

Research Article

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Numerical solution of two dimensional time fractional-order biological population model

DOI 10.1515/phys-2016-0021

Received Oct 08, 2015; accepted May 04, 2016

Abstract: In this work, we provide an approximate solution of a parabolic fractional degenerate problem emerging in a spatial diffusion of biological population model using a fractional variational iteration method (FVIM). Four test illustrations are used to show the proficiency and accuracy of the projected scheme. Comparisons between exact solutions and numerical solutions are presented for different values of fractional order α .

Keywords: Biological population model; Mittag-leffler function; Caputo fractional derivative; Lagrange multiplier; parabolic equation

PACS: 02.30.Jr; 02.60.-x; 04.20.Fy; 04.20.Jb.

1 Introduction

Fractional calculus is an integral branch of mathematics for arbitrary derivatives and integrals. Fractional differential equations are vitally important due to their proved and diverse uses in engineering and every part of science. Fractional differential equations are the best for modeling various processes in engineering and physical sciences. Since many standard models with integer-order derivatives involving nonlinear models, do not work sufficiently well in many cases. At the outset of this century, there has been an increase in the role of fractional calculus in countless arenas such as economics, mechanics, bioinformatics, chemistry, electricity, control analysis, signal and image processing, fluid flow, propagation of seismic waves etc. Many fields of academics in various branches involve unusual diffusion, control and vibration, random walk

with continuous time, Brownian motion and fractional Brownian motion, fractional kinetic model with neutron point, power law, Riesz potential etc and hence fractional calculus serves an important role in every part of science and technology. Many researchers developed various numerical methods to solve nonlinear phenomenon since analytic solution do not exist in every situation. In order to maintain precise and consistent solutions, many techniques have been applied to find the solution of the differential equations with fractional order derivative. Some of the neoteric numerical methods are finite element method, finite difference method, fractional differential transform method and the adomain decomposition method etc.

The Fractional variational iteration method (FVIM) is one of these novel approaches to solve nonlinear phenomenon of various fields. He [1–3] first proposed and solved fractional differential equations [4] with the help of VIM. Odibat *et al.* [5] used VIM to solve problems in fluid mechanics. Yulita *et al.* [6] solved Zakharov–Kuznetsov equations of fractional order. Inc [7] solved the space- and time-fractional Burgers differential equations. Freeborn *et al.* [8] applied a nonlinear least square fitting to extract the double -dispersion cole bioimpedance model. Lu [9] applied a variational iteration method to obtain approximate solution of the Fornberg–Whitham equation. Sakar *et al.* [10] and Prakash *et al.* [11] have made use of the variational iteration method to find numerical solutions of time-fractional Fornberg–Whitham equation and fractional coupled Burger’s equation respectively. Shakeri *et al.* [12] solved the two dimensional biological population model for the standard case when $\alpha = 1$ by a variational iteration method. Recently, Srivastava *et al.* [13] studied two dimensional time fractional-order biological population model by Fractional reduced differential transform method. The biological population model is also studied by Cheng *et al.* [14] for the standard case when $\alpha = 1$ by using the element-free kp-Ritz method and Kumar *et al.* [15] applied the Homotopy Analysis method to solve the Biological Population model.

The basic goal of the present effort is to make use of the Fractional Variational Iteration Method (FVIM) to find the solution of a time-fractional degenerate partial differential

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equation of parabolic type arising in the diffusion of the spatial type of biological model:

$$p_t^\alpha = p_{xx}^2 + p_{yy}^2 + f(p), \quad t \geq 0, \quad x, y \in IR, \quad (1)$$

subject to condition at initial stage $p(x, y, 0)$, where p is the density of the population and function f is the supply of population (births and deaths in that region).

Biologists have a sustained and strong belief that immigration or dispersal are important factors for the regulation of some specie's population. Gurtin and Maccamy [16] described the scattering of a race in a biological model in zone B assuming the functions of spot $X = (x, y)$ in B and time t ; where $p(x, y, t)$ is the density of population, $v(x, y, t)$ is the velocity of diffusion and $f(x, y, t)$ is the cause of population.

The function $p(x, y, t)$ indicates counting of characters on the field at location X and time t , per unit volume and total number of individuals at the sub region R of the region B can be found out by integrating over sub region R at any time t . Function $f(x, y, t)$ brings forth the ratio of fellows that are brought inly at the positions X per unit volume by births and deaths. $v(x, y, t)$ denotes the velocity of diffusion from one position to other position in the flow of population of those fellows which at any time t occupy the place X . The functions p , v and f must satisfy the law of population balance: for each sub region R of B at any time t as:

$$\frac{d^\alpha}{dt^\alpha} \int_R p dv + \int_{\partial R} p \mathbf{v} \cdot \hat{n} dA = \int_R f dV. \quad (2)$$

In this relation $\hat{n}$ is the normal unit vector outward side of boundary of the region R . This law presents that the changing rate of population in the region R plus the rate at which animals leave the boundary of R should be equal to the rate at which animals are right away to region R . Gurtin and Maccamy [16] proved it by assuming the conditions:

$$f = f(p), \quad v = -k(p)\nabla p, \quad (3)$$

where $k(p) > 0$ for $p > 0$. The ensuing nonlinear fractional partial differential equation with the density of p is attained as:

$$p_t^\alpha = \phi(p)_{xx} + \phi(p)_{yy} + f(p), \quad t \geq 0, \quad x, y \in IR. \quad (4)$$

