

Research Article

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Oscillation criteria for nonlinear fractional differential equation with damping term

DOI 10.1515/phys-2016-0012

Received November 07, 2015; accepted March 03, 2016

Abstract: In this paper, we study the oscillation of solutions to a non-linear fractional differential equation with damping term. The fractional derivative is defined in the sense of the modified Riemann-Liouville derivative. By using a variable transformation, a generalized Riccati transformation, inequalities, and integration average technique we establish new oscillation criteria for the fractional differential equation. Several illustrative examples are also given.

Keywords: oscillation; oscillation criteria; fractional derivative; modified Riemann-Liouville derivative; damping term

PACS: 02.30.Hq, 02.30.Sa, 02.60.Lj, 02.90.+p

1 Introduction

Fractional differential equations are generalizations of classical differential equations of integer order and have recently proved to be valuable tools in the modelling of many phenomena in various fields of science and engineering. Apart from diverse areas of mathematics, fractional differential equations arise in rheology, viscoelasticity, chemical physics, electrical networks, fluid flows, control, dynamical processes in self-similar and porous structures, *etc.*; see, for example, [1–6]. Fractional derivatives have appeared in lots of work where they are used for better descriptions of material properties. Mathematical modelling based on enhanced rheological models naturally leads to differential equations of fractional order and to the necessity of the formulation of initial conditions to

such equations. This growth in use has been caused by the intensive development of the theory of fractional calculus itself and its applications. The books on the subject of fractional integrals and fractional derivatives by Diethelm [7], Miller and Ross [8], Podlubny [9] and Kilbas *et al.* [10] summarise and organise much of the field of fractional calculus including many of the theories and applications of fractional differential equations. Many papers have studied some aspects of fractional differential equations. Most have focused on the existence of, methods for defining or stability of the solutions (or positive solutions) to nonlinear initial (or boundary) value problems for fractional differential equations (or systems) using nonlinear analysis techniques (fixed-point theorems, Leray-Schauder theory). We refer to [11–21] and the references cited therein.

Recently, research on the oscillation of various equations including differential equations, difference equations and dynamic equations on time scales, has been a hot topic the literature. A lot of effort has been committed to establishing new oscillation criteria for these equations; see the monographs [22, 23]. In these investigations, we notice that very little attention has been paid to the oscillation of fractional differential equations.

In 2006, a definition for a fractional derivative called the modified Riemann-Liouville derivative, was suggested by Jumarie [24] and its application have subsequently been studied by many researchers [25–28].

Recently, Qin and Zheng [29] established oscillation criteria for linear fractional differential equations with damping term of the form:

$$D_t^\alpha (a(t) D_t^\alpha (r(t) D_t^\alpha x(t))) + p(t) D_t^\alpha (r(t) D_t^\alpha x(t)) + q(t) x(t) = 0 \quad t \geq t_0 > 0, 0 < \alpha < 1,$$

where $D_t^\alpha (\cdot)$ denotes the modified Riemann-Liouville derivative with respect to variable t .

Now, in this paper, we are concerned with the oscillation of fractional differential equations with damping term in the form of:

$$D_t^\alpha (a(t) D_t^\alpha (r(t) D_t^\alpha x(t))) + p(t) D_t^\alpha (r(t) D_t^\alpha x(t)) + q(t) f(x(t)) = 0, \quad t \geq t_0 > 0, 0 < \alpha < 1, \quad (1)$$

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where: $D_t^\alpha(\cdot)$ denotes the modified Riemann-Liouville derivative with respect to the variable t ; the function $a \in C^\alpha([t_0, \infty), R_+)$; $r \in C^{2\alpha}([t_0, \infty), R_+)$; $p, q \in C([t_0, \infty), R_+)$; the function of f belongs to $C(R, R)$; $f(x)/x \geq k > 0$ for all $x \neq 0$ and C^α denotes a continuous derivative of order α .

Some of the key properties of Jumarie's modified Riemann-Liouville derivative of order α are listed as follows:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t f(t, \xi, \alpha) d\xi, & 0 < \alpha < 1 \\ (f^{(n)}(t))^{(\alpha-n)}, & 1 \leq n \leq \alpha < n+1 \end{cases} \quad (2)$$

$$D_t^\alpha (f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t) \quad (3)$$

$$D_t^\alpha f[g(t)] = f'_g[g(t)]D_t^\alpha g(t) = D_t^\alpha f[g(t)](g'(t))^\alpha \quad (4)$$

$$D_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, \quad (5)$$

where $f(t, \xi, \alpha) = (t - \xi)^{-\alpha} (f(\xi) - f(0))$.

As usual, a solution $x(t)$ of (1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

In the rest of this paper, we denote for the sake of convenience:

$$\begin{aligned} \xi &= t^\alpha / \Gamma(1+\alpha); \xi_i = t_i^\alpha / \Gamma(1+\alpha); i = 0, 1, 2, 3, 4, 5; \\ a(t) &= \tilde{a}(\xi); r(t) = \tilde{r}(\xi); p(t) = \tilde{p}(\xi); q(t) = \tilde{q}(\xi); \\ \delta_1(\xi, \xi_i) &= \int_{\xi_i}^\xi (1/A(s)\tilde{a}(s)) ds; \delta_1(t, t_i) = \delta_1(\xi, \xi_i); \\ \delta_2(\xi, \xi_i) &= \int_{\xi_i}^\xi (\tilde{\delta}_1(s, \xi_i)/\tilde{r}(s)) ds; \delta_2(t, t_i) = \delta_2(\xi, \xi_i); \\ A(\xi) &= \exp(\int_{\xi_0}^\xi (\tilde{p}(s)/\tilde{a}(s) ds)). \end{aligned}$$

Let $h_1, h_2, H \in C([\xi_0, \infty), R)$ satisfy

$$H(\xi, \xi) = 0, \quad H(\xi, s) > 0, \quad \xi > s \geq \xi_0 \quad (6)$$

H has continuous partial derivatives $\partial H(\xi, s)/\partial \xi$ and $\partial H(\xi, s)/\partial s$ on $[\xi_0, \infty)$ such that

$$\frac{\partial H(\xi, s)}{\partial \xi} = -h_1(\xi, s) \sqrt{H(\xi, s)} \quad (7)$$

$$\frac{\partial H(\xi, s)}{\partial s} = -h_2(\xi, s) \sqrt{H(\xi, s)}, \quad \xi > s \geq \xi_0. \quad (8)$$

This paper is organized next as follows: in Section 2, we establish new oscillation criteria for (1) using the Riccati transformation, inequalities and the integration average technique and in Section 3, we present some examples that apply the results established. Finally, we give a conclusion.

2 Oscillatory criteria

Lemma 1. Assume $x(t)$ is an eventually positive solution of (1), and

$$\int_{\xi_0}^{\infty} \frac{1}{A(s)\tilde{a}(s)} ds = \infty \quad (9)$$

$$\int_{t_0}^{\infty} \frac{\alpha t^{\alpha-1}}{\Gamma(1+\alpha)r(t)} dt = \infty \quad (10)$$

$$\int_{\xi_0}^{\infty} \frac{1}{\tilde{r}(\xi)} \int_{\xi}^{\infty} \frac{1}{A(\tau)\tilde{a}(\tau)} \int_{\tau}^{\infty} A(s)\tilde{q}(s) ds d\tau d\xi = \infty. \quad (11)$$

Then, there exist a sufficiently large T such that $D_t^\alpha(r(t)D_t^\alpha x(t)) > 0$ on $[T, \infty)$ and either $D_t^\alpha x(t) > 0$ on $[T, \infty)$ or $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof. Suppose $x(t)$ is an eventually positive solution of (1). Let $a(t) = \tilde{a}(\xi)$, $r(t) = \tilde{r}(\xi)$, $x(t) = \tilde{x}(\xi)$, $p(t) = \tilde{p}(\xi)$, $q(t) = \tilde{q}(\xi)$ where $\xi = t^\alpha / \Gamma(1+\alpha)$. Then by using (5), we obtain $D_t^\alpha \xi(t) = 1$, and furthermore, by use of the first equality in (4), we have

$$D_t^\alpha a(t) = D_t^\alpha \tilde{a}(\xi) = \tilde{a}'(\xi) D_t^\alpha \xi(t) = \tilde{a}'(\xi). \quad (12)$$

Similarly we have $D_t^\alpha r(t) = \tilde{r}'(\xi)$, $D_t^\alpha x(t) = \tilde{x}'(\xi)$, $D_t^\alpha p(t) = \tilde{p}'(\xi)$, $D_t^\alpha q(t) = \tilde{q}'(\xi)$. So, (1) can be transformed into the following form:

$$\begin{aligned} \left[\tilde{a}(\xi) \left(\tilde{r}(\xi) \tilde{x}'(\xi) \right) \right]' + \tilde{p}(\xi) \left(\tilde{r}(\xi) \tilde{x}'(\xi) \right)' \\ + \tilde{q}(\xi) f(\tilde{x}(\xi)) = 0, \quad \xi \geq \xi_0 > 0. \end{aligned} \quad (13)$$

Then $\tilde{x}(\xi)$ is an eventually positive solution of (13), and there exists $\xi_1 > \xi_0$ such that $\tilde{x}(\xi) > 0$ on $[\xi_1, \infty)$. So, $f(\tilde{x}(\xi)) > 0$ and we have

$$\begin{aligned} \left[A(\xi) \tilde{a}(\xi) \left(\tilde{r}(\xi) \tilde{x}'(\xi) \right) \right]' &= A(\xi) \left[\tilde{a}(\xi) \left(\tilde{r}(\xi) \tilde{x}'(\xi) \right) \right]' \\ &+ A'(\xi) \tilde{a}(\xi) \left(\tilde{r}(\xi) \tilde{x}'(\xi) \right)' \\ &= A(\xi) \left\{ \left[\tilde{a}(\xi) \left(\tilde{r}(\xi) \tilde{x}'(\xi) \right) \right]' \right. \\ &\left. + \tilde{p}(\xi) \left(\tilde{r}(\xi) \tilde{x}'(\xi) \right)' \right\}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} [A(\xi) \tilde{a}(\xi) \left(\tilde{r}(\xi) \tilde{x}'(\xi) \right)]' &= \\ -A(\xi) \tilde{q}(\xi) f(\tilde{x}(\xi)) &< 0, \quad \xi \geq \xi_1. \end{aligned} \quad (14)$$

Then, $A(\xi) \tilde{a}(\xi) (\tilde{r}(\xi) \tilde{x}'(\xi))'$ is strictly decreasing on $[\xi_1, \infty)$, thus we know that $(\tilde{r}(\xi) \tilde{x}'(\xi))'$ is eventually of one sign. For $\xi_2 > \xi_1$ is sufficiently large, we claim $(\tilde{r}(\xi) \tilde{x}'(\xi))' > 0$ on $[\xi_2, \infty)$. Otherwise, assume that there exists a sufficiently large $\xi_3 > \xi_2$ such that $(\tilde{r}(\xi) \tilde{x}'(\xi))' < 0$ on $[\xi_3, \infty)$. Thus, $\tilde{r}(\xi) \tilde{x}'(\xi)$ is strictly decreasing on $[\xi_3, \infty)$, and we get that

$$\begin{aligned} \tilde{r}(\xi) \tilde{x}'(\xi) - \tilde{r}(\xi_3) \tilde{x}'(\xi_3) &= \int_{\xi_3}^{\xi} \frac{A(s) \tilde{a}(s) (\tilde{r}(s) \tilde{x}'(s))'}{A(s) \tilde{a}(s)} ds \\ &\leq A(\xi_3) \tilde{a}(\xi_3) (\tilde{r}(\xi_3) \tilde{x}'(\xi_3))' \int_{\xi_3}^{\xi} \frac{1}{A(s) \tilde{a}(s)} ds. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \tilde{r}(\xi) \tilde{x}'(\xi) &\leq \tilde{r}(\xi_3) \tilde{x}'(\xi_3) \\ &+ A(\xi_3) \tilde{a}(\xi_3) (\tilde{r}(\xi_3) \tilde{x}'(\xi_3))' \int_{\xi_3}^{\xi} \frac{1}{A(s) \tilde{a}(s)} ds. \end{aligned} \quad (15)$$

By (9), we have $\lim_{\xi \rightarrow \infty} \tilde{r}(\xi) \tilde{x}'(\xi) = -\infty$. So there exists a sufficiently large $\xi_4 > \xi_3$ such that $\tilde{x}'(\xi) < 0$, $\xi \in [\xi_4, \infty)$. Then, we have

$$\begin{aligned} \tilde{x}(\xi) - \tilde{x}(\xi_4) &= \int_{\xi_4}^{\xi} \tilde{x}'(s) ds = \int_{\xi_4}^{\xi} \frac{\tilde{r}(s)}{\tilde{r}(s)} \tilde{x}'(s) ds \\ &\leq \tilde{r}(\xi_4) \tilde{x}'(\xi_4) \int_{\xi_4}^{\xi} \frac{1}{\tilde{r}(s)} ds \end{aligned}$$

and so,

$$\tilde{x}(\xi) \leq \tilde{r}(\xi_4) \tilde{x}'(\xi_4) \int_{\xi_4}^{\xi} \frac{\alpha t^{\alpha-1}}{\Gamma(1+\alpha)r(t)} dt. \quad (16)$$

By (10), we deduce that $\lim_{\xi \rightarrow \infty} \tilde{x}(\xi) = -\infty$, which contradicts the fact that $\tilde{x}(\xi)$ is an eventually positive solution of (13). Thus, $(\tilde{r}(\xi) \tilde{x}'(\xi))' > 0$ on $[\xi_2, \infty)$, and then $D_t^\alpha(r(t) D_t^\alpha x(t)) > 0$ on $[t_2, \infty)$. So, $D_t^\alpha x(t) = \tilde{x}'(\xi)$ is eventually of one sign. Now we assume $\tilde{x}'(\xi) < 0$ on $[\xi_5, \infty)$ where $\xi_5 > \xi_4$ is sufficiently large. Since $\tilde{x}(\xi) > 0$, we have $\lim_{\xi \rightarrow \infty} \tilde{x}(\xi) = \beta \geq 0$. We claim $\beta = 0$. Otherwise, assume $\beta > 0$. Then $\tilde{x}(\xi) \geq \beta$ on $[\xi_5, \infty)$, $f(x(\xi)) \geq k \cdot x(\xi) > M$ for $M \in \mathbb{R}_+$ and by (14) we have

$$\begin{aligned} \left[A(\xi) \tilde{a}(\xi) (\tilde{r}(\xi) \tilde{x}'(\xi))' \right]' &= -A(\xi) \tilde{q}(\xi) f(\tilde{x}(\xi)) \\ &\leq -A(\xi) \tilde{q}(\xi) M. \end{aligned} \quad (17)$$

Substituting ξ with s in (17), and integrating it with respect to s from ξ to ∞ yields

$$\begin{aligned} \int_{\xi}^{\infty} \left[A(s) \tilde{a}(s) (\tilde{r}(s) \tilde{x}'(s))' \right]' ds &\leq -M \int_{\xi}^{\infty} A(s) \tilde{q}(s) ds \\ &- A(\xi) \tilde{a}(\xi) (\tilde{r}(\xi) \tilde{x}'(\xi))' \\ &\leq -\lim_{\xi \rightarrow \infty} \left[A(\xi) \tilde{a}(\xi) (\tilde{r}(\xi) \tilde{x}'(\xi))' \right] - M \int_{\xi}^{\infty} A(s) \tilde{q}(s) ds \\ &< -M \int_{\xi}^{\infty} A(s) \tilde{q}(s) ds \end{aligned} \quad (18)$$

which means

$$(\tilde{r}(\xi) \tilde{x}'(\xi))' > M \frac{1}{A(\xi) \tilde{a}(\xi)} \int_{\xi}^{\infty} A(s) \tilde{q}(s) ds \quad (19)$$

substituting ξ with τ in (19), and integrating it with respect to τ from ξ to ∞ yields

$$\begin{aligned} \int_{\xi}^{\infty} (\tilde{r}(s) \tilde{x}'(s))' ds &> M \int_{\xi}^{\infty} \frac{1}{\varkappa} \int_{\xi}^{\infty} A(s) \tilde{q}(s) ds d\tau \\ \lim_{\xi \rightarrow \infty} \tilde{r}(\xi) \tilde{x}'(\xi) - \tilde{r}(\xi) \tilde{x}'(\xi) &> M \int_{\xi}^{\infty} \frac{1}{\varkappa} \int_{\xi}^{\infty} A(s) \tilde{q}(s) ds d\tau \\ -\tilde{r}(\xi) \tilde{x}'(\xi) &> -\lim_{\xi \rightarrow \infty} \tilde{r}(\xi) \tilde{x}'(\xi) + M \int_{\xi}^{\infty} \frac{1}{\varkappa} \int_{\xi}^{\infty} A(s) \tilde{q}(s) ds d\tau \\ -\tilde{r}(\xi) \tilde{x}'(\xi) &> M \int_{\xi}^{\infty} \frac{1}{A(\tau) \tilde{a}(\tau)} \int_{\xi}^{\infty} A(s) \tilde{q}(s) ds d\tau, \end{aligned} \quad (20)$$

where $\varkappa = A(\tau) \tilde{a}(\tau)$. That is,

$$\tilde{x}'(\xi) < -M \frac{1}{\tilde{r}(\xi)} \int_{\xi}^{\infty} \frac{1}{\varkappa} \int_{\xi}^{\infty} A(s) \tilde{q}(s) ds d\tau \quad (21)$$

substituting ξ with ζ in (21), and integrating it with respect to ζ from ξ_5 to ξ yields

$$\begin{aligned} \int_{\xi_5}^{\xi} \tilde{x}'(s) ds &< -M \int_{\xi_5}^{\xi} \frac{1}{\tilde{r}(\zeta)} \int_{\xi}^{\infty} \frac{1}{\varkappa} \int_{\xi}^{\infty} A(s) \tilde{q}(s) ds d\tau d\zeta \\ \tilde{x}(\xi) - \tilde{x}(\xi_5) &< -M \int_{\xi_5}^{\xi} \frac{1}{\tilde{r}(\zeta)} \int_{\xi}^{\infty} \frac{1}{\varkappa} \int_{\xi}^{\infty} A(s) \tilde{q}(s) ds d\tau d\zeta \end{aligned}$$

$$\tilde{x}(\xi) < \tilde{x}(\xi_5) - M \int_{\xi_5}^{\xi} \frac{1}{\tilde{r}(\zeta)} \int_{\xi}^{\infty} \frac{1}{\tilde{r}(\zeta)} \int_{\xi}^{\infty} A(s) \tilde{q}(s) ds d\tau d\zeta.$$

By (11), we have $\lim_{t \rightarrow \infty} \tilde{x}(\xi) = -\infty$, which causes a contradiction. So, the proof is complete. $\square$

Lemma 2. Assume that $x(t)$ is an eventually positive solution of (1) such that

$$D_t^\alpha (r(t) D_t^\alpha x(t)) > 0, \quad D_t^\alpha x(t) > 0 \quad (22)$$

on $[t_1, \infty)$, where $t_1 > t_0$ is sufficiently large. Then, for $t \geq t_1$, we have

$$D_t^\alpha x(t) \geq \frac{A(\xi) \delta_1(t, t_1) a(t) D_t^\alpha (r(t) D_t^\alpha x(t))}{r(t)} \quad (23)$$

$$x(t) \geq A(\xi) \delta_2(t, t_1) a(t) D_t^\alpha (r(t) D_t^\alpha x(t)). \quad (24)$$

Proof. Assume that x is an eventually positive solution of (1). So, by (14), we obtain that $A(\xi) \tilde{a}(\xi) (\tilde{r}(\xi) \tilde{x}'(\xi))'$ is strictly decreasing on $[\xi_1, \infty)$. Then,

$$\begin{aligned} \tilde{r}(\xi) \tilde{x}'(\xi) &\geq \tilde{r}(\xi) \tilde{x}'(\xi) - \tilde{r}(\xi_1) \tilde{x}'(\xi_1) \\ &= \int_{\xi_1}^{\xi} \frac{A(s) \tilde{a}(s) (\tilde{r}(s) \tilde{x}'(s))'}{A(s) \tilde{a}(s)} ds \\ &\geq A(\xi) \tilde{a}(\xi) (\tilde{r}(\xi) \tilde{x}'(\xi))' \int_{\xi_1}^{\xi} \frac{1}{A(s) \tilde{a}(s)} ds \\ &= A(\xi) \tilde{a}(\xi) (\tilde{r}(\xi) \tilde{x}'(\xi))' \tilde{\delta}_1(\xi, \xi_1) \end{aligned} \quad (25)$$

and so,

$$r(t) D_t^\alpha x(t) \geq A(\xi) \delta_1(t, t_1) a(t) D_t^\alpha (r(t) D_t^\alpha x(t)) \quad (26)$$

multiplying both sides of (26) by $1/r(t)$, we obtain

$$D_t^\alpha x(t) \geq \frac{A(\xi) \delta_1(t, t_1) a(t) D_t^\alpha (r(t) D_t^\alpha x(t))}{r(t)}.$$

On the other hand, we have

$$\tilde{x}(\xi) \geq \tilde{x}(\xi) - \tilde{x}(\xi_1) = \int_{\xi_1}^{\xi} \tilde{x}'(s) ds = \int_{\xi_1}^{\xi} \frac{\tilde{r}(s) \tilde{x}'(s)}{\tilde{r}(s)} ds.$$

Using (26), we obtain

$$\begin{aligned} \tilde{x}(\xi) &\geq \int_{\xi_1}^{\xi} \frac{A(s) \tilde{a}(s) (\tilde{r}(s) \tilde{x}'(s))' \tilde{\delta}_1(s, \xi_1)}{\tilde{r}(s)} ds, \\ \tilde{x}(\xi) &\geq A(\xi) \tilde{a}(\xi) (\tilde{r}(\xi) \tilde{x}'(\xi))' \int_{\xi_1}^{\xi} \frac{\tilde{\delta}_1(s, \xi_1)}{\tilde{r}(s)} ds \\ &= A(\xi) \tilde{\delta}_2(\xi, \xi_1) \tilde{a}(\xi) (\tilde{r}(\xi) \tilde{x}'(\xi))'. \end{aligned} \quad (27)$$

That is

$$x(t) \geq A(\xi) \delta_2(t, t_1) a(t) D_t^\alpha (r(t) D_t^\alpha x(t)).$$

So, the proof is complete. $\square$

Lemma 3. [30]: Assume that A and B are nonnegative real numbers. Then,

$$\lambda AB^{\lambda-1} - A^\lambda \leq (\lambda - 1) B^\lambda \quad (28)$$

for all $\lambda > 1$.

Theorem 4. Assume that (9)-(11) hold and $f(x)/x \geq k > 0$ for all $x \neq 0$. If there exists $\phi \in C^\alpha([t_0, \infty), \mathbb{R}_+)$ such that for any sufficiently large $T \geq \xi_0$, there exist a, b, c with $T \leq a < c < b$ satisfying

$$\begin{aligned} &\frac{1}{H(b, c)} \int_c^b H(b, s) k A(s) \tilde{\phi}(s) \tilde{q}(s) ds \\ &+ \frac{1}{H(c, a)} \int_a^c H(s, a) k A(s) \tilde{\phi}(s) \tilde{q}(s) ds \\ &> \frac{1}{H(b, c)} \int_c^b \frac{\tilde{r}(s) \tilde{\phi}(s)}{4\delta_1(s, \xi_2)} Q_2^2(b, s) ds \\ &+ \frac{1}{H(c, a)} \int_a^c \frac{\tilde{r}(s) \tilde{\phi}(s)}{4\delta_1(s, \xi_2)} Q_1^2(s, a) ds, \end{aligned} \quad (29)$$

where $k \in \mathbb{R}_+$, $\tilde{\phi}(\xi) = \phi(t)$, $Q_1(s, \xi) = h_1(s, \xi) - (\tilde{\phi}'(s)/\tilde{\phi}(s)) \sqrt{H(s, \xi)}$, $Q_2(\xi, s) = h_2(\xi, s) - (\tilde{\phi}'(s)/\tilde{\phi}(s)) \sqrt{H(\xi, s)}$; then, (1) is oscillatory or satisfies $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof. Suppose the contrary that $x(t)$ is non-oscillatory solution of (1). Then without loss of generality, we may assume that there is a solution $x(t)$ of (1) such that $x(t) > 0$ on $[t_1, \infty)$, where t_1 is sufficiently large. By Lemma 1, we have $D_t^\alpha (r(t) D_t^\alpha x(t)) > 0$, $t \in [t_2, \infty)$, where $t_2 > t_1$ is sufficiently large, and either $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$ or $\lim_{t \rightarrow \infty} x(t) = 0$. If we take $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$. Define the following generalized Riccati function:

$$\omega(t) = \phi(t) \frac{A(\xi) a(t) D_t^\alpha (r(t) D_t^\alpha x(t))}{x(t)}. \quad (30)$$

For $t \in [t_2, \infty)$, we have

$$\begin{aligned} D_t^\alpha \omega(t) &= D_t^\alpha \phi(t) \frac{A(\xi) a(t) D_t^\alpha (r(t) D_t^\alpha x(t))}{x(t)} \\ &+ \phi(t) D_t^\alpha \left\{ \frac{A(\xi) a(t) D_t^\alpha (r(t) D_t^\alpha x(t))}{x(t)} \right\}. \end{aligned}$$

So,

$$\begin{aligned}
 D_t^\alpha \omega(t) &= D_t^\alpha \phi(t) \frac{\omega(t)}{\phi(t)} \\
 &+ \frac{\phi(t) x(t) D_t^\alpha (A(\xi) a(t) D_t^\alpha (r(t) D_t^\alpha x(t)))}{x^2(t)} \\
 &- \phi(t) \frac{D_t^\alpha x(t) A(\xi) a(t) D_t^\alpha (r(t) D_t^\alpha x(t))}{x^2(t)} \\
 &= D_t^\alpha \phi(t) \frac{\omega(t)}{\phi(t)} \\
 &+ \phi(t) \frac{[D_t^\alpha A(\xi) a(t) D_t^\alpha (r(t) D_t^\alpha x(t))]}{x(t)} \\
 &+ \phi(t) \frac{[A(\xi) D_t^\alpha (a(t) D_t^\alpha (r(t) D_t^\alpha x(t)))]}{x(t)} \\
 &- \phi(t) \frac{D_t^\alpha x(t) A(\xi) a(t) D_t^\alpha (r(t) D_t^\alpha x(t))}{x^2(t)}. \quad (31)
 \end{aligned}$$

If we use $D_t^\alpha \xi = 1$ and (23), we obtain

$$\begin{aligned}
 D_t^\alpha \omega(t) &\leq D_t^\alpha \phi(t) \frac{\omega(t)}{\phi(t)} \\
 &+ \phi(t) \frac{[A(\xi) \frac{p(t)}{a(t)} a(t) D_t^\alpha (r(t) D_t^\alpha x(t))]}{x(t)} \\
 &+ \phi(t) \frac{[A(\xi) D_t^\alpha (a(t) D_t^\alpha (r(t) D_t^\alpha x(t)))]}{x(t)} \\
 &- \phi(t) \frac{\delta_1(t, t_2)}{\phi^2(t) r(t)} \omega^2(t)
 \end{aligned}$$

and so,

$$\begin{aligned}
 D_t^\alpha \omega(t) &\leq D_t^\alpha \phi(t) \frac{\omega(t)}{\phi(t)} \\
 &+ \phi(t) \frac{A(\xi) [p(t) D_t^\alpha (r(t) D_t^\alpha x(t))]}{x(t)} \\
 &+ \phi(t) \frac{A(\xi) [D_t^\alpha (a(t) D_t^\alpha (r(t) D_t^\alpha x(t)))]}{x(t)} \\
 &- \phi(t) \frac{\delta_1(t, t_2)}{\phi^2(t) r(t)} \omega^2(t) \\
 &= D_t^\alpha \phi(t) \frac{\omega(t)}{\phi(t)} - \frac{A(\xi) q(t) f(x(t)) \phi(t)}{x(t)} \\
 &- \frac{\delta_1(t, t_2)}{\phi(t) r(t)} \omega^2(t).
 \end{aligned}$$

Using $f(x(t))/x(t) \geq k$,

$$\begin{aligned}
 D_t^\alpha \omega(t) &\leq D_t^\alpha \phi(t) \frac{\omega(t)}{\phi(t)} - kA(\xi) q(t) \phi(t) \\
 &- \frac{\delta_1(t, t_2)}{\phi(t) r(t)} \omega^2(t). \quad (32)
 \end{aligned}$$

Now, let $\omega(t) = \tilde{\omega}(\xi)$. Then we have $D_t^\alpha \omega(t) = \tilde{\omega}'(\xi)$ and $D_t^\alpha \phi(t) = \tilde{\phi}'(\xi)$. Thus (32) is transformed into

$$\begin{aligned}
 \tilde{\omega}'(\xi) &\leq \frac{\phi'(\xi)}{\phi(\xi)} \tilde{\omega}(\xi) - kA(\xi) \tilde{q}(\xi) \tilde{\phi}(\xi) \\
 &- \frac{\tilde{\delta}_1(\xi, \xi_2)}{\tilde{\phi}(\xi) \tilde{r}(\xi)} \tilde{\omega}^2(\xi), \quad \xi \geq \xi_2. \quad (33)
 \end{aligned}$$

We can choose a, b, c arbitrary in $[\xi_2, \infty)$ with $b > c > a$. Substituting ξ with s , we multiply both sides of (33) by $H(\xi, s)$ and integrating it with respect to s from c to ξ for $\xi \in [c, b]$, then we get that

$$\begin{aligned}
 \int_c^\xi H(\xi, s) kA(s) \tilde{q}(s) \tilde{\phi}(s) ds &\leq - \int_c^\xi H(\xi, s) \tilde{\omega}'(s) ds \\
 &+ \int_c^\xi H(\xi, s) \frac{\tilde{\phi}'(s)}{\tilde{\phi}(s)} \tilde{\omega}(s) ds - \int_c^\xi H(\xi, s) \frac{\tilde{\delta}_1(s, \xi_2)}{\tilde{\phi}(s) \tilde{r}(s)} \tilde{\omega}^2(s) ds
 \end{aligned}$$

using the method of integration by parts

$$\begin{aligned}
 \int_c^\xi H(\xi, s) kA(s) \tilde{q}(s) \tilde{\phi}(s) ds &\leq H(\xi, c) \tilde{\omega}(c) \\
 &- \int_c^\xi \left[\left(\frac{H(\xi, s)}{\vartheta} \right)^{1/2} \tilde{\omega}(s) + \frac{1}{2} (\vartheta)^{1/2} Q_2(\xi, s) \right]^2 ds \\
 &+ \int_c^\xi \frac{\vartheta}{4} Q_2^2(\xi, s) ds,
 \end{aligned}$$

where $\vartheta = \frac{\tilde{r}(s)\tilde{\phi}(s)}{\tilde{\delta}_1(s, \xi_2)}$ and therefore,

$$\begin{aligned}
 \int_c^\xi H(\xi, s) kA(s) \tilde{q}(s) \tilde{\phi}(s) ds \\
 \leq H(\xi, c) \tilde{\omega}(c) + \int_c^\xi \frac{\vartheta}{4} Q_2^2(\xi, s) ds. \quad (34)
 \end{aligned}$$

Letting $\xi \rightarrow b^-$ in (34) and dividing both sides by $H(\xi, c)$, we obtain,

$$\begin{aligned}
 \frac{1}{H(b, c)} \int_c^b H(b, s) kA(s) \tilde{q}(s) \tilde{\phi}(s) ds \\
 \leq \tilde{\omega}(c) + \frac{1}{H(b, c)} \int_c^b \frac{\vartheta}{4} Q_2^2(b, s) ds. \quad (35)
 \end{aligned}$$

On the other hand, substituting ξ with s , multiplying both sides of (33) by $H(s, \xi)$ and integrating it with respect to s from ξ to c for $\xi \in (a, c]$, we deduce that

$$\int_{\xi}^c H(s, \xi) kA(s) \tilde{q}(s) \tilde{\phi}(s) ds \leq -H(c, \xi) \tilde{\omega}(c) + \int_{\xi}^c \frac{\tilde{r}(s) \tilde{\phi}(s)}{4\tilde{\delta}_1(s, \xi_2)} Q_1^2(s, \xi) ds. \quad (36)$$

Letting $\xi \rightarrow a^+$ in (36) and dividing both sides of it by $H(c, \xi)$ and we obtain

$$\frac{1}{H(c, a)} \int_a^c H(s, a) kA(s) \tilde{q}(s) \tilde{\phi}(s) ds \leq -\tilde{\omega}(c) + \frac{1}{H(c, a)} \int_a^c \frac{\tilde{r}(s) \tilde{\phi}(s)}{4\tilde{\delta}_1(s, \xi_2)} Q_1^2(s, a) ds. \quad (37)$$

A combination of (35) and (37) yields the inequality

$$\begin{aligned} & \frac{1}{H(b, c)} \int_c^b H(b, s) kA(s) \tilde{q}(s) \tilde{\phi}(s) ds \\ & + \frac{1}{H(c, a)} \int_a^c H(s, a) kA(s) \tilde{q}(s) \tilde{\phi}(s) ds \\ & \leq \frac{1}{H(b, c)} \int_c^b \frac{\tilde{r}(s) \tilde{\phi}(s)}{4\tilde{\delta}_1(s, \xi_2)} Q_2^2(b, s) ds \\ & + \frac{1}{H(c, a)} \int_a^c \frac{\tilde{r}(s) \tilde{\phi}(s)}{4\tilde{\delta}_1(s, \xi_2)} Q_1^2(s, a) ds \end{aligned} \quad (38)$$

which contradicts (29). Thus, the proof is complete. $\square$

Theorem 5. Under the conditions of Theorem 4, if for any sufficiently large $l \geq \xi_0$,

$$\limsup_{\xi \rightarrow \infty} \int_l^{\xi} \left[H(s, l) kA(s) \tilde{q}(s) \tilde{\phi}(s) - \frac{\theta}{4} Q_1^2(s, l) \right] ds > 0, \quad (39)$$

$$\limsup_{\xi \rightarrow \infty} \int_l^{\xi} \left[H(\xi, s) kA(s) \tilde{q}(s) \tilde{\phi}(s) - \frac{\theta}{4} Q_2^2(\xi, s) \right] ds > 0, \quad (40)$$

then (1) is oscillatory.

Proof. For any sufficiently large $T \geq \xi_0$, let $a = T$. If we choose $l = a$ in (39), then there exists $c > a$ such that

$$\int_a^c \left[H(s, a) kA(s) \tilde{q}(s) \tilde{\phi}(s) - \frac{\theta}{4} Q_1^2(s, a) \right] ds > 0. \quad (41)$$

If we choose $l = c > a$ in (40), then there exists $b > c$ such that

$$\int_c^b \left[H(b, s) kA(s) \tilde{q}(s) \tilde{\phi}(s) - \frac{\theta}{4} Q_2^2(b, s) \right] ds > 0. \quad (42)$$

Finally, we combine (41) and (42), to obtain (29). Thus, the proof is complete from Theorem 4. $\square$

If we choose $H(\xi, s) = (\xi - s)^\lambda$, $\xi \geq s \geq \xi_0$, where $\lambda > 1$ is a constant in Theorem 4 and Theorem 5, then we obtain the following corollaries.

Corollary 1. Under the conditions of Theorem 4, if for any sufficiently large $T \geq \xi_0$, there exist a, b, c with $T \leq a < c < b$ satisfying

$$\begin{aligned} & \frac{1}{(c-a)^\lambda} \int_a^c (s-a)^\lambda kA(s) \tilde{q}(s) \tilde{\phi}(s) ds \\ & + \frac{1}{(b-c)^\lambda} \int_c^b (b-s)^\lambda kA(s) \tilde{q}(s) \tilde{\phi}(s) ds \\ & > \frac{1}{(c-a)^\lambda} \int_a^c \frac{\theta}{4} (s-a)^{\lambda-2} \left(\lambda + \frac{\tilde{\phi}'(s)}{\tilde{\phi}(s)} (s-a) \right)^2 ds \\ & + \frac{1}{(b-c)^\lambda} \int_c^b \frac{\theta}{4} (b-s)^{\lambda-2} \left(\lambda - \frac{\tilde{\phi}'(s)}{\tilde{\phi}(s)} (b-s) \right)^2 ds \end{aligned} \quad (43)$$

then (1) is oscillatory.

Corollary 2. Under the conditions of Theorem 5, if for any sufficiently large $l \geq \xi_0$,

$$\begin{aligned} & \limsup_{\xi \rightarrow \infty} \int_l^{\xi} \left[(s-l)^\lambda kA(s) \tilde{q}(s) \tilde{\phi}(s) - \frac{\theta}{4} (s-l)^{\lambda-2} \left(\lambda + \frac{\tilde{\phi}'(s)}{\tilde{\phi}(s)} (s-l) \right)^2 \right] ds > 0, \\ & \limsup_{\xi \rightarrow \infty} \int_l^{\xi} \left[(\xi-s)^\lambda kA(s) \tilde{q}(s) \tilde{\phi}(s) - \frac{\theta}{4} (\xi-s)^{\lambda-2} \left(\lambda - \frac{\tilde{\phi}'(s)}{\tilde{\phi}(s)} (\xi-s) \right)^2 \right] ds > 0, \end{aligned} \quad (44)$$

then (1) is oscillatory.

Theorem 6. If (9)-(11) hold, ϕ is defined as in Theorem 4 and

$$\int_{\xi_0}^{\infty} \left[kA(s) \tilde{q}(s) \tilde{\phi}(s) - \frac{\tilde{r}(s) [\tilde{\phi}'(s)]^2}{4\tilde{\delta}_1(s, \xi_2) \tilde{\phi}(s)} \right] ds = \infty. \quad (45)$$

Then every solution of (1) is oscillatory or satisfies $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof. Suppose the contrary that $x(t)$ is a non-oscillatory solution of (1). Then without loss of generality, we may assume that there is a solution $x(t)$ of (1) such that $x(t) > 0$ on $[t_1, \infty)$, where t_1 is sufficiently large. By Lemma 1, we have $D_t^\alpha (r(t) D_t^\alpha x(t)) > 0$, $t \in [t_2, \infty)$, where $t_2 > t_1$ is sufficiently large, and either $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$ or $\lim_{t \rightarrow \infty} x(t) = 0$. Now we assume that $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$. Let $\omega(t)$, $\tilde{\omega}(\xi)$ be defined as in Theorem 1. Thus, we obtain (33). So,

$$\begin{aligned} \tilde{\omega}'(\xi) &\leq -kA(\xi) \tilde{q}(\xi) \tilde{\phi}(\xi) - \frac{\tilde{\delta}_1(\xi, \xi_2)}{\tilde{r}(\xi) \tilde{\phi}(\xi)} \tilde{\omega}^2(\xi) \\ &\quad + \frac{\tilde{\phi}'(\xi)}{\tilde{\phi}(\xi)} \tilde{\omega}(\xi) \\ &= -kA(\xi) \tilde{q}(\xi) \tilde{\phi}(\xi) + \frac{1}{4} \frac{\tilde{r}(\xi) [\tilde{\phi}'(\xi)]^2}{\tilde{\delta}_1(\xi, \xi_2) \tilde{\phi}(\xi)} \\ &\quad - \left[\left(\frac{\tilde{\delta}_1(\xi, \xi_2)}{\tilde{r}(\xi) \tilde{\phi}(\xi)} \right)^{1/2} \tilde{\omega}(\xi) - \frac{1}{2} \left(\frac{\tilde{r}(\xi) \tilde{\phi}(\xi)}{\tilde{\delta}_1(\xi, \xi_2)} \right)^{1/2} \frac{\tilde{\phi}'(\xi)}{\tilde{\phi}(\xi)} \right]^2 \\ &\leq -kA(\xi) \tilde{q}(\xi) \tilde{\phi}(\xi) + \frac{1}{4} \frac{\tilde{r}(\xi) [\tilde{\phi}'(\xi)]^2}{\tilde{\delta}_1(\xi, \xi_2) \tilde{\phi}(\xi)}, \quad \xi \geq \xi_2 \end{aligned}$$

and thus,

$$kA(\xi) \tilde{q}(\xi) \tilde{\phi}(\xi) - \frac{1}{4} \frac{\tilde{r}(\xi) [\tilde{\phi}'(\xi)]^2}{\tilde{\delta}_1(\xi, \xi_2) \tilde{\phi}(\xi)} \leq -\tilde{\omega}'(\xi). \quad (46)$$

Substituting ξ with s in (46) and integrating it with respect to s from ξ_2 to ξ , then we get that

$$\begin{aligned} \int_{\xi_2}^{\xi} \left[kA(s) \tilde{q}(s) \tilde{\phi}(s) - \frac{1}{4} \frac{\tilde{r}(s) [\tilde{\phi}'(s)]^2}{\tilde{\delta}_1(s, \xi_2) \tilde{\phi}(s)} \right] ds \\ \leq \tilde{\omega}(\xi_2) - \tilde{\omega}(\xi) \leq \tilde{\omega}(\xi_2) < \infty \end{aligned} \quad (47)$$

which contradicts (45). So, the proof is complete. $\square$

Theorem 7. Assume (9)-(11) hold, and there exists a function $G \in C([\xi_0, \infty), \mathbb{R})$ such that $G(\xi, \xi) = 0$, for $\xi \geq \xi_0$, $G(\xi, s) > 0$, for $\xi > s \geq \xi_0$, and G has a non-positive continuous partial derivative $G_s(\xi, s)$. If $\tilde{\phi}$ is defined as in Theorem 4 and

$$\limsup_{\xi \rightarrow \infty} \frac{1}{G(\xi, \xi_0)} \left\{ \int_{\xi_0}^{\xi} G(\xi, s) \left\{ \varpi - \frac{1}{4} \varrho \right\} ds \right\} = \infty, \quad (48)$$

where $\varrho = \frac{\tilde{r}(s) [\tilde{\phi}'(s)]^2}{\tilde{\delta}_1(s, \xi_2) \tilde{\phi}(s)}$ and $\varpi = kA(s) \tilde{q}(s) \tilde{\phi}(s)$. Then every solution of (1) is oscillatory or satisfies $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof. Suppose the contrary that $x(t)$ is a non-oscillatory solution of (1). Then without loss of generality, we may assume that there is a solution $x(t)$ of (1) such that $x(t) > 0$ on $[t_1, \infty)$, where t_1 is sufficiently large. By Lemma 1, we have $D_t^\alpha (r(t) D_t^\alpha x(t)) > 0$, $t \in [t_2, \infty)$, where $t_2 > t_1$ is sufficiently large, and either $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$ or $\lim_{t \rightarrow \infty} x(t) = 0$. Now we assume that $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$. Let $\omega(t)$, $\tilde{\omega}(\xi)$ be defined as in Theorem 4. Thus we have (46). So,

$$\varpi - \frac{\varrho}{4} \leq -