Rapid Communication Open Access

Bin Ye*, Tingzhong Zhang, Liang Qiu, and Xuesong Wang*

Quantum Discord and Entanglement in Grover Search Algorithm with Static Imperfections

DOI 10.1515/phys-2016-0020
Received November 15, 2015; accepted April 25, 2016

Abstract: Imperfections and noise in realistic quantum computers may seriously affect the accuracy of quantum algorithms. In this article we explore the impact of static imperfections on quantum entanglement as well as non-entangled quantum correlations in Grover's search algorithm. Using the metrics of concurrence and geometric quantum discord, we show that both the evolution of entanglement and quantum discord in Grover algorithm can be restrained with the increasing strength of static imperfections. For very weak imperfections, the quantum entanglement and discord exhibit periodic behavior, while the periodicity will most certainly be destroyed with stronger imperfections. Moreover, entanglement sudden death may occur when the strength of static imperfections is greater than a certain threshold.

Keywords: quantum computation; concurrence; quantum correlation; grover algorithm; static imperfection

PACS: 03.67.Mn; 03.67.Ac

1 Introduction

Quantum algorithms, both in certainty and probability, have great advantages over classical algorithm in the potential of computing. It is believed that quantum algorithm provides speedup depending largely on quantum entanglement and other quantum phenomenon. Entanglement - a special kind of quantum state, is used to describe a compound system which contains two or more members in the quantum system. However, recent studies have demonstrated that there are other quantum correlations which cannot be captured by entanglement [1, 2]. In order to quantify and describe these non-entangled quantum correlations, quantum discord was proposed by Zurek et al [3–5].

In realistic situations, a quantum computing system is not an ideally closed system and thus external environments will have an impact on it. Even if a quantum computing system is isolated, there will be unavoidable interqubit couplings in the hardware of quantum computers - which is called static imperfections [6, 7]. These static imperfections may lead to many-body quantum chaos, which strongly modifies the hardware properties of realistic quantum computers. Therefore, studying the behavior of quantum entanglement and quantum discord under static imperfections has far-reaching consequences.

Grover's quantum search algorithm plays an important role in the advance of quantum computation and quantum information because it provides a proof that quantum computers are faster than classical ones in database searching. The algorithm uses $O(\sqrt{N})$ (N is the size of search space) evaluations to find a unique element in an unstructured database, while classical algorithms need $O(N)$ steps [8]. Thus Grover search algorithm achieves a quadratic speedup over the classical algorithm in unsorted database searching. For the global stability diagram of reliable operability of the above mentioned algorithm the reader is referred to [9]. Using the methods of quantum trajectories or perturbative approach, the effects of dissipative decoherence on the accuracy of the Grover quantum search algorithm have been analyzed in Refs. [10–12]. For the unitary noise resulting from tiny fluctuations and drift in the parameters of the quantum components performing the Grover search algorithm, see [13]. The amount of entanglement of the multiqubit quantum states was calculated in Ref. [14] and the scale invariance property of entanglement dynamics was also exhibited.

As quantum discord is a novel and not yet fully understood measure of quantum correlations, we will mainly fo-

***Corresponding Author: Bin Ye:** School of Information and Electrical Engineering, China University of Mining and Technology, Xuzhou, 221116, China, E-mail: yebin@cumt.edu.cn
Tingzhong Zhang: School of Information and Electrical Engineering, China University of Mining and Technology, Xuzhou, 221116, China, E-mail: 513553268@qq.com
Liang Qiu: School of Science, China University of Mining and Technology, Xuzhou, 221116, China, E-mail: lqiu@cumt.edu.cn
***Corresponding Author: Xuesong Wang:** School of Information and Electrical Engineering, China University of Mining and Technology, Xuzhou, 221116, China, E-mail: wangxuesongcumt@163.com

cus on the evolution of it as well as quantum entanglement in the disturbed Grover search algorithm. Specifically, we will examine the behaviors of entanglement and quantum discord in the presence of static interqubit couplings. The paper is organized as follows. In Sec.2, we briefly review the Grover search algorithm. In Sec.3, we introduce quantum entanglement, quantum correlations and their measurements. The effects of static imperfections on quantum entanglement and quantum discord in Grover search algorithm are discussed in Sec.4. We conclude the paper with some discussions in Sec.5.

