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A time-delay equation: well-posedness to optimal control

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Abstract: In this paper, well-posedness, controllability and optimal control for a time-delay beam equation are studied. The equation of motion is modeled as a time-delayed distributed parameter system(DPS) and includes Heaviside functions and their spatial derivatives due to the finite size of piezoelectric patch actuators used to suppress the excessive vibrations based on displacement and moment conditions. The optimal control problem is defined with the performance index including a weighted quadratic functional of the displacement and velocity which is to be minimized at a given terminal time and a penalty term defined as the control voltage used in the control duration. Optimal control law is obtained by using Maximum principle and hence, the optimal control problem is transformed the into a boundary-, initial and terminal value problem.The explicit solution of the control problem is obtained by eigenfunction expansions of the state and adjoint variables. Numerical results are presented to show the effectiveness and applicability of the piezoelectric control.

Keywords: Wellposedness; Time-delay; Vibration; Control; Maximum Principle

PACS: 64.30.Ef, 62.20.dj, 62.20.fg, 62.30.+d

1 Introduction

Time-delay is a widespread phenomenon in applied sciences, *e.g.* space engineering, chemical kinetics, fluid dynamics, etc. Such delays occur as a result of the finite time-response of actuators used in the implementation of control law [11]. Since time delay leads to a decrement in the performance of the actuator, the state variable cannot reflect the changes in the system [18]. Active vibration control of mechanical systems, which are modeled as DPS without time delays, has been excessively studied in the literature by several authors, such as, but not limited to [3, 7, 12–16]; however, the optimal control of DPS with time delay has not received considerable attention yet. Some studies of the control of DPS with time delay can be summarized by [4–6, 11, 18]. Although undamped free vibrations in mechanical systems are often studied from an optimal control point of view, controlling other forms of vibrations caused by the thermal effect or an external excitation has not received enough attention. Vibrations caused by the thermal effect and external excitation are also modeled as displacement or moment boundary conditions. The contribution of material damping to the piezoelectric control of a system without time delay in control function has been studied in the [8] and it was observed that neglecting natural damping leads to under-estimation of the structural behavior [12].

In the literature, according to author's best knowledge, there is no study dealing with the well-posedness, controllability and optimal control of a time-delay beam equation. Therefore, this paper presents an original contribution to the literature of the study of the well-posedness, controlability and optimal control of a time-delay beam equation is studied and results are simulated using MATLAB. The contribution of material damping to the system is also considered. Moreover, in order to achieve a better observation for the beam, three other conditions(time delay, internal damping and boundary effects) are taken into account.

This paper studies the dynamic response of a beam, modeled as an Euler-Bernoulli beam with Kelvin-Voigt damping and a time delay in the control function, subject to displacement and moment boundary conditions. The control is applied by means of piezoelectric actuators bonded onto the beam.

Due to the presence of the damping term, the governing equation does not admit a variational principle in the classical sense, making an approach based on variational calculus impractical. Also, the equation of motion includes Heaviside functions and their spatial derivatives due to the finite size of the piezoelectric patch actuators

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used. Because of this, it is not sensible to use the optimal linear quadratic regulator or linear quadratic Gaussian control algorithms for solving the system. Therefore, optimal control law is obtained by using the Maximum principle and hence, the optimal control problem is transformed into a boundary-, initial and terminal value problem. The performance index, to be minimized with minimum expenditure of voltage applied to the piezoelectric patch actuators, consists of a weighted quadratic function of displacement and velocity of the beam. It also includes a penalty function of control voltage. The solution of the control problem is sought by the eigenfunction expansion method. To illustrate the capability and efficiency of the control law, introduced, two numerical examples are presented.

2 Mathematical Formulation of the Control Problem

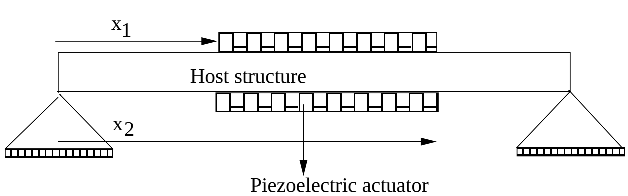

Figure 1: Beam diagram with piezoelectric pathces

The time-delay optimal control problem of a smart beam is considered. The beam shown in Fig. 1, initially un-deformed, has internal damping modeled as Kelvin-Voigt damping and is subject to displacement and moment boundary conditions. The equation of motion of the beam is controlled by piezoelectric patch actuators, bonded on both sides of the beam, with a control voltage $\bar{C}(\bar{t}-\tau)$ is given by [1]

$$\vartheta \upsilon_{\bar{t}\bar{t}} + 2\bar{\xi}\rho\upsilon_{\bar{t}\bar{x}\bar{x}\bar{x}\bar{x}} + \rho\upsilon_{\bar{x}\bar{x}\bar{x}\bar{x}} = \bar{C}(\bar{t}-\tau)\left(\mathrm{H}''(\bar{x}-\bar{x}_1) - \mathrm{H}''(\bar{x}-\bar{x}_2)\right) \quad (1)$$