Gurney *et al.* [17] applied a particular case of $\phi(p)$ for the field of population of animals. Movements in the field are due to either grown-up creatures compelled out by invaders or by youthful creatures just reaching maturity moving out of their ancestral homes to establish reproduction province of their own. In these two cases, it can

be inferred that these will be in a neighboring unoccupied zone. In the model at hand, drive will take place almost merely "down" the population density gradient, and this model will be more applicable in a high density population place than a low population density place. In this model, they assumed a rectangular place, in which a cattle may either dwell at its existing place or may change its position from high density population to low density population. The probability distribution in these two situations can be categorised with the aid of the magnitude of the population density gradient at the grid site concerned. This model leads to (1) with $\phi(p) = p^2$, i.e. the following equation:

$$p_t^\alpha = p_{xx}^2 + p_{yy}^2 + f(p), \quad t \geq 0, \quad x, y \in IR, \quad (5)$$

subject to the initial condition $p(x, y, 0)$. Some properties of equation (2) such as Holder estimates of its solutions are studied by Y.G. Lu [18]. Two models of constitutive equations for f are the Malthusian law

$$f = \mu p, \quad (\mu = \text{constant}), \quad (6)$$

and the Verhulst law

$$f = \mu p - \gamma p^2, \quad (\mu, \gamma = \text{constant}). \quad (7)$$

We study a more general form of f as $f(p) = hp^a(1 - rp^b)$ which yields

$$p_t^\alpha = p_{xx}^2 + p_{yy}^2 + hp^a(1 - rp^b), \quad t \geq 0, \quad x, y \in IR, \quad (8)$$

where α, β, h and r are real numbers. It can be noted that the Malthusian law and Verhulst law are particular situations obtained when $h = \mu, \alpha = 1, r = 0$ and $h = \mu, \alpha = \beta = 1, r = \frac{\gamma}{\mu}$ respectively.

2 Preliminaries

2.1 Definition

A real function $f(t)$, $t > 0$ is said to be in the space C_α , $\alpha \in R$ if there exists a real number $p(> \alpha)$, such that $f(t) = t^p f_1(t)$, where $f_1 \in C[0, \infty]$ clearly $C_\alpha \subset C_\beta$ if $\beta \leq \alpha$ [19–21].

2.2 Definition

A function $f(t)$, $t > 0$ is said to be in the space C_α^m , $m \in N \cup \{0\}$, if $f^{(m)} \in C_\alpha$ [19–21].

2.3 Definition

The left sided Riemann-Liouville fractional integral of order $\mu > 0$, of a function $f \in C_\alpha$, $\alpha \geq -1$ is defined as [22, 23]

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\mu}} d\tau = \frac{1}{\Gamma(1+\mu)} \int_0^t f(\tau) (d\tau)^\mu$$

where $I^0 f(t) = f(t)$.

2.4 Definition

The (left sided) Caputo fractional derivative of $f, f \in C_{-1}^m$, $m \in \mathbb{N} \cup \{0\}$ [22, 23],

$$D_t^\mu f(t) = \begin{cases} I^{m-\mu} f^{(m)}(t), & m-1 < \mu < m, m \in \mathbb{N} \\ \frac{d^m(f(t))}{dt^m}, & \mu = m. \end{cases}$$

Note that [20–24]

- (a) $I_t^\alpha f(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x, s) ds, \alpha, t > 0$.
- (b) $D_t^\alpha u(x, t) = I_t^{m-\alpha} \frac{\partial^m u(x, t)}{\partial t^m} f(t), m-1 < \alpha < m$.
- (c) $I^\mu t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\mu)} t^{\gamma+\mu}$.

2.5 Definition

The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by the following series representation, valid in the whole complex plane [24, 25]: $E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+n\alpha)}, \alpha > 0, z \in \mathbb{C}$.

3 Fractional variational iteration method for fractional order biological population model

Consider the following two dimensional time fractional order biological population model:

$$p_t^\alpha = p_{xx}^2 + p_{yy}^2 + hp^\alpha (1 - rp^b), \quad t \geq 0, x, y \in \mathbb{R}. \quad (9)$$

According to the FVIM, the correction functional for this equation is as follows [3]:

$$p_{n+1}(x, y, t) = p_n + \frac{1}{\Gamma(1+\alpha)} \int_0^t \lambda \left(\frac{\partial^\alpha p_n}{\partial \tau^\alpha} - \frac{\partial^2 \tilde{p}_n^2}{\partial x^2} - \frac{\partial^2 \tilde{p}_n^2}{\partial y^2} - h \left(\tilde{p}_n^a (1 - r\tilde{p}_n^b) \right) \right) (d\tau)^\alpha. \quad (10)$$

Now by variational theory λ , must satisfy $\frac{\partial^\alpha \lambda}{\partial \tau^\alpha} = 0$ and $\lambda + 1|_{\tau=t} = 0$. From these equations, we obtain that $\lambda = -1$ and

a new correction functional emerges:

$$p_{n+1}(x, y, t) = p_n(x, y, t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^\alpha p_n(x, y, \tau)}{\partial \tau^\alpha} - \frac{\partial^2 p_n^2}{\partial x^2} - \frac{\partial^2 p_n^2}{\partial y^2} - h \left(p_n^a (1 - rp_n^b) \right) \right) (d\tau)^\alpha. \quad (11)$$

We can build consecutive approximations p_n , $n \geq 0$ by using λ , a common Lagrange's multiplier, that can be obtained by variational theory. The functions $\tilde{p}_n$ are restricted variation i.e. $\delta \tilde{p}_n = 0$ Consequently, first we elect the Lagrange multiplier λ , which can be obtained using integration by parts. In this way we can obtain sequences $p_{n+1}(x, y, t)$, $n \geq 0$ of the solution and finally we can obtain the solution as $p(x, y, t) = \lim_{n \rightarrow \infty} p_n(x, y, t)$.

4 Test Examples

In the present section, we use the projected technique on some test examples.

Example 1. Consider the ensuing time fractional-order biological population model $\frac{\partial^\alpha p}{\partial t^\alpha} = \frac{\partial^2 p^2}{\partial x^2} + \frac{\partial^2 p^2}{\partial y^2} + hp$, which can be obtained by putting $a = 1, r = 0, b = 1$ in (8), subject to the given condition $p(x, y, 0) = \sqrt{xy}$. Now by the given initial condition:

$$\begin{aligned} p_0(x, y, t) &= \sqrt{xy} \\ p_1(x, y, t) &= \sqrt{xy} \left(1 + \frac{ht^\alpha}{\Gamma(1+\alpha)} \right) \\ p_2(x, y, t) &= \sqrt{xy} \left(1 + \frac{ht^\alpha}{\Gamma(1+\alpha)} + \frac{h^2 t^{2\alpha}}{\Gamma(1+2\alpha)} \right) \\ p_3(x, y, t) &= \sqrt{xy} \left(1 + \frac{ht^\alpha}{\Gamma(1+\alpha)} + \frac{h^2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{h^3 t^{3\alpha}}{\Gamma(1+3\alpha)} \right) \\ p_n(x, y, t) &= \sqrt{xy} \left(1 + \frac{ht^\alpha}{\Gamma(1+\alpha)} + \frac{h^2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{h^3 t^{3\alpha}}{\Gamma(1+3\alpha)} \right. \\ &\quad \left. + \dots + \frac{h^n t^{n\alpha}}{\Gamma(1+n\alpha)} \right) \\ p(x, y, t) &= \lim_{n \rightarrow \infty} p_n(x, y, t) = (E_\alpha(ht^\alpha))\sqrt{xy}, \end{aligned}$$

which is the exact solution.