2 The Grover Search Algorithm

The search problem for an unstructured database is a fundamental and practical problem which appears in many different fields. It was first shown by Grover that searching a database using quantum mechanics can be substantially faster than any classical search of unsorted data.

Grover's search algorithm begins with a quantum register of n qubits, where n is the number of qubits necessary to represent the search space of size $N = 2^n$, all initialized to $|0\rangle$. By applying the Hadamard operation $\mathbf{H}^{\otimes n}$, the qubits are put into an equal superposition state $|\varphi_0\rangle$:

$$|\varphi_0\rangle = \mathbf{H}^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}}\sum_{i=0}^{2^n-1}|i\rangle \tag{1}$$

We then repeat the Grover transformation a number of times in order to enlarge the probability amplitude of the searched state $|\tau\rangle$ while restraining the probability amplitudes of other states. The Grover transformation operator $\mathbf{G}$ is composed of two operators: $\mathbf{G} = \mathbf{DO}$. Here the quantum oracle $\mathbf{O}$ is a quantum black-box that will rotate the phase of the state by π radians if the system is indeed in the searched state while do nothing otherwise. The diffusion operator $\mathbf{D}$ is independent of the searched state: $\mathbf{D}_{ii} = -1 + 2/N$ and $\mathbf{D}_{ij} = 2/N(i \neq j)$.

For the equal superposition state $|\varphi_0\rangle$, t applications of the Grover transformation operator $\mathbf{G}$ give

$$\begin{aligned} |\varphi(t)\rangle &= \mathbf{G}^t|\varphi_0\rangle \\ &= \sin((t+1/2)\omega_G)|\tau\rangle + \cos((t+1/2)\omega_G)|\eta\rangle \end{aligned} \tag{2}$$

where the Grover frequency $\omega_G = 2\arcsin(\sqrt{1/N})$ and $|\eta\rangle = \sum_{i\neq\tau}|i\rangle/\sqrt{N-1}$. Hence, the ideal algorithm gives a rotation in the two-dimensional plane $(|\tau\rangle, |\eta\rangle)$ [9]. If the quantum state is measured at the moment, the searched state $|\tau\rangle$ will yield with very high probability. Although Grover's algorithm is probabilistic, the error truly becomes negligible as N grows large.

3 Quantum Entanglement and Quantum Discord

3.1 Quantum Entanglement and Its Measurement

Quantum entanglement is a quantum mechanical phenomenon in which the quantum states of two or more objects have to be described with reference to each other, even though the individual objects may be spatially separated. Entangled states display strong correlations that are impossible in classical mechanics. It is believed to be an essential resource for quantum computation, quantum communication etc. To quantify entanglement, a number of measures have been proposed, such as entanglement of formation, entanglement of distillation, entanglement cost, etc [15, 16]. Nevertheless, most proposed measures of entanglement involve extremizations which are difficult to handle analytically. For that reason we choose concurrence to investigate the behavior of entanglement because it is well justified and mathematically tractable.

Suppose ρ_{AB} to be the density matrix of a quantum system that is composed of two qubits, q_A and q_B. The concurrence of the two-qubit density matrix $C(\rho_{AB})$ is computed as:

$$C(\rho_{AB}) = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \tag{3}$$

where the λ_i's are the square root of the eigenvalues of $\rho_{AB}\tilde{\rho}_{AB}$. And $\tilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y)\rho^{\star}_{AB}(\sigma_y \otimes \sigma_y)$, where σ_y denotes the Pauli-Y matrix $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ and $\rho^{\star}_{AB}$ denotes the complex conjunction of ρ_{AB} in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The concurrence for a pure state is zero and it takes the maximum value of one for a completely entangled state.