where υ is the transversal displacement, $\bar{x} \in (0,\ell)$ is the space variable, ℓ is the length of the beam, $\bar{t} \in (0,\bar{t}_f)$ is the time variable, $\bar{t_f}$ is the terminal time, $\tau \geq 0$ is the known constant delay in the control function, ϑ is the mass per unit area, $\bar{\xi}$ is the damping coefficient, H is the Heaviside function and $(\bar{x}_1,\bar{x}_2)$ is the location of a piezoelectric actuator. The flexural stiffness of the beam, ρ, is defined as follows:

$$\rho = \frac{Eh^3}{12(1-\varrho^2)}$$

in which E is Young's modulus, h is the elastic thickness of the plate and ϱ is Poisson's ratio. Eq. (1) is subject to the following boundary conditions:

$$\upsilon(0,\bar{t}) = \upsilon(\ell,\bar{t}) = \bar{\zeta}(\bar{t}), \quad \upsilon_{\bar{x}\bar{x}}(0,\bar{t}) = \upsilon_{\bar{x}\bar{x}}(\ell,\bar{t}) = \bar{\eta}(\bar{t}) \quad (2)$$

where $\zeta(\bar{t})$ is the displacement boundary condition, $\eta(\bar{t})$ is the moment boundary condition and the initial conditions are:

$$\upsilon(\bar{x},0) = \upsilon_0(\bar{x}), \quad \upsilon_{\bar{t}}(\bar{x},0) = \upsilon_1(\bar{x}). \quad (3)$$

Theorem 1. ***(Second Order Linear Picard-Lindelof Existence-Uniqueness Theorem:)*** *Let the coefficients $a(x), b(x), c(x), f(x)$ be continuous on an interval J containing $x = x0$. Assume $a(x) \neq 0$ on J. Let g_1 and g_2 be real constants. The initial value problem*

$$a(x)y'' + b(x)y' + c(x)y = f(x), \quad y(x_0) = g_1, \quad y'(x_0) = g_2 \quad (4)$$

has a solution and this solution is unique [20].

The system defined by Eqs. (1–3) can be reduced to an ordinary differential equation by using the Galerkin expansion method like in Eq. (42) section 3. Please note that all coefficients and functions are analytic functions in Eq. (42) near the 0. Then, Eq. (42) with initial conditions satisfies the Second Order Linear Picard-Lindelof Existence-Uniqueness theorem. Namely, the system has a unique solution.

The uniqueness of the optimal control function will be discussed in the next section of this paper. For convenience, let us introduce non-dimensional variables as

$$u = \frac{\upsilon}{\ell}, \quad x = \frac{\bar{x}}{\ell}, \quad t = \frac{1}{\ell^2}\sqrt{\frac{\rho}{\vartheta}}\bar{t}, \quad \xi = \frac{\bar{\xi}}{\ell^2}\sqrt{\frac{\rho}{\vartheta}},$$
$$C(t-\tau) = \frac{\ell}{\rho}C(\bar{t}-\tau), \quad (x,t) \in \mathcal{S} = (0,1)\times(0,t_f)$$

where t_f is the terminal time. Substituting new parameters into Eq. (1), one obtains the non-dimensional equation of the motion subject to the following boundary and initial conditions, respectively.

$$u_{tt} + 2\xi u_{txxxx} + u_{xxxx} = C(t-\tau)\left(\mathrm{H}''(x-x_1) - \mathrm{H}''(x-x_2)\right) \quad (5)$$

$$u(0,t) = u(1,t) = \zeta(t), \quad u_{xx}(0,t) = u_{xx}(1,t) = \eta(t), \quad (6)$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \quad (7)$$

The aim of the optimal control problem is to determine an optimum voltage function $C(t-\tau)$ to minimize the performance index of the beam at t_f with the minimum expenditure of control voltage. Therefore, the performance index is defined by the weighted dynamic response of the beam and the expenditure of control over $(0, t_f)$. The set of admissible control functions is given by

$$C_{ad} = \{C(t) | C(t) \in L^2(0, t_f), \quad |C(t)| \le c_0 < \infty\} \tag{8}$$

and the performance index of the controlled system is defined as follows;

$$\mathcal{J}(C(t)) = \int_0^1 [\mu_1 u^2(x, t_f) + \mu_2 u_t^2(x, t_f)]dx + \int_0^{t_f} \mu_3 C^2(t)dt \tag{9}$$

where $\mu_1, \mu_2 \ge 0, \quad \mu_1 + \mu_2 \ne 0$ and $\mu_3 > 0$ are weighting constants. The first integral in Eq. (9) is the modified dynamic response of the beam and last integral represents the measure of the total voltage energy that accumulates over $(0, t_f)$. The optimal control of a beam with a time delay in the control function is expressed as