Fig. 1 demonstrates the comparison between exact solution and approximate solution for different values of fractional order $\alpha = 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}$ when $h = \frac{1}{5}$. Fig. 1(a) shows the comparison when $t = 0.1$, Fig. 1(b) when $t = 0.2$ and Fig. 1(c) when $t = 0.3$. Fig. 2 represents the graphical solution for $\alpha = 1$ when $t = 0.1, 0.2$ and 0.3 respectively. It can be observed from Fig. 1 that the exact and numerical

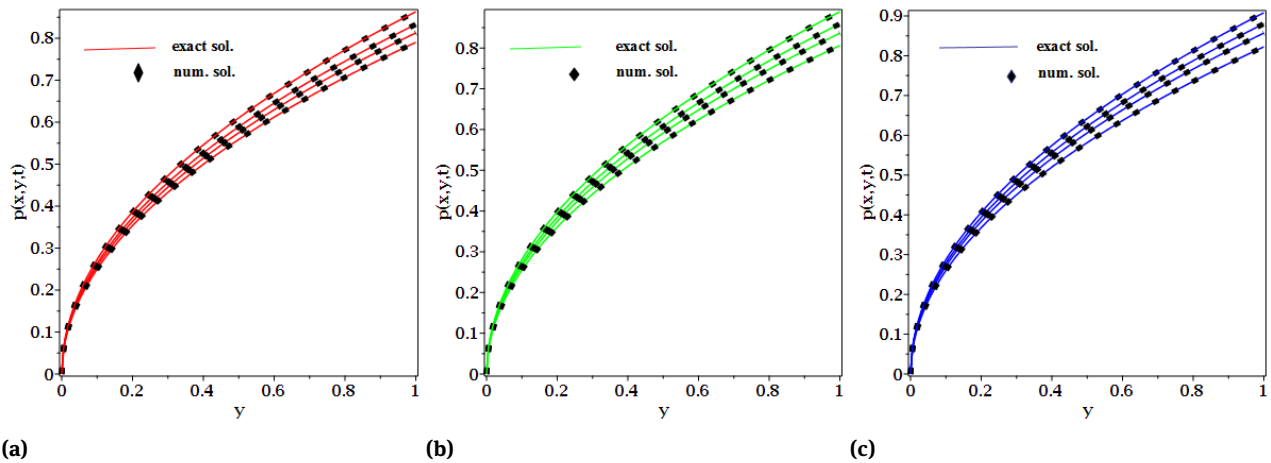


Figure 1: Comparison between exact and approximate solution for different values of fractional order α for example 1.

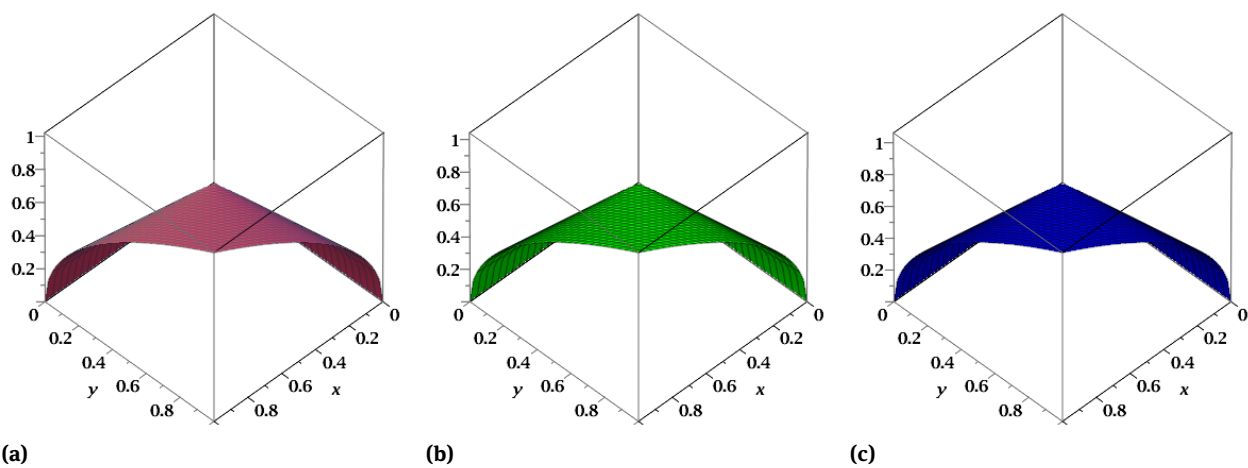


Figure 2: 3D-plot for different values of $t = 0.1, 0.2, 0.3$ when $\alpha = 1$ for example 1.

solutions are almost identical for different values of fractional order α . Table 1 shows the comparison of absolute error between the exact solution and the tenth approximate solution for different value of fractional order α when $t = 0.1, 0.2$ and 0.3 respectively.