3.2 Quantum Discord

If measurement of a single quantum subsystem alters global correlations, then the subsystems are quantum correlated. Entanglement is not the only kind of genuine quantum correlation. Compared with entanglement in composite quantum systems, quantum correlations are more general and fundamental. Recent studies have demonstrated that quantum entanglement is just one form of quantum correlation and there are other forms of correlations even in separable states [17, 18].

The use of quantum discord as a measure of quantum correlations has attracted increasing interest. Here geometric quantum discord is used to describe and quantify non-entangled quantum correlation. This measure is significant in capturing quantum correlations from a geometric perspective. The geometric measure of quantum discord is defined as [19, 20]

$$D_G(\rho_{AB}) = \min_{\chi_{AB}} \|\rho_{AB} - \chi_{AB}\|^2, \tag{4}$$

where the minimum is over the set of zero-discord states and the geometric quantity $\|\rho_{AB} - \chi_{AB}\|^2 = \mathrm{Tr}(\rho_{AB} - \chi_{AB})^2$ is the squared Hilbert-Schmidt distance between the Hermitian operators ρ_{AB} and χ_{AB}.

Specifically, the density matrix ρ_{AB} for a two-qubit state can be written as

$$\begin{aligned} \rho_{AB} &= \frac{1}{4}(I_A \otimes I_B + \sum_{i=1}^{3} x_i \sigma_i \otimes I_B \\ &+ \sum_{i=1}^{3} I_A \otimes y_i \sigma_i + \sum_{i,j=1}^{3} w_{ij} \sigma_i \otimes \sigma_j), \end{aligned} \tag{5}$$

where I is the identity matrix, $x_i = \mathrm{Tr}\rho(\sigma_i \otimes I)$ and $y_i = \mathrm{Tr}\rho(I \otimes \sigma_i)$ are the components of the local Bloch vectors, $\sigma_i\,(i = 1, 2, 3)$ are the Pauli spin matrices $\{\sigma_x, \sigma_y, \sigma_z\}$, $w_{ij} = \mathrm{Tr}\rho(\sigma_i \otimes \sigma_j)$. As a consequence, the geometric quantum discord of ρ_{AB} is evaluated as [21]

$$D_G(\rho_{AB}) = \frac{1}{4}(\|\mathbf{x}\|^2 + \|\mathbf{W}\|^2 - \lambda_{max}), \tag{6}$$

where $\mathbf{x} = (x_1, x_2, x_3)^T$, $\mathbf{W}$ is a matrix consisted by w_{ij} and λ_{max} is the largest eigenvalue of the matrix $\mathbf{K} = \mathbf{x}\mathbf{x}^T + \mathbf{W}\mathbf{W}^T$ (the superscript T denotes the transpose). The geometric measure of quantum discord for an arbitrary state has been evaluated in Ref. [20] and an explicit and tight lower bound has been obtained.

4 The Effect of Static Imperfections

To study the effects of static imperfections on quantum algorithms, we use the standard generic quantum computer model proposed in Refs. [6, 7, 9]. The quantum computer model is defined as a two-dimensional lattice of qubits with static imperfections in the individual qubit energies and residual short-range interqubit couplings. The Hamiltonian of this model is

$$\begin{aligned} \mathbf{H} &= \mathbf{H}_s + \sum_i \frac{\Delta_0}{2}\sigma_z^{(i)}, \\ \mathbf{H}_s &= \sum_i \alpha_i \sigma_z^{(i)} + \sum_{i<l} \beta_{il} \sigma_x^{(i)} \sigma_x^{(l)} \end{aligned} \tag{7}$$

