
$< g^7 + u(c_0 + c_1x)g^3 + u^2c_2, ug^5 + u^2c_3, u^2g >, c_0, c_1, c_2, c_3 \in \mathbb{Z}_3.$	2	8
$< g^7 + u(c_0 + c_1x)g^3, ug^5, u^2 >, c_0, c_1 \in \mathbb{Z}_3.$	1	9
$< g^7 + u(c_0 + c_1x)g^2 + u^2(c_2 + c_3x + c_4x^2 + c_5x^3), ug^4 + u^2(c_6 + c_7x + c_8x^2)g, u^2g^4 >, c_i \in \mathbb{Z}_3, 0 \le i \le 8.$	4	5
$< g^7 + u(c_0 + c_1x)g^2 + u^2(c_2 + c_3x + c_4x^2), ug^4 + u^2(c_5 + c_6x + c_7x^2), u^2g^3 >, c_i \in \mathbb{Z}_3, 0 \le i \le 7.$	2	6
$< g^7 + u(c_0 + c_1x)g^2 + u^2(c_2 + c_3x), ug^4 + u^2(c_4 + c_5x), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 5.$	2	7
$< g^7 + u(c_0 + c_1x)g^2 + u^2c_2, ug^4 + u^2c_3, u^2g >, c_0, c_1, c_2, c_3 \in \mathbb{Z}_3.$	2	8

Table 9: Non free module negacyclic codes of length 9 over $\mathbb{Z}_3 + u\mathbb{Z}_3 + u^2\mathbb{Z}_3$, $u^3 = 0$.

Non-zero generator polynomials	$d(c)$	Ranks
$< g^7 + u(c_0 + c_1x)g^2, ug^4, u^2 >, c_0, c_1 \in \mathbb{Z}_3.$	1	9
$< g^7 + u(c_0 + c_1x)g + u^2(c_2 + c_3x + c_4x^2), ug^3 + u^2(c_5 + c_6x + c_7x^2), u^2g^3 >, c_i \in \mathbb{Z}_3, 0 \le i \le 7.$	2	6
$< g^7 + u(c_0 + c_1x)g + u^2(c_2 + c_3x), ug^3 + u^2(c_4 + c_5x), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 5.$	2	7
$< g^7 + u(c_0 + c_1x)g + u^2c_2, ug^3 + u^2c_3, u^2g >, c_0, c_1, c_2, c_3 \in \mathbb{Z}_3.$	2	8
$< g^7 + u(c_0 + c_1x)g, ug^3, u^2 >, c_0, c_1 \in \mathbb{Z}_3.$	1	9
$< g^7 + u(c_0 + c_1x) + u^2(c_2 + c_3x), ug^2 + u^2(c_4 + c_5x), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 5.$	2	7
$< g^7 + u(c_0 + c_1x) + u^2c_2, ug^2 + u^2c_3, u^2g >, c_0, c_1, c_2, c_3 \in \mathbb{Z}_3.$	2	8
$< g^7 + u(c_0 + c_1x), ug^2, u^2 >, c_0, c_1 \in \mathbb{Z}_3.$	1	9
$< g^7 + uc_0 + u^2c_1, ug + u^2c_2, u^2g >, c_0, c_1, c_2 \in \mathbb{Z}_3.$	2	8
$< g^7 + uc_0, ug, u^2 >, c_0 \in \mathbb{Z}_3.$	1	9
$< g^8 + uc_0g^6 + u^2(c_1 + c_2x + c_3x^2)g^4, ug^7 + u^2(c_4 + c_5x)g^5, u^2g^7 >, c_i \in \mathbb{Z}_3, 0 \le i \le 5.$	6	2
$< g^8 + uc_0g^6 + u^2(c_1 + c_2x + c_3x^2)g^3, ug^7 + u^2(c_4 + c_5x)g^4, u^2g^6 >, c_i \in \mathbb{Z}_3, 0 \le i \le 5.$	8	3
$< g^8 + uc_0g^6 + u^2(c_1 + c_2x + c_3x^2)g^2, ug^7 + u^2(c_4 + c_5x)g^3, u^2g^5 >, c_i \in \mathbb{Z}_3, 0 \le i \le 5.$	6	4
$< g^8 + uc_0g^6 + u^2(c_1 + c_2x + c_3x^2)g, ug^7 + u^2(c_4 + c_5x)g^2, u^2g^4 >, c_i \in \mathbb{Z}_3, 0 \le i \le 5.$	4	5
$< g^8 + uc_0g^6 + u^2(c_1 + c_2x + c_3x^2), ug^7 + u^2(c_4 + c_5x)g, u^2g^3 >, c_i \in \mathbb{Z}_3, 0 \le i \le 5.$	2	6
$< g^8 + uc_0g^6 + u^2(c_1 + c_2x), ug^7 + u^2(c_3 + c_4x), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 4.$	2	7
$< g^8 + uc_0g^6 + u^2c_1, ug^7 + u^2c_2, u^2g >, c_0, c_1, c_2 \in \mathbb{Z}_3.$	2	8
$< g^8 + uc_0g^6, ug^7, u^2 >, c_0 \in \mathbb{Z}_3.$	1	9
$< g^8 + uc_0g^5 + u^2(c_1 + c_2x + c_3x^2 + c_4x^3)g^2, ug^6 + u^2(c_5 + c_6x + c_7x^2)g^3, u^2g^6 >, c_i \in \mathbb{Z}_3, 0 \le i \le 7.$	8	3
$< g^8 + uc_0g^5 + u^2(c_1 + c_2x + c_3x^2 + c_4x^3)g, ug^6 + u^2(c_5 + c_6x + c_7x^2)g^2, u^2g^5 >, c_i \in \mathbb{Z}_3, 0 \le i \le 7.$	6	4
$< g^8 + uc_0g^5 + u^2(c_1 + c_2x + c_3x^2 + c_4x^3), ug^6 + u^2(c_5 + c_6x + c_7x^2)g, u^2g^4 >, c_i \in \mathbb{Z}_3, 0 \le i \le 7.$	4	5