Example 2. Consider the ensuing time fractional order biological population model $\frac{\partial^\alpha p}{\partial t^\alpha} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + p$, which can be obtained by putting $h = 1, a = 1, r = 0, b = 1$ in (8), subject to given condition $p(x, y, 0) = \sqrt{\sin(x) \sinh(y)}$. Now by the given initial condition:

$$p_0(x, y, t) = \sqrt{\sin(x) \sinh(y)}$$

$$p_1(x, y, t) = p_0(x, y, t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^\alpha p(x, y, \tau)}{\partial \tau^\alpha} \right)$$

$$\begin{aligned} & - \frac{\partial^2 p^2(x, y, \tau)}{\partial x^2} - \frac{\partial^2 p^2(x, y, \tau)}{\partial y^2} \\ & - p(x, y, \tau) \Big) (d\tau)^\alpha \\ & = \left(1 + \frac{t^\alpha}{\Gamma(1+\alpha)} \right) \sqrt{(\sin(x))(\sinh(y))} \\ p_2(x, y, t) &= \left(1 + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \right) \sqrt{\sin(x) \sinh(y)} \\ p_3(x, y, t) &= \left(1 + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \right. \\ & \quad \left. + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right) \sqrt{\sin(x) \sinh(y)} \\ p_n(x, y, t) &= \left(1 + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right. \\ & \quad \left. + \dots + \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \right) \sqrt{\sin(x) \sinh(y)} \end{aligned}$$

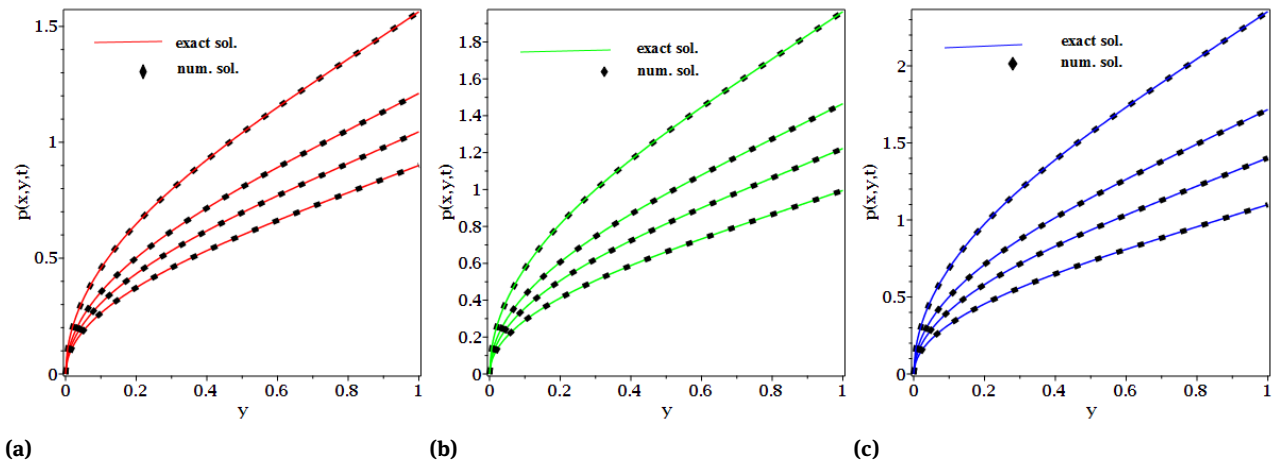


Figure 3: Comparison between exact and approximate solution for different values of fractional order α for example 2.

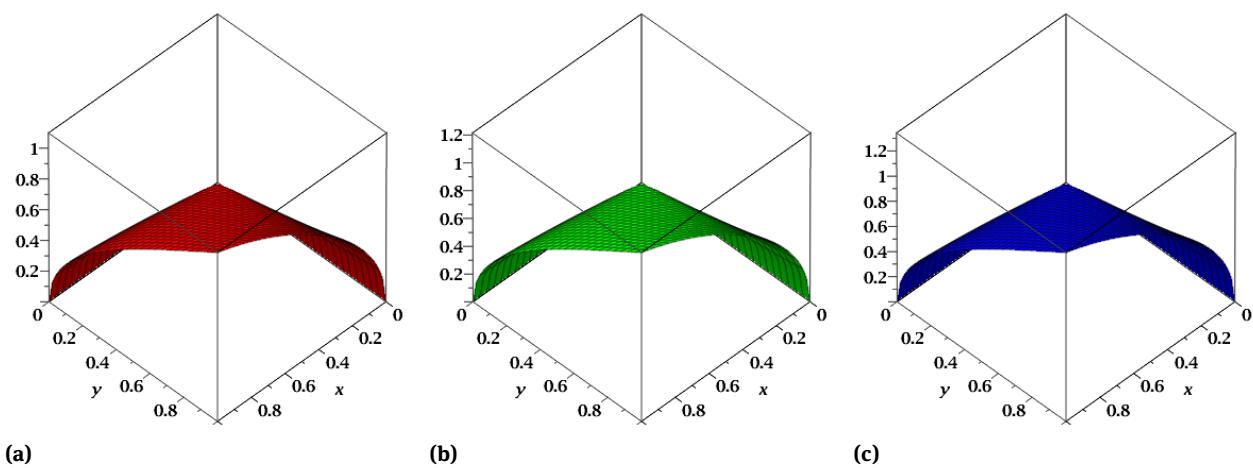


Figure 4: 3D-plot for different values of $t = 0.1, 0.2, 0.3$ when $\alpha = 1$ for example 2.

$$p(x, y, t) = \lim_{n \rightarrow \infty} p_n(x, y, t) = (E_\alpha(t^\alpha)) \sqrt{\sin(x) \sinh(y)},$$

which is the exact solution.

Fig. 3 demonstrates the comparison between exact solution and approximate solution for different values of fractional order $\alpha = 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}$. Fig. 3(a) shows the comparison when $t = 0.1$, Fig. 3(b) when $t = 0.2$ and Fig. 3(c) when $t = 0.3$. Fig. 4 represents graphical solution for $\alpha = 1$ when $t = 0.1, 0.2$ and 0.3 respectively. It can be observed from Fig. 3 that the exact and numerical solutions are almost identical for different values of fractional order α . Table 2. shows the comparison of absolute error between the exact solution and the tenth approximate solution for different value of fractional order α when $t = 0.1, 0.2$ and 0.3 respectively.