where $\sigma^{(i)}_{x(y,z)}$ denotes the Pauli spin operators $\sigma_{x(y,z)}$ for the ith qubit, Δ_0 is the average spacing between the two states of one qubit, α_i denotes the fluctuation in the energy spacing of the ith qubit, and β_{il} denotes the inter-qubit couplings over nearest-neighbor qubit pairs with periodic boundary conditions applied. Since the model in Eq. (7) catches the main physics of different experimental proposals, it can be considered as a generic quantum computer. Following [6, 7, 9, 22] we assume that the average spacing Δ_0 is compensated by specially applied laser pulses so that the static imperfections act as an additional unitary operator $\mathbf{U}_s = e^{-i\mathbf{H}_s}$ between any two consecutive quantum gates in the quantum circuit. The parameters α_i and β_{il} arise generally as a result of imperfections and are chosen to be random numbers with initial values uniformly distributed in $[-\alpha, \alpha]$ and $[-\beta, \beta]$, respectively. α_i and β_{il} remain unchanged for one iteration of the Grover algorithm. Without loss of generality, we will consider the coupling strength $\alpha = \beta = \epsilon$.

In our numerical experiments, we first defined a 3×3 square lattice which contains 9 qubits. As the implementation of Grover algorithm with the elementary quantum gates requires an ancillary qubit, the size of the search space becomes $N = 2^8$. As a result, we can find the searched state after $\pi\sqrt{N}/4 - 1/2 \approx 12$ iterations of Grover algorithm. Similarly, a two-dimensional lattice model containing 4×3 qubits was generated. The size of the search space is correspondingly $N = 2^{11}$ and it takes approximately 35 iterations to find the searched state.

According to Eq. (3) and Eq. (5), we have numerically calculated the concurrence and the geometric quantum discord with iterations of Grover search algorithm in the presence of imperfections. The experimental results of the concurrence C and the geometric quantum discord D_G for the 3×3 lattice model are shown in Fig. 1. For the 4×3 lattice model they are shown in Fig. 2. In Fig. 1 and 2, the concurrence C exhibits periodical behavior when the strength of the static imperfections become very weak. Furthermore, the period is almost equal to the number of iterations that we will find the searched state $|\tau\rangle$ with highest probability (it is approximately 12 in Fig. 1 and around 35 in Fig. 2). As the strength of the imperfections increase, the amplitude decreases and the periodical behavior of entanglement is destroyed gradually. Hence the quantum search algorithm is unable to function properly.

Although the geometric quantum discord D_G exhibit similar dynamical behaviors with concurrence C for small or moderate imperfections, they show different behaviors for large imperfections. When the strength of static imperfections exceeds a certain value, we see the sudden death of entanglement (as depicted in the left bottom panels of Fig. 1 and Fig. 2), while this phenomenon cannot be observed in dynamics of geometric quantum discord. Our