$$\mathcal{J}(C^\circ(t)) = \min_{C(t) \in C_{ad}} \mathcal{J}(C(t)) \tag{10}$$

subject to the Eqs. 5—7. In order to achieve the maximum principle, let us introduce an adjoint variable $v(x, t)$ satisfying the following equation

$$v_{tt} - 2\xi v_{txxxx} + v_{xxxx} = 0 \tag{11}$$

and subject to the following homogeneous boundary conditions

$$v(x, t) = v_{xx}(x, t) = 0 \quad \text{at} \quad x = 0, 1 \tag{12}$$

and terminal conditions

$$v_t(x, t) - 2\xi v_{xxxx}(x, t) = 2\mu_1 u(x, t), \quad v(x, t) = -2\mu_2 u_t(x, t) \quad \text{at} \quad t = t_f. \tag{13}$$

A maximum principle, in terms of the Hamiltonian function is derived as a necessary condition for the optimal control function. It is proved in [3] that under some convexity assumption, which is satisfied by Eq. (9), on the performance index function, the maximum principle is also a sufficient condition for the optimal control function. Note that υ is the unique solution to the system defined by Eqs. 1—3. By considering uniqueness of the solution, it can be concluded that when υ° is the unique solution to the system, the corresponding control function $\bar{C}^\circ$ must be unique. In this situation, system has a unique control function and solution and the system is called observable. The Hilbert uniqueness method proved that observable implies the system under consideration is controllable [21, 22]. The maximum principle gives an explicit expression for the optimal control function and implicitly relates optimal control to the state variable. Then, the maximum principle can be given as follows:

Theorem 2. *(Maximum principle) The maximization problem states that if*

$$\mathcal{H}[t; v^\circ, C^\circ(t)] = \max_{C(t) \in C_{ad}} \mathcal{H}[t; v, C(t)] \tag{14}$$

in which $v = v(x, t)$ satisfies the adjoint system given by Eqs. 11—13 and the Hamiltonian function is defined by

$$\mathcal{H}[t; v, C(t)] = C(t)[v_x(x_2, t) - v_x(x_1, t)] - \mu_3 C^2(t), \tag{15}$$

then

$$\mathcal{J}[C^\circ(t)] \le \mathcal{J}[C(t)], \quad \forall C(t) \in C_{ad} \tag{16}$$

where $C^\circ(t)$ is the optimal control function.

Proof. Before starting the proof, let us introduce an operator and its adjoint operator as follows:

$$Y(u) = u_{tt} + u_{xxxx} + 2\xi u_{txxxx}, \quad Y^*(v) = v_{tt} + v_{xxxx} - 2\xi v_{txxxx}. \tag{17}$$

The deviations are given by $\Delta u = u - u^\circ$, $\Delta u_t = u_t - u_t^\circ$ in which u° is the optimal displacement. The operator $Y(\Delta u) = \Delta C(t)(\mathrm{H}''(x - x_1) - \mathrm{H}''(x - x_2))$ is subject to the following boundary conditions

$$\Delta u(x, t) = \Delta u_{xx}(x, t) = 0 \quad \text{at} \quad x = 0, 1 \tag{18}$$

and initial conditions

$$\Delta u(x, t) = \Delta u_t(x, t) = 0 \quad \text{at} \quad t = 0. \tag{19}$$

Consider the following functional

$$\int_0^1 \int_0^{t_f} \left\{ vY(\Delta u) - \Delta u Y^*(v) \right\} dtdx = \int_0^1 \int_0^{t_f} \left\{ v\Delta C(t)(\mathrm{H}''(x - x_1) - \mathrm{H}''(x - x_2)) \right\} dtdx. \tag{20}$$

Integrating the left side of Eq. (20) twice integration by parts with respect to t and four times integration by parts with respect to x, using Eqs. (18–19), one observes the following relation:

$$\int_0^1 \int_0^{t_f} \left\{ vY(\Delta u) - \Delta u Y^*(v) \right\} dtdx$$

$$= \int_0^1 \Big(v(x,t_f)\Delta u_t(x,t_f) - \Delta u(x,t_f)\big[v_t(x,t_f) - 2\xi v_{xxxx}(x,t_f)\big]\Big)dx. \quad (21)$$

In view of Eq. (13), Eq. (21) becomes

$$\int_0^1\int_0^{t_f} \Big\{ vY(\Delta u) - \Delta u Y^\star(v)\Big\} dt dx = -2\int_0^1 \big(\mu_1 u(x,t_f)\Delta u(x,t_f) + \mu_2 u_t(x,t_f)\Delta u_t(x,t_f)\big)dx. \quad (22)$$