Example 3. Consider the ensuing time fractional- order biological population model $\frac{\partial^\alpha p}{\partial t^\alpha} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + p(1 - \lambda p)$ which can be obtained by putting $h = 1, a = 1, r = \lambda, b = 1$ in (8) subject to given condition

$$p(x, y, 0) = e^{\frac{\sqrt{\lambda}(x+y)}{2\sqrt{2}}}. \text{ Now by the given initial condition:}$$

$$p_0(x, y, t) = e^{\frac{\sqrt{\lambda}(x+y)}{2\sqrt{2}}}$$

$$p_1(x, y, t) = p_0 - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^\alpha p_0}{\partial \tau^\alpha} - \frac{\partial^2 p_0^2}{\partial x^2} - \frac{\partial^2 p_0^2}{\partial y^2} \right. \\ \left. - p_0(1 - \lambda p_0) \right) (d\tau)^\alpha$$

$$= \left(1 + \frac{t^\alpha}{\Gamma(1+\alpha)} \right) e^{\frac{\sqrt{\lambda}(x+y)}{2\sqrt{2}}}$$

$$p_2(x, y, t) = \left(1 + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \right) e^{\frac{\sqrt{\lambda}(x+y)}{2\sqrt{2}}}$$

Table 1: Absolute error $|p - p_{10}|$.

x	y	$t = 0.1$			$t = 0.2$			$t = 0.3$		
		1	2/3	1/2	1/3	1	2/3	1/2	1/3	1/3
0.1	0.2	0	0	0	4.4×10^{-14}	2.7×10^{-17}	0	1.5×10^{-15}	5.8×10^{-13}	2.7×10^{-17}
0.1	0.6	0	0	0	7.7×10^{-14}	5.5×10^{-17}	0	2.6×10^{-15}	1.0×10^{-12}	5.5×10^{-17}
0.1	1.0	0	0	0	1.0×10^{-13}	5.5×10^{-17}	0	3.4×10^{-15}	1.2×10^{-12}	5.5×10^{-17}
0.5	0.2	0	0	0	1.0×10^{-13}	5.5×10^{-17}	0	3.4×10^{-15}	1.2×10^{-12}	5.5×10^{-17}
0.5	0.6	0	0	0	1.7×10^{-13}	1.1×10^{-16}	0	5.9×10^{-15}	2.2×10^{-12}	1.1×10^{-16}
0.5	1.0	0	0	0	2.2×10^{-13}	2.2×10^{-16}	0	7.6×10^{-15}	2.9×10^{-12}	1.1×10^{-16}
0.9	0.2	0	0	0	1.3×10^{-13}	1.1×10^{-16}	0	4.6×10^{-15}	1.7×10^{-12}	1.1×10^{-16}
0.9	0.6	0	0	0	2.3×10^{-13}	2.2×10^{-16}	0	7.9×10^{-15}	3.0×10^{-12}	1.1×10^{-16}
0.9	1.0	0	0	0	3.0×10^{-13}	2.2×10^{-16}	0	1.0×10^{-14}	3.8×10^{-12}	2.2×10^{-16}

Table 2: Absolute error $|p - p_{10}|$.

x	y	$t = 0.1$			$t = 0.2$			$t = 0.3$		
		1	2/3	1/2	1/3	1	2/3	1/2	1/3	1/3
0.1	0.2	2.8×10^{-17}	7.0×10^{-13}	1.8×10^{-9}	2.9×10^{-6}	5.6×10^{-17}	1.2×10^{-10}	8.6×10^{-8}	4.1×10^{-5}	6.4×10^{-15}
0.1	0.6	5.6×10^{-17}	1.2×10^{-12}	3.2×10^{-9}	5.1×10^{-6}	1.1×10^{-16}	2.1×10^{-10}	1.5×10^{-7}	7.3×10^{-5}	1.1×10^{-14}
0.1	1.0	5.6×10^{-17}	1.7×10^{-12}	4.3×10^{-9}	7.0×10^{-6}	1.1×10^{-16}	2.8×10^{-10}	2.1×10^{-7}	9.9×10^{-5}	1.5×10^{-14}
0.5	0.2	5.6×10^{-17}	1.5×10^{-12}	3.9×10^{-9}	6.3×10^{-6}	1.7×10^{-16}	2.6×10^{-10}	1.9×10^{-7}	8.9×10^{-5}	1.4×10^{-14}
0.5	0.6	1.1×10^{-16}	2.7×10^{-12}	6.9×10^{-9}	1.1×10^{-5}	2.2×10^{-16}	4.5×10^{-10}	3.3×10^{-7}	1.6×10^{-4}	2.5×10^{-14}
0.5	1.0	1.1×10^{-16}	3.7×10^{-12}	9.4×10^{-9}	1.5×10^{-5}	3.3×10^{-16}	6.2×10^{-10}	4.5×10^{-7}	2.2×10^{-4}	3.4×10^{-14}
0.9	0.2	5.6×10^{-17}	2.0×10^{-12}	5.0×10^{-9}	8.6×10^{-6}	1.7×10^{-16}	3.3×10^{-10}	2.4×10^{-7}	1.1×10^{-4}	1.8×10^{-14}
0.9	0.6	1.1×10^{-17}	3.5×10^{-12}	8.9×10^{-9}	1.4×10^{-5}	3.3×10^{-16}	5.8×10^{-10}	4.3×10^{-7}	2.0×10^{-4}	3.2×10^{-14}
0.9	1.0	2.2×10^{-17}	4.7×10^{-12}	1.2×10^{-8}	2.0×10^{-5}	4.4×10^{-16}	7.9×10^{-10}	5.8×10^{-7}	2.8×10^{-4}	4.4×10^{-14}