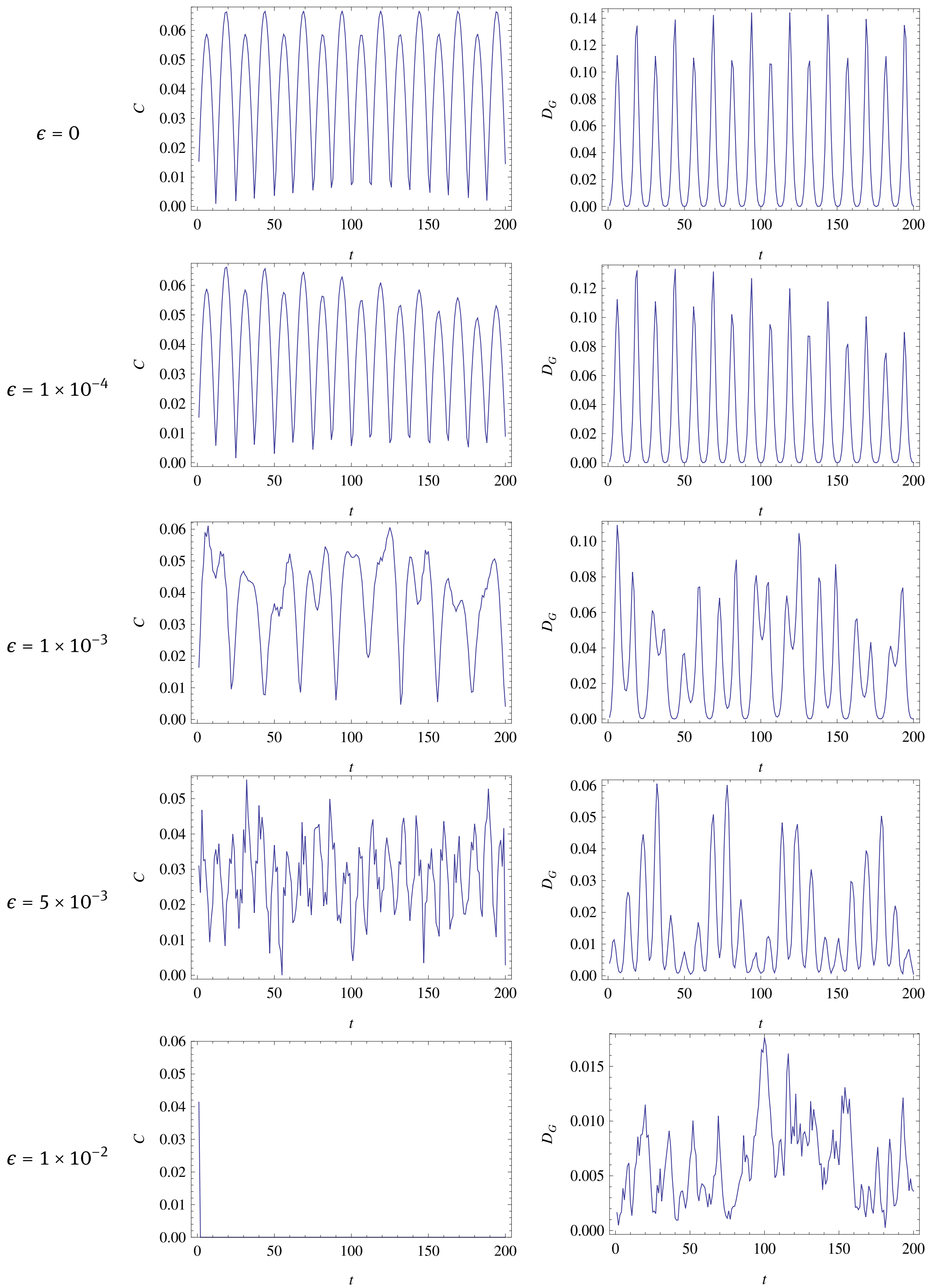

Figure 1: The concurrence C and the geometric quantum discord D_G with iterations of the perturbed Grover search algorithm for a 3×3 lattice model. The strength of the imperfection are $\epsilon = 0$, 1×10^{-4}, 1×10^{-3}, 5×10^{-3}, 1×10^{-2} from top to bottom.