Table 10: Non free module negacyclic codes of length 9 over $\mathbb{Z}_3 + u\mathbb{Z}_3 + u^2\mathbb{Z}_3$, $u^3 = 0$.

Non-zero generator polynomials	$d(c)$	Ranks
$< g^8 + uc_0g^5 + u^2(c_1 + c_2x + c_3x^2), ug^6 + u^2(c_4 + c_5x + c_6x^2), u^2g^3 >, c_i \in \mathbb{Z}_3, 0 \le i \le 6.$	2	6
$< g^8 + uc_0g^5 + u^2(c_1 + c_2x), ug^6 + u^2(c_3 + c_4x), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 4.$	2	7
$< g^8 + uc_0g^5 + u^2c_1, ug^6 + u^2c_2, u^2g >, c_0, c_1, c_2 \in \mathbb{Z}_3.$	2	8
$< g^8 + uc_0g^5, ug^6, u^2 >, c_0 \in \mathbb{Z}_3.$	1	9
$< g^8 + uc_0g^4 + u^2(c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4), ug^5 + u^2(c_6 + c_7x + c_8x^2)g, u^2g^5 >, c_i \in \mathbb{Z}_3, 0 \le i \le 8.$	6	4
$< g^8 + uc_0g^4 + u^2(c_1 + c_2x + c_3x^2 + c_4x^3), ug^5 + u^2(c_5 + c_6x + c_7x^2 + c_8x^3), u^2g^4 >, c_i \in \mathbb{Z}_3, 0 \le i \le 8.$	4	5
$< g^8 + uc_0g^4 + u^2(c_1 + c_2x + c_3x^2), ug^5 + u^2(c_4 + c_5x + c_6x^2), u^2g^3 >, c_i \in \mathbb{Z}_3, 0 \le i \le 6.$	2	6
$< g^8 + uc_0g^4 + u^2(c_1 + c_2x), ug^5 + u^2(c_3 + c_4x), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 4.$	2	7
$< g^8 + uc_0g^4 + u^2c_1, ug^5 + u^2c_2, u^2g >, c_0, c_1, c_2 \in \mathbb{Z}_3.$	2	8
$< g^8 + uc_0g^4, ug^5, u^2 >, c_0 \in \mathbb{Z}_3.$	1	9
$< g^8 + uc_0g^3 + u^2(c_1 + c_2x + c_3x^2 + c_4x^3), ug^4 + u^2(c_5 + c_6x + c_7x^2 + c_8x^3), u^2g^4 >, c_i \in \mathbb{Z}_3, 0 \le i \le 8.$	4	5
$< g^8 + uc_0g^3 + u^2(c_1 + c_2x + c_3x^2), ug^4 + u^2(c_4 + c_5x + c_6x^2), u^2g^3 >, c_i \in \mathbb{Z}_3, 0 \le i \le 6.$	2	6
$< g^8 + uc_0g^3 + u^2(c_1 + c_2x), ug^4 + u^2(c_3 + c_4x), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 4.$	2	7
$< g^8 + uc_0g^3 + u^2c_1, ug^4 + u^2c_2, u^2g >, c_0, c_1, c_2 \in \mathbb{Z}_3.$	2	8
$< g^8 + uc_0g^3, ug^4, u^2 >, c_0 \in \mathbb{Z}_3.$	1	9
$< g^8 + uc_0g^2 + u^2(c_1 + c_2x + c_3x^2), ug^3 + u^2(c_4 + c_5x + c_6x^2), u^2g^3 >, c_i \in \mathbb{Z}_3, 0 \le i \le 6.$	2	6
$< g^8 + uc_0g^2 + u^2(c_1 + c_2x), ug^3 + u^2(c_3 + c_4x), u^2g^2 >, c_i \in \mathbb{Z}_3. 0 \le i \le 4.$	2	7
$< g^8 + uc_0g^2 + u^2c_1, ug^3 + u^2c_2, u^2g >, c_0, c_1, c_2 \in \mathbb{Z}_3.$	2	8
$< g^8 + uc_0g^2, ug^3, u^2 >, c_0 \in \mathbb{Z}_3.$	1	9
$< g^8 + uc_0g + u^2(c_1 + c_2x), ug^2 + u^2(c_3 + c_4x), u^2g^2 >, c_i \in \mathbb{Z}_3, 0 \le i \le 4.$	2	7
$< g^8 + uc_0g + u^2c_1, ug^2 + u^2c_2, u^2g >, c_0, c_1, c_2 \in \mathbb{Z}_3.$	2	8
$< g^8 + uc_0g, ug^2, u^2 >, c_0 \in \mathbb{Z}_3.$	1	9