For the right side of Eq. (20), recall the properties of dirac-delta function

$$H''(x-\theta) = \delta'(x-\theta), \quad \int_0^1 \delta'(x-\theta)\varsigma(x)dx = -\varsigma'(\theta), \quad \theta \in (0,1). \quad (23)$$

In the light of Eq. (23), the right side of Eq. (20) is obtained as follows:

$$\int_0^{t_f}\int_0^1 v(x,t)\Delta C(t)\big[\delta'(x-x_1) - \delta'(x-x_2)\big]dxdt = \int_0^{t_f} \Delta C(t)\big(v_x(x_2,t) - v_x(x_1,t)\big)dt. \quad (24)$$

Consider the difference of the performance index

$$\Delta\mathcal{J}[C(t)] = \mathcal{J}[C(t)] - \mathcal{J}[C^\circ(t)] = \int_0^1 \Big\{\mu_1[u^2(x,t_f) - u^{\circ^2}(x,t_f)] + \mu_2[u_t^2(x,t_f) - u_t^{\circ^2}(x,t_f)]\Big\}dx + \int_0^{t_f} \mu_3[C^2(t) - C^{\circ^2}(t)]dt \quad (25)$$

Let us expand $u^2(x,t_f)$ and $u_t^2(x,t_f)$ to Taylor series around $u^{\circ^2}(x,t_f)$ and $u_t^{\circ^2}(x,t_f)$, respectively. Then, one observes the following

$$u^2(x,t_f) - u^{\circ^2}(x,t_f) = 2u^\circ(x,t_f)\Delta u(x,t_f) + r, \quad (26a)$$

$$u_t^2(x,t_f) - u_t^{\circ^2}(x,t_f) = 2u_t^\circ(x,t_f)\Delta u_t(x,t_f) + r_t \quad (26b)$$

where $r = 2(\Delta u)^2$ + higher order terms > 0 and $r_t = 2(\Delta u_t)^2$ + higher order terms > 0. Substituting Eq. (26) into Eq. (25) gives

$$\Delta\mathcal{J}[C(t)] = \int_0^1 \Big\{\mu_1[2u^\circ(x,t_f)\Delta u(x,t_f) + r] + \mu_2[2u_t^\circ(x,t_f)\Delta u_t(x,t_f) + r_t]\Big\}dx + \int_0^{t_f} \mu_3[C(t)^2 - C(t)^{\circ^2}]dt. \quad (27)$$

From Eq. (22) and because of $\mu_1 r + \mu_2 r_t > 0$, one obtains

$$\Delta\mathcal{J}[C(t)] \geq -\int_0^{t_f} \Big\{\Big(C(t)\big[v_x(x_2,t) - v_x(x_1,t)\big] + \mu_3 C^2(t)\Big) - \Big(C^\circ(t)\big[v_x^\circ(x_2,t) - v_x^\circ(x_1,t)\big] + \mu_3 C^{\circ^2}(t)\Big)\Big\}dt \geq 0 \quad (28)$$

which leads to

$$C(t)[v_x(x_1,t) - v_x(x_2,t)] + \mu_3 C(t)^2 \geq C^\circ(t)[v_x^\circ(x_1,t) - v_x^\circ(x_2,t)] + \mu_3 C^{\circ^2}(t), \quad (29)$$

that is,

$$\mathcal{H}[t; v^\circ, C] \geq \mathcal{H}[t; v, C].$$

Hence, we obtain

$$\mathcal{J}[C] \geq \mathcal{J}[C^\circ], \quad \forall C \in C_{ad}$$

Therefore, the optimal control function is given by

$$C(t) = \frac{v_x^\circ(x_2,t) - v_x^\circ(x_1,t)}{2\mu_3}. \quad (30)$$

□

The existence and uniqueness of the solution to the adjoint system, defined by Eqs. (11–13), can be obtained in a similar way to Eqs. (1–3). Then, the state system given by Eqs. (1–3) is controllable.

3 Solution Method

The solution of the optimal control problem is sought as follows: Let the adjoint variable $v(x,t)$ satisfying Eqs. (11–13) be expanded in Fourier sine series as

$$v(x,t) = \sum_{n=1}^{\infty} Z_n(t)\varphi_n(x) \quad (31)$$

where the orthonormal eigenfunctions

$$\varphi_n(x) = \sqrt{2}\sin(\lambda_n x), \quad \lambda_n = n\pi \tag{32}$$

satisfy boundary conditions given by Eq. (12). Substituting Eq. (31) into Eq. (11), multiplying both sides with $\varphi_n(x)$ and integrating both sides over $(0, 1)$ leads to the following lumped parameter system(LPS) in time

$$\ddot{Z}_n - 2\xi\lambda_n^4\dot{Z}_n + \lambda_n^4 Z_n = 0, \quad \text{for } n = 1, 2, \ldots. \tag{33}$$