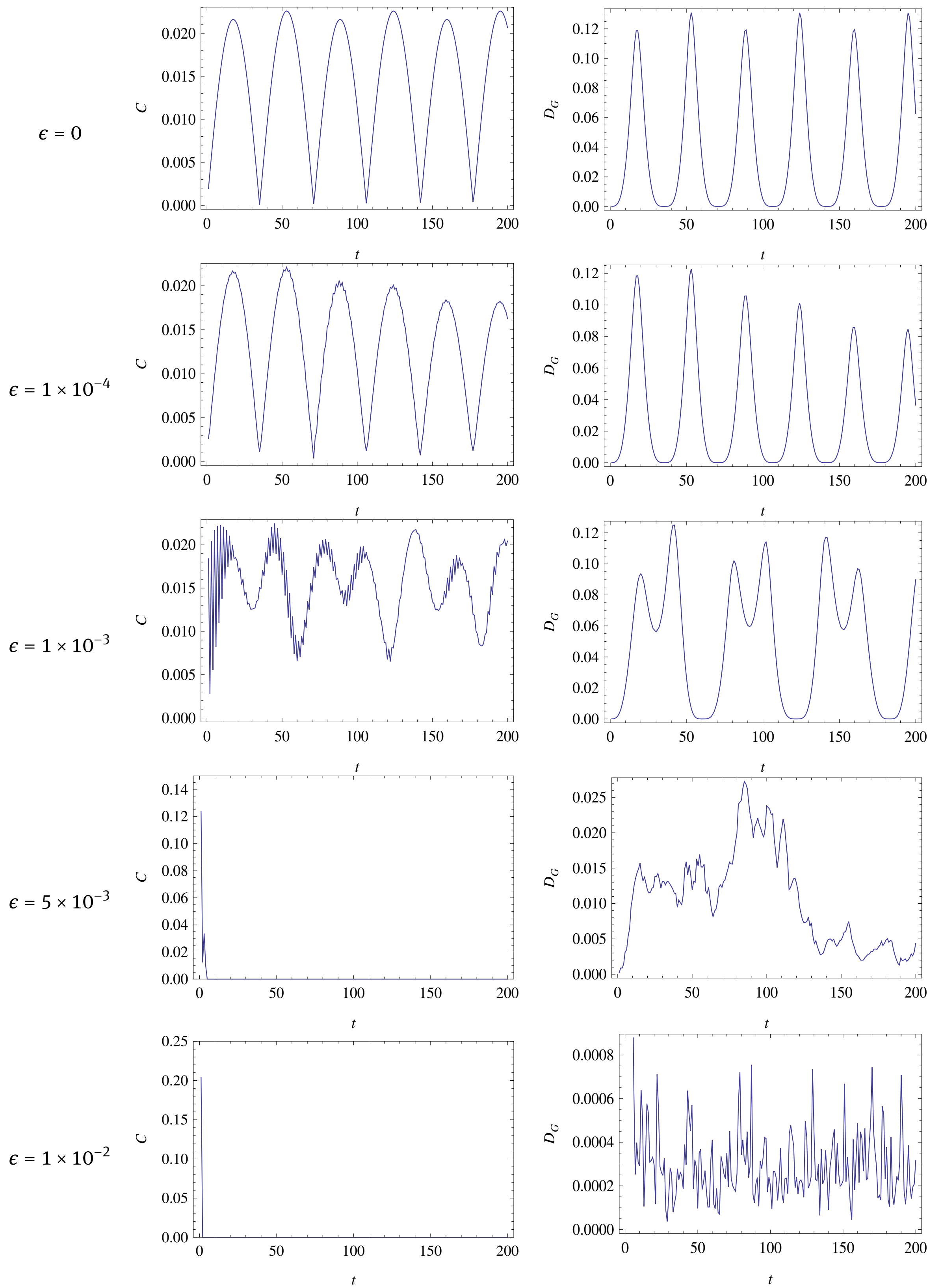

Figure 2: The concurrence C and the geometric quantum discord D_G with iterations of the perturbed Grover search algorithm for a 4×3 lattice model. The strength of the imperfection are $\epsilon = 0, 1 \times 10^{-4}, 1 \times 10^{-3}, 5 \times 10^{-3}, 1 \times 10^{-2}$ from top to bottom.

numerical results suggests that quantum discord in quantum algorithms is more robust to static imperfections than quantum entanglement.

5 Conclusions

Quantum entanglement and non-entangled quantum correlations are valuable resources in quantum information processing, but at the same time, they are very fragile. In this paper we have numerically studied the dynamics of quantum entanglement and quantum discord in quantum computing systems with the static interqubit couplings. Static imperfections in quantum computing systems can break quantum correlations and seriously destroy operability, and thus lead to unpredictable computation results. Although the evolutions of both entanglement and quantum discord will be suppressed by static imperfections in the same way, quantum discord is slightly more robust against imperfections in comparison with quantum entanglement.