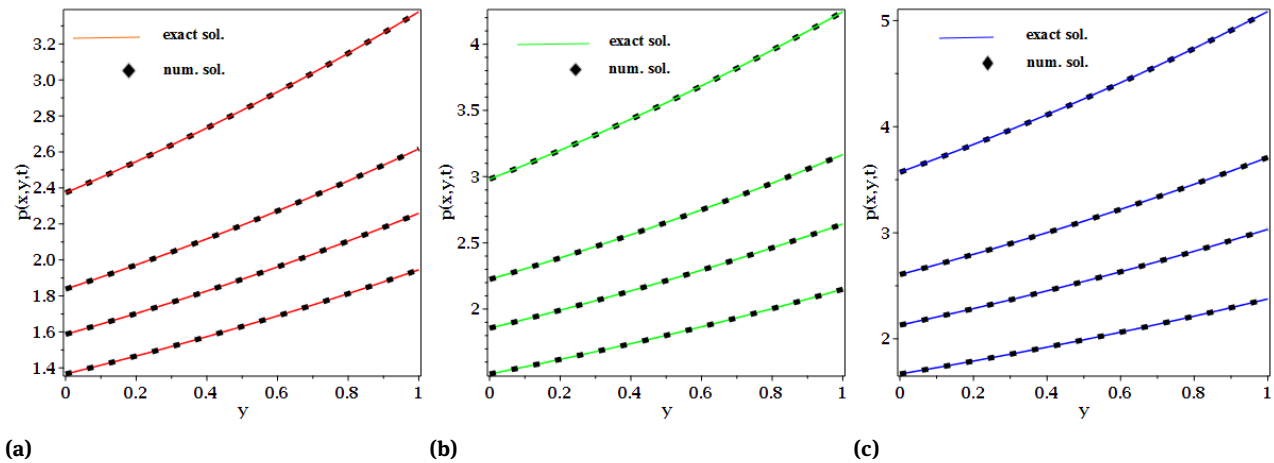


Figure 5: Comparison between exact and approximate solution for different values of fractional order α for example 3.

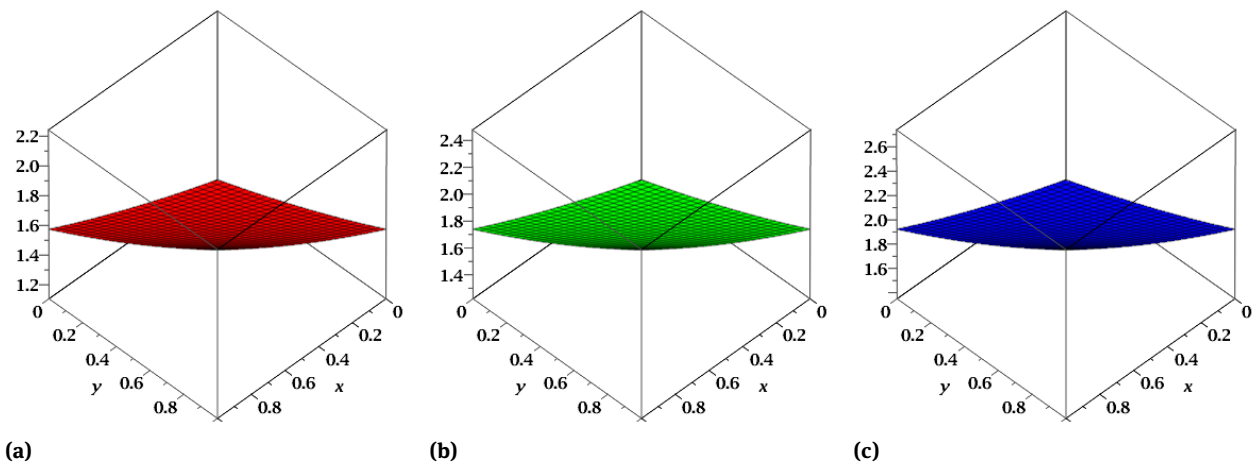


Figure 6: 3D-plot for different values of $t = 0.1, 0.2, 0.3$ when $\alpha = 1$ for example 3.

$$p_3(x, y, t) = \left(1 + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right) e^{\frac{\sqrt{\lambda}(x+y)}{2\sqrt{2}}}$$

$$p_n(x, y, t) = \left(1 + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \dots + \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \right) e^{\frac{\sqrt{\lambda}(x+y)}{2\sqrt{2}}}$$

$$p(x, y, t) = \lim_{n \rightarrow \infty} p_n(x, y, t) = (E_\alpha(t^\alpha)) e^{\frac{\sqrt{\lambda}(x+y)}{2\sqrt{2}}},$$

which is the exact solution.

Fig. 5 demonstrates the comparison between exact solution and approximate solution for different values of fractional order $\alpha = 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}$ when $\lambda = 1$. Fig. 5(a) shows the comparison when $t = 0.1$, Fig. 5(b) when $t = 0.2$ and Fig. 5(c) when $t = 0.3$. Fig. 6 represents graphical solu-

tion for $\alpha = 1$ when $t = 0.1, 0.2$ and 0.3 respectively. It can be observed from Fig. 5 that the exact and numerical solutions are almost identical for different values of fractional order α . Table 3. shows the comparison of absolute error between the exact solution and the tenth approximate solution for different value of fractional order α when $t = 0.1, 0.2$ and 0.3 respectively.

Example 4. Consider the ensuing linear time fractional-order biological population model $\frac{\partial^\alpha p}{\partial t^\alpha} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} - p(\frac{8p}{9} + 1)$ which can be obtained by putting $h = 1, a = 1, r = \frac{8}{9}, b = 1$ (8) subject to given condition $p(x, y, 0) = e^{\frac{x+y}{3}}$. Now by the given initial condition:

$$p_0(x, y, t) = e^{\frac{x+y}{3}}$$

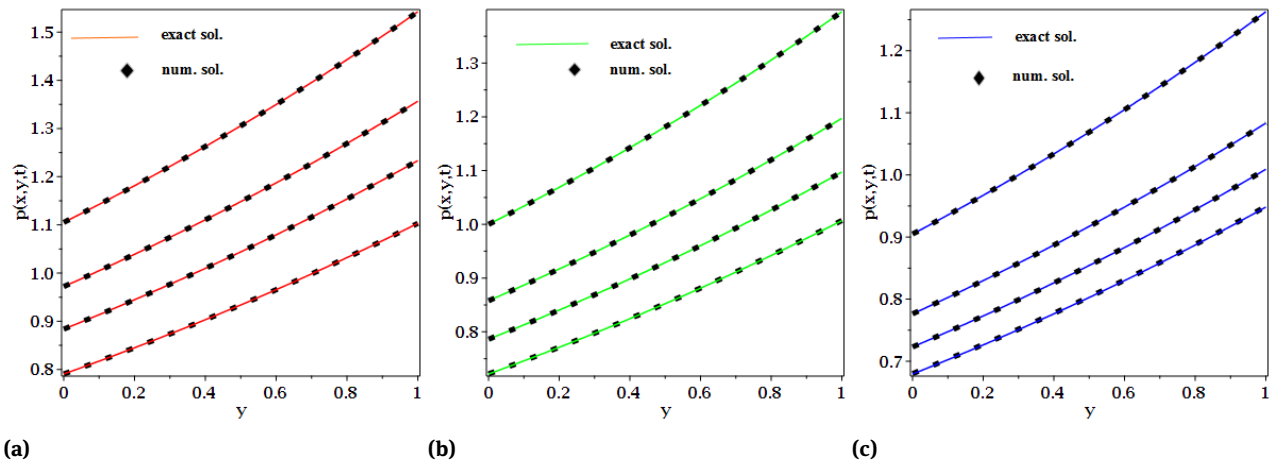


Figure 7: Comparison between exact and approximate solution for different values of fractional order α for example 4.

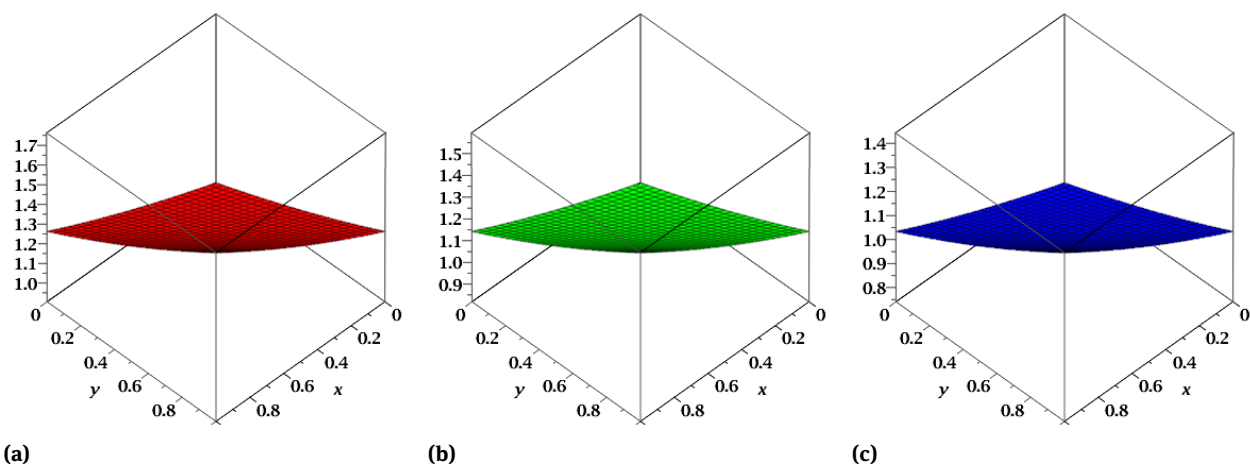


Figure 8: 3D-plot for different values of $t = 0.1, 0.2, 0.3$ when $\alpha = 1$ for example 4.

$$\begin{aligned}
 p_1(x, y, t) &= p_0 - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^\alpha p_0}{\partial \tau^\alpha} - \frac{\partial^2 p_0^2}{\partial x^2} - \frac{\partial^2 p_0^2}{\partial y^2} \right. \\
 &\quad \left. + p_0 \left(\frac{8p_0}{9} + 1 \right) \right) (d\tau)^\alpha \\
 &= \left(1 - \frac{t^\alpha}{\Gamma(1+\alpha)} \right) e^{\frac{x+y}{3}} \\
 p_2(x, y, t) &= \left(1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \right) e^{\frac{x+y}{3}} \\
 p_3(x, y, t) &= \left(1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right) e^{\frac{x+y}{3}} \\
 p_n(x, y, t) &= \left(1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right. \\
 &\quad \left. + \dots + \frac{(-1)^n t^{n\alpha}}{\Gamma(1+n\alpha)} \right) e^{\frac{x+y}{3}} \\
 p(x, y, t) &= \lim_{n \rightarrow \infty} p_n(x, y, t) = (E_\alpha(-t^\alpha)) e^{\frac{x+y}{3}},
 \end{aligned}$$

which is the exact solution.

Fig. 7 demonstrates the comparison between the exact solution and the approximate solution for different values of fractional order $\alpha = 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}$ when $\lambda = 1$. Fig. 7(a) shows the comparison when $t = 0.1$, Fig. 7(b) when $t = 0.2$ and Fig. 7(c) when $t = 0.3$. Fig. 8 represents graphical solution for $\alpha = 1$ when $t = 0.1, 0.2$ and 0.3 respectively. It can be observed from Fig. 7 that the exact and numerical solutions are almost identical for different values of fractional order α . Table 4 shows the comparison of absolute error between the exact solution and the approximate solution for different value of fractional order α when $t = 0.1, 0.2$ and 0.3 respectively.

Table 3: Absolute error $|p - p_{10}|$.