The general solution of the LPS given by Eq. (33) is given by

$$Z_n(t) = a_n\kappa_n(t) + b_n\iota_n(-t), \tag{34}$$

where

$$\kappa_n(t) = \exp\left((\varpi_n + \xi\lambda_n^4)t\right), \quad \iota_n(t) = \exp\left((\varpi_n - \xi\lambda_n^4)t\right),$$
$$\varpi_n = \lambda_n^2\sqrt{\xi^2\lambda_n^4 - 1} \tag{35}$$

and a_n and b_n are constants to be determined. Next, we solve the equation of the optimal motion. In order to convert the non-homogeneous boundary conditions to homogeneous ones, let us define the following relation

$$w = u - \zeta(t) - \beta(x)\eta(t), \quad \beta(x) = \frac{x^2 - x}{2}. \tag{36}$$

Then, the system given by Eqs. (5–7) becomes

$$w_{tt} + 2\xi w_{txxxx} + w_{xxxx} = C(t-\tau)\left(\mathrm{H}''(x - x_1) - \mathrm{H}''(x - x_2)\right) - \zeta''(t) - \beta(x)\eta''(t) \tag{37}$$

subject to the new homogeneous boundary conditions

$$w(x, t) = w_{xx}(x, t) = 0 \quad \text{at} \quad x = 0, 1 \tag{38}$$

and initial conditions

$$w(x, 0) = u_0(x) - \zeta(0) - \beta(x)\eta(0),$$
$$w_t(x, 0) = u_1(x) - \zeta'(0) - \beta(x)\eta'(0). \tag{39}$$

Due to Eq. (36), one observes the terminal conditions of the adjoint equation Eq. (13) as follows:

$$v_t(x, t_f) - 2\xi v_{xxxx}(x, t_f) = 2\mu_1[w(x, t_f) + \zeta(t_f) + \beta(x)\eta(t_f)], \tag{40a}$$

$$v(x, t_f) = -2\mu_2[w_t(x, t_f) + \zeta_t(t_f) + \beta(x)\eta_t(t_f)]. \tag{40b}$$

Now, let us obtain the solution of the motion equation by using Fourier sine series

$$w(x, t) = \sum_{n=1}^{\infty}\Omega_n(t)\varphi_n(x) \tag{41}$$

in which $\Omega_n(t)$ satisfies the following LPS

$$\ddot{\Omega}_n + 2\xi\lambda_n^4\dot{\Omega}_n + \lambda_n^4\Omega_n = \gamma_1\zeta''(t) + \gamma_2\eta''(t) + \gamma_3 C(t-\tau), \tag{42}$$

where

$$\gamma_1 = -\int_0^1 \varphi_n(x)dx, \quad \gamma_2 = -\int_0^1 \beta(x)\varphi_n(x)dx,$$
$$\gamma_3 = \int_0^1 (\mathrm{H}''(x - x_1) - \mathrm{H}''(x - x_2))\varphi_n(x)dx.$$

The general solution of Eq. (42) is given by

$$\Omega_n(t) = c_n\kappa_n(-t) + d_n\iota_n(t) + \frac{1}{2\varpi_n}\int_0^t (\iota_n(t-s) - \kappa_n(s-t))(\gamma_1\zeta''(s) + \gamma_2\eta''(s))ds + \frac{1}{2\varpi_n}\int_\tau^t (\iota_n(t-s) - \kappa_n(s-t))(\gamma_3 C(s-\tau))ds \tag{43}$$

in which c_n and d_n are constants to be determined by means of Eq. (39). The remaining unknown constants a_n and b_n appearing in Eq. (43) are evaluated by using the terminal conditions given by Eq. (40).

4 Numerical Results and Discussions

In this section, the theoretical results obtained in the previous sections are illustrated to show the effectiveness and capability of the proposed control algorithm for controlling the dynamic response of a smart beam with Kelvin-Voigt damping and a time delay in a minimum level control voltage to be applied to the piezoelectric patch actuator. Due to the selection of the damping coefficient (ξ), three cases: over damping, critical damping and under damping are studied in the literature. Under damping is the most important case due to the insufficient internal damping of the beam. Therefore, in this paper, under damping is studied. In this case, $\xi < \frac{1}{\lambda_n^2}$ and two complex roots are obtained from Eq. (5). Also, taking the one term solution, when $\xi < \frac{1}{\lambda_n^2}$, the stability of the system defined by Eqs. (5–7) is guaranteed [9]. The location of the piezoelectric patch actuators and predetermined terminal time are specified as $(x_1, x_2) = (0.25, 0.75)$ and $t_f = 5$, respectively. The time delay in the control voltage function applied to the piezoelectric patch actuator τ is evaluated as 0.002 and the damping coefficient is taken into account as 0.001. The weighting coefficients in the calculations are taken as $\mu_3 = 10^{-2}$ and $\mu_1 = \mu_2 = 1$ for the controlled case. Displacement boundary conditions, moment boundary conditions and initial conditions are evaluated as follows;

$$\text{For the case a, } \zeta(t) = 0,\ \eta(t) = e^{5-t},\ u_0 = \sqrt{2}\cos(\pi x) \text{ and } u_1 = \sqrt{2}\sin(\pi x),$$

$$\text{For the case b, } \zeta(t) = 5 - t,\ \eta(t) = 0,\ u_0 = \sqrt{2}\sin(\pi x), \text{ and } u_1 = \pi\sqrt{2}\cos(\pi x).$$

Let us define the dynamic response of the beam and the control voltage as follows, respectively;

$$\mathcal{D}(t_f) = \int_0^1 [u^2(x, t_f) + u_t^2(x, t_f)]dx, \quad \mathcal{C} = \int_0^{t_f} C^2(t)dt \quad (44)$$

The dynamic response of the beam is presented in table 1 for case a and b. Observing table 1, it can be seen that for both cases with/out delay in the control function, the penalty μ_3 on the expenditure of voltage decreases as the dynamic response of the beam decreases corresponding to an increase in the voltage. Also, it can be observed from table 1 that in the cases of $\tau = 0$, the dynamic response of the beam is less than the cases in which $\tau = 0.002$. In case a, the vibrations in the beam are induced by larger external/thermal excitations than case b. Therefore, it seems from table 1 that the value of the dynamic response of the beam corresponding to case b is less than the dynamic response of the beam corresponding to case a. Also, as a parallel result to this, the difference between dynamic responses with/out delay for case a is larger than the corresponding difference for case b. A comparison of the controlled and uncontrolled dynamic responses with/out delay in table 1 demonstrates a substantial decrease as a result of the proposed control method. In table 2, some numerical results are given for the dynamic response of the beam with/out delay at $\xi = 0$. Due to the absence of internal damping in the system, the dynamic response is much more than the case with internal damping. Also, similarly to table 1, the dynamic response of the beam without delay is less than the case with delay for the undamped case.

In table 3, the dynamic response of the time-delayed controlled beam is given together with the control voltage spent for case a. As stated above, the difference between the dynamic responses in table 3 is an effect of the internal damping. The same observation is valid for the spent voltage to suppress the vibrations in the time-delayed controlled beam. In the case of the system with delay and $\xi = 0.001$, the control voltage spent in $(0, t_f)$ is less than the case without damping. Parallel results to the ones observed in table 1–3 are obtained from table 4 for case b in the system with delay.

The un/controlled displacements and velocities are plotted at the midpoint of the beam at $x = 0.5$ since their maximum occurs at this point at $t = 0$ owing to the displacement and moment boundary conditions. Therefore, the midpoint gives an idea about the transient behavior of the time-delayed controlled damped beam. By observing Figs. 2–3, it seems that the uncontrolled displacement displays a steady harmonic motion while the controlled displacement gradually decreases in case a. The same observations is valid for the uncontrolled velocity plotted in Figs. 4–5 where the velocity is effectively suppressed be-

Table 1: The values of $\mathcal{D}(t_f)$ at $\xi = 0.001$ for different values of μ_3 in case a and b.

μ_3	$\mathcal{D}^a_{(\tau=0.002)}$	$\mathcal{D}^a_{(\tau=0.000)}$	μ_3	$\mathcal{D}^b_{(\tau=0.002)}$	$\mathcal{D}^b_{(\tau=0.000)}$
10^2	3203.52	3202.10	10^2	231.91	231.80
10^1	214.65	212.38	10^1	10.62	10.45
10^0	29.00	27.90	10^0	1.82	1.74
10^{-1}	1.42	1.36	10^{-1}	0.813	0.808

Table 2: The values of $\mathcal{D}(t_f)$ at $\xi = 0$ for different values of μ_3 in case a and b.

μ_3	$\mathcal{D}^a_{(\tau=0.002)}$	$\mathcal{D}^a_{(\tau=0.000)}$	μ_3	$\mathcal{D}^b_{(\tau=0.002)}$	$\mathcal{D}^b_{(\tau=0.000)}$
10^2	3401.34	3399.82	10^2	246.2	246.0
10^1	223.227	220.73	10^1	11.1	10.9
10^0	30.643	29.462	10^0	1.92	1.83
10^{-1}	1.439	1.383	10^{-1}	0.8141	0.809

Table 3: The values of $\mathcal{D}(t_f)$ at $\tau = 0.002$ for different values of μ_3 in case a.

μ_3	$\mathcal{D}^a_{(\xi=0.0001)}$	$\mathcal{C}$	μ_3	$\mathcal{D}^a_{(\xi=0)}$	$\mathcal{C}$
10^2	3203.52	29.26	10^2	3401	32.55
10^1	214.65	140.2	10^1	223	149
10^0	29.00	203.2	10^0	30	215
10^{-1}	1.42	297.2	10^{-1}	1.44	314

Table 4: The values of $\mathcal{D}(t_f)$ at $\tau = 0.002$ for different values of μ_3 in case b.

μ_3	$\mathcal{D}^b_{(\xi=0.0001)}$	$\mathcal{C}$	μ_3	$\mathcal{D}^b_{(\xi=0)}$	$\mathcal{C}$
10^2	231.9	2.4	10^2	246.1	2.64
10^1	10.6	11.4	10^1	11.08	12.09
10^0	1.8	16	10^0	1.91	16.87
10^{-1}	0.81	21.6	10^{-1}	0.81	22.8

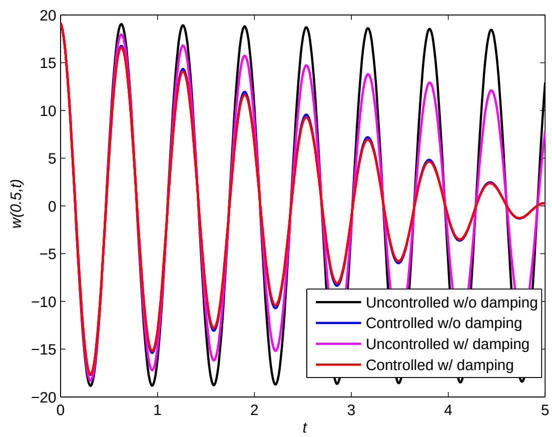

Figure 2: Controlled and uncontrolled displacements with delay at (0.5) for case a.

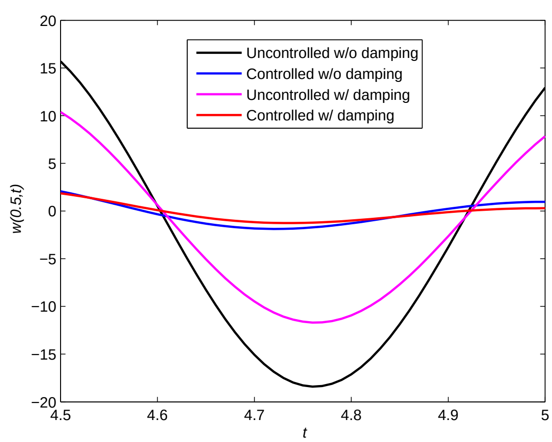

Figure 3: For $4.5 \leq t \leq 5$, controlled and uncontrolled displacements with delay at (0.5) for case a.

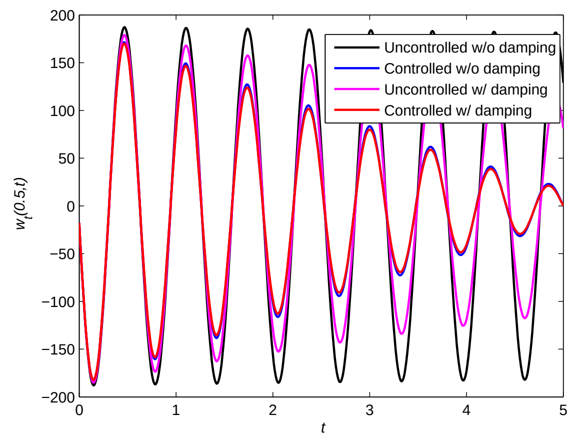

Figure 4: Controlled and uncontrolled velocities with delay at (0.5) for case a.

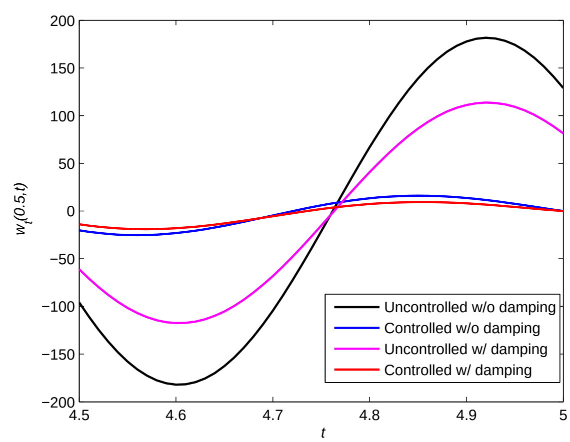

Figure 5: For $4.5 \leq t \leq 5$, controlled and uncontrolled velocities with delay at (0.5) for case a.

cause of control. For case b, the displacement and velocity of the beam are plotted in Figs. 6–7 and Figs. 8–9, respectively, illustrating the effect of the control and internal damping are illustrated. It can be concluded from Figs. 6–9 that the uncontrolled displacements and velocities are damped at a given terminal time $t_f = 5$ as a result of control actuation. Let us focus on the bandwidths in Figs. 2–5 and Figs. 6–9. Because the external excitation/thermal effect is larger in case a than case b, the bandwiths of the Figs. 2–3 and Figs. 4–5 is larger than the bandwidths of Figs. 6–7 and Figs. 8–9. By taking into consideration all tables and figures, it can be concluded that the control method introduced for the beam, with Kelvin-Voigt damping and a time delay in the control voltage, is effective and applicable for the beam with/out damping.

5 Conclusion

In this paper, the optimal control algorithm to suppress undesirable vibrations in a smart beam, with Kelvin-Voigt damping and a time delay in the control voltage function, subject to the displacement and moment boundary conditions, is investigated by using the maximum principle. The performance index of the control problem consists of a weighted quadratic functional of the dynamic responses of the beam to be minimized at a predetermined terminal time and a penalty term defined as the control voltage spent in the control process. By means of the maximum principle, the optimal control problem is transformed into a coupled system of partial differential equations in terms of state, adjoint and control variables subject to the boundary, initial and terminal conditions. The explicit solution

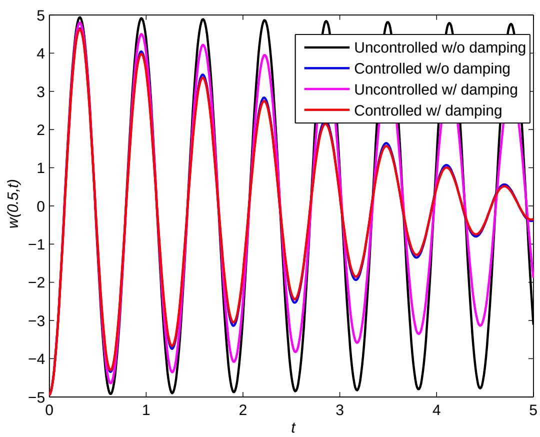

Figure 6: Controlled and uncontrolled displacements with delay at (0.5) for case b.

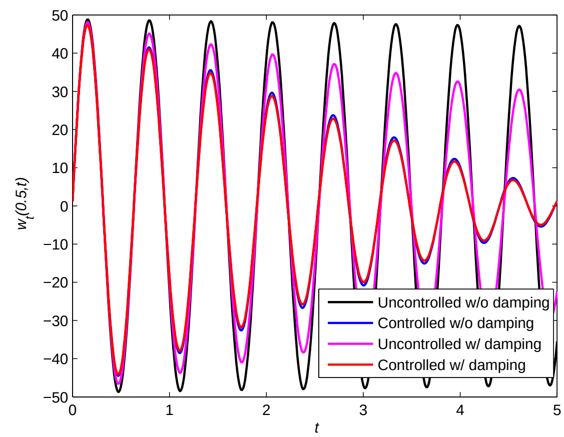

Figure 8: Controlled and uncontrolled velocities with delay at (0.5) for case b.

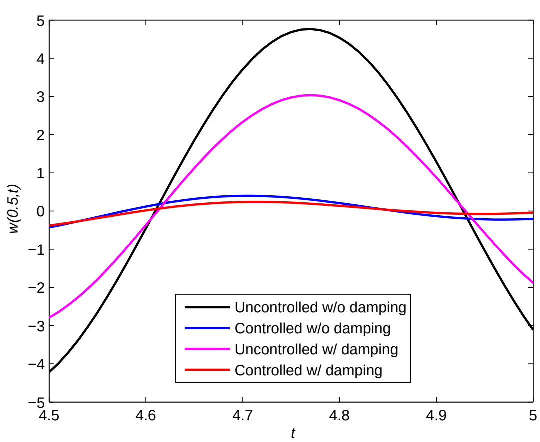

Figure 7: For $4.5 \le t \le 5$, controlled and uncontrolled displacements with delay at (0.5) for case b.

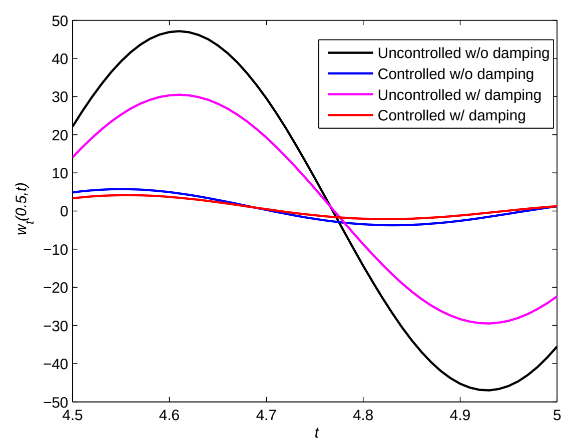

Figure 9: For $4.5 \le t \le 5$, controlled and uncontrolled velocities with delay at (0.5) for case b.

of the problem is sought by the eigenfunction expansion method. By using MATLAB, numerical results are given to demonstrate the robustness and efficiency of the proposed control algorithm.

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