Acknowledgement: The authors wish to express their gratitude to the Advanced Analysis and Computation Center of CUMT for the award of CPU hours to accomplish this work. This work is supported by the Fundamental Research Funds for the Central Universities (Grant No. 2015QNA67).

References

[1] Horodecki R., Horodecki P., Horodecki M., Horodecki K., Quantum entanglement, Rev. Mod. Phys., 2009, 81, 865–942.
[2] Datta A., Shaji A., Quantum discord and quantum computing - an appraisal, Int. J. Quantum Inf., 2011, 9, 1787–1805.
[3] Zurek W. H., Einselection and decoherence from an information theory perspective, Ann. Phys., 2000, 9, 855–864.
[4] Ollivier H., Zurek W. H., Quantum discord: a measure of the quantumness of correlations, Phys. Rev. Lett., 2001, 88, 017901.
[5] Henderson L., Vedral V., Classical quantum and total correlations, J. Phys. A, 2001, 34, 6899–6905.
[6] Georgeot B., Shepelyansky D. L., Emergence of quantum chaos in the quantum computer core and how to manage it, Phys. Rev. E, 2000, 62, 6366–6375.
[7] Georgeot B., Shepelyansky D. L., Quantum chaos border for quantum computing, Phys. Rev. E, 2000, 62, 3504–3507.
[8] Grover L. K., From Schrödinger's equation to quantum search algorithm, Am. J. Phys., 2001, 69, 769–777.
[9] Pomeransky A. A., Zhirov O. V., Shepelyansky D. L., Phase diagram for the Grover algorithm with static imperfections, Eur. Phys. J. D, 2004, 31, 131–135.
[10] Zhirov O. V., Shepelyansky D. L., Dissipative decoherence in the Grover algorithm, Eur. Phys. J. D, 2006, 38, 405–408.
[11] Azuma H., Decoherence in Grover's quantum algorithm: Perturbative approach, Phys. Rev. A, 2002, 65, 042311.
[12] Song P. H., Kim I., Computational leakage: Grover's algorithm with imperfections, Eur. Phys. J. D, 2003, 23, 299-303.
[13] Shapira D., Mozes S., Biham O., Effect of unitary noise on Grover's quantum search algorithm, Phys. Rev. A, 2003, 67, 042301.
[14] Roosi M., Bruß D., Macchiavello C., Scale invariance of entanglement dynamics in Grover's quantum search algorithm, Phys. Rev. A, 2013, 87, 022331.
[15] Bennett C. H., DiVincenzo D. P., Smolin J. A., Wootters W. K., Mixed-state entanglement and quantum error correction, Phys. Rev. A, 1996, 54, 3824.
[16] Bennett C. H., Bernstein H. J., Popescu S., Schumacher B., Concentrating partial entanglement by local operations, Phys. Rev. A, 1996, 53, 2046.
[17] Dorner R., Vedral V., Correlations in quantum physics, Int. J. Mod. Phys. B, 2013, 27, 1345017.
[18] Sadhukhan D., et al., Quantum discord length is enhanced while entanglement length is not by introducing disorder in a spin chain, Phys. Rev. E, 2016, 93, 012131.
[19] Dakić B., Vedral V., Brukner C., Necessary and sufficient condition for nonzero quantum discord, Phys. Rev. Lett., 2010, 105, 190502.
[20] Luo S. L., Fu S. S., Geometric measure of quantum discord, Phys. Rev. A, 2010, 82, 034302.
[21] Chen H., Fu Y. Q., Fang J. X., Geometric discord of non-X-structured state under decoherence channels, Int. J. Theor. Phys., 2014, 53, 2967–2979.
[22] Ye B., Qiu L., 1/f noise in Ising quantum computers, Fluct. Noise Lett., 2014, 13, 1450006.