x	y	t = 0.1				t = 0.2				t = 0.3			
		1	2/3	1/2	1/3	1	2/3	1/2	1/3	1	2/3	1/2	1/3
$\alpha \rightarrow$	0.1	2.2×10^{-16}	5.5×10^{-12}	1.4×10^{-8}	2.3×10^{-5}	4.4×10^{-16}	9.1×10^{-10}	6.7×10^{-7}	3.2×10^{-4}	5.0×10^{-14}	1.8×10^{-8}	6.6×10^{-6}	1.5×10^{-3}
	0.1	4.4×10^{-16}	6.3×10^{-12}	1.6×10^{-8}	2.6×10^{-5}	4.4×10^{-16}	1.1×10^{-9}	7.7×10^{-7}	3.7×10^{-4}	5.8×10^{-14}	2.1×10^{-8}	7.6×10^{-6}	1.8×10^{-3}
	0.1	2.2×10^{-16}	7.3×10^{-12}	1.9×10^{-8}	3.0×10^{-5}	6.7×10^{-16}	1.2×10^{-9}	8.9×10^{-7}	4.2×10^{-4}	6.7×10^{-14}	2.4×10^{-8}	8.7×10^{-6}	2.0×10^{-3}
	0.5	4.4×10^{-16}	6.3×10^{-12}	1.6×10^{-8}	2.6×10^{-5}	4.4×10^{-16}	1.1×10^{-9}	7.7×10^{-7}	3.7×10^{-4}	5.8×10^{-14}	2.1×10^{-8}	7.6×10^{-6}	1.8×10^{-3}
	0.5	2.2×10^{-16}	7.3×10^{-12}	1.9×10^{-8}	3.0×10^{-5}	6.7×10^{-16}	1.2×10^{-9}	8.9×10^{-7}	4.2×10^{-4}	6.7×10^{-14}	2.4×10^{-8}	8.7×10^{-6}	2.0×10^{-3}
	0.5	4.4×10^{-16}	8.4×10^{-12}	2.1×10^{-8}	3.5×10^{-5}	8.9×10^{-16}	1.4×10^{-9}	1.0×10^{-7}	4.9×10^{-4}	7.7×10^{-14}	2.8×10^{-8}	1.0×10^{-5}	2.3×10^{-3}
	0.9	2.2×10^{-16}	7.3×10^{-12}	1.9×10^{-8}	3.0×10^{-5}	6.7×10^{-16}	1.2×10^{-9}	8.9×10^{-7}	4.2×10^{-4}	6.7×10^{-14}	2.4×10^{-8}	8.7×10^{-6}	2.0×10^{-3}
	0.5	4.4×10^{-16}	8.4×10^{-12}	2.1×10^{-8}	3.5×10^{-5}	8.9×10^{-16}	1.4×10^{-9}	1.0×10^{-7}	4.9×10^{-4}	7.7×10^{-14}	2.8×10^{-8}	1.0×10^{-5}	2.3×10^{-3}
	1.0	4.4×10^{-16}	9.7×10^{-12}	2.5×10^{-8}	4.0×10^{-5}	8.9×10^{-16}	1.6×10^{-9}	1.2×10^{-7}	5.6×10^{-4}	8.9×10^{-14}	3.2×10^{-8}	1.2×10^{-5}	2.7×10^{-3}

Table 4: Absolute error $|p - p_{10}|$.

x	y	t = 0.1			t = 0.2			t = 0.3					
		1	2/3	1/2	1/3	1	2/3	1/2	1/3	1	2/3	1/2	1/3
$\alpha \rightarrow$	0.1	0	4.9×10^{-12}	1.1×10^{-8}	1.3×10^{-5}	5.6×10^{-16}	7.7×10^{-10}	4.7×10^{-7}	1.5×10^{-4}	4.8×10^{-14}	1.5×10^{-8}	4.2×10^{-6}	6.4×10^{-4}
	0.2	0	5.6×10^{-12}	1.2×10^{-8}	1.4×10^{-5}	6.7×10^{-16}	8.8×10^{-10}	5.3×10^{-7}	1.7×10^{-4}	5.5×10^{-14}	1.7×10^{-8}	4.8×10^{-6}	7.3×10^{-4}
	0.6	0	6.4×10^{-12}	1.4×10^{-8}	1.6×10^{-5}	6.7×10^{-16}	1.0×10^{-9}	6.1×10^{-7}	2.0×10^{-4}	6.3×10^{-14}	1.9×10^{-8}	5.5×10^{-6}	8.4×10^{-4}
	1.0	0	5.6×10^{-12}	1.2×10^{-8}	1.4×10^{-5}	6.7×10^{-16}	8.8×10^{-10}	5.3×10^{-7}	1.7×10^{-4}	5.5×10^{-14}	1.7×10^{-8}	4.8×10^{-6}	7.3×10^{-4}
0.5	0.2	0	6.4×10^{-12}	1.4×10^{-8}	1.6×10^{-5}	6.7×10^{-16}	1.0×10^{-9}	6.1×10^{-7}	2.0×10^{-4}	6.3×10^{-14}	1.9×10^{-8}	5.5×10^{-6}	8.4×10^{-4}
	0.6	0	7.3×10^{-12}	1.6×10^{-8}	1.9×10^{-5}	8.9×10^{-16}	1.1×10^{-9}	6.9×10^{-7}	2.3×10^{-4}	7.2×10^{-14}	2.2×10^{-8}	6.2×10^{-6}	9.6×10^{-4}
	1.0	0	6.4×10^{-12}	1.4×10^{-8}	1.6×10^{-5}	6.7×10^{-16}	1.0×10^{-9}	6.1×10^{-7}	2.0×10^{-4}	6.3×10^{-14}	1.9×10^{-8}	5.5×10^{-6}	8.4×10^{-4}
	0.9	0	7.3×10^{-12}	1.6×10^{-8}	1.9×10^{-5}	8.9×10^{-16}	1.1×10^{-9}	6.9×10^{-7}	2.3×10^{-4}	7.2×10^{-14}	2.2×10^{-8}	6.2×10^{-6}	9.6×10^{-4}
0.9	0.6	0	8.4×10^{-12}	1.8×10^{-8}	2.1×10^{-5}	1.1×10^{-15}	1.3×10^{-9}	7.9×10^{-7}	2.6×10^{-4}	8.2×10^{-14}	2.5×10^{-8}	7.1×10^{-6}	1.1×10^{-4}
	1.0	0	8.4×10^{-12}	1.8×10^{-8}	2.1×10^{-5}	1.1×10^{-15}	1.3×10^{-9}	7.9×10^{-7}	2.6×10^{-4}	8.2×10^{-14}	2.5×10^{-8}	7.1×10^{-6}	1.1×10^{-4}

5 Conclusion

In this paper, two dimensional time fractional-order biological population model is investigated, using the Fractional variational iteration method (FVIM). The numerical results are in excellent agreement with exact solutions. Numerical results shows that FVIM is a powerful and reliable algorithm for solving the approximate solution of the two dimensional time fractional-order biological population model as sequences of the approximate solution converge to the exact solution rapidly. The main advantage of this technique over other methods is that this technique does not require discretization of variables, perturbation and any other restrictive assumptions. Therefore FVIM is easy to implement and computationally very attractive over other methods.

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