Table 11: Non free module negacyclic codes of length 9 over $\mathbb{Z}_3 + u\mathbb{Z}_3 + u^2\mathbb{Z}_3$, $u^3 = 0$.

Non-zero generator polynomials	$d(c)$	Ranks
$< g^8 + uc_0 + u^2c_1, ug + u^2c_2, u^2g >, c_0, c_1, c_2 \in \mathbb{Z}_3.$	2	8
$< g^8 + uc_0, ug, u^2 >, c_0 \in \mathbb{Z}_3.$	1	9
$< ug^i, u^2 >, 1 \le i \le 8.$	1	9
$< ug^i + u^2c_0, u^2g >, c_0 \in \mathbb{Z}_3, 1 \le i \le 8.$	2	8
$< ug^i + u^2(c_0 + c_1x), u^2g^2 >, c_0, c_1 \in \mathbb{Z}_3, 2 \le i \le 7.$	2	7
$< ug^i + u^2(c_0 + c_1x + c_2x^2), u^2g^3 >, c_0, c_1, c_2 \in \mathbb{Z}_3, 3 \le i \le 6.$	2	6
$< ug^i + u^2(c_0 + c_1x + c_2x^2 + c_3x^3), u^2g^4 >, c_0, c_1, c_2, c_3 \in \mathbb{Z}_3, i = 4, 5.$	4	5
$< ug^8 + u^2c_0g^1, u^2g^2 >, c_0 \in \mathbb{Z}_3.$	2	7
$< ug^8 + u^2c_0g^2, u^2g^3 >, c_0 \in \mathbb{Z}_3.$	2	6
$< ug^8 + u^2c_0g^3, u^2g^4 >, c_0 \in \mathbb{Z}_3.$	4	5
$< ug^8 + u^2c_0g^4, u^2g^5 >, c_0 \in \mathbb{Z}_3.$	6	4
$< ug^8 + u^2c_0g^5, u^2g^6 >, c_0 \in \mathbb{Z}_3.$	8	3
$< ug^8 + u^2c_0g^6, u^2g^7 >, c_0 \in \mathbb{Z}_3.$	6	2
$< ug^8 + u^2c_0g^7, u^2g^8 >, c_0 \in \mathbb{Z}_3.$	9	1
$< ug^7 + u^2(c_0 + c_1x)g^1, u^2g^3 >, c_0, c_1 \in \mathbb{Z}_3.$	2	6
$< ug^7 + u^2(c_0 + c_1x)g^2, u^2g^4 >, c_0, c_1 \in \mathbb{Z}_3.$	4	5
$< ug^7 + u^2(c_0 + c_1x)g^3, u^2g^5 >, c_0, c_1 \in \mathbb{Z}_3.$	6	4
$< ug^7 + u^2(c_0 + c_1x)g^4, u^2g^6 >, c_0, c_1 \in \mathbb{Z}_3.$	8	3
$< ug^7 + u^2(c_0 + c_1x)g^5, u^2g^7 >, c_0, c_1 \in \mathbb{Z}_3.$	6	2
$< ug^6 + u^2(c_0 + c_1x + c_2x^2)g^1, u^2g^4 >, c_0, c_1, c_2 \in \mathbb{Z}_3.$	4	5
$< ug^6 + u^2(c_0 + c_1x + c_2x^2)g^2, u^2g^5 >, c_0, c_1, c_2 \in \mathbb{Z}_3.$	6	4
$< ug^6 + u^2(c_0 + c_1x + c_2x^2)g^3, u^2g^6 >, c_0, c_1, c_2 \in \mathbb{Z}_3.$	8	3
$< ug^5 + u^2(c_0 + c_1x + c_2x^2 + c_3x^3)g, u^2g^5 >, c_0, c_1, c_2, c_3 \in \mathbb{Z}_3.$	6	4

7 Conclusion

In this paper, the algebraic structure of the ring $R_{k,p} = \mathbb{Z}_p[u]/\langle u^k\rangle$, where p is prime and negacyclic codes over this ring are studied. To find the generator of these code, we discussed the rank and hamming distance over $R_{k,p}$. We also presented some example of these codes including the rank and distance. For future study, it would be interesting to investigate the algebraic structure of negacyclic codes over $R_{k,p}$, where p is not prime and their rank and minimum distance. Another question is to find new negacyclic codes of general length over that ring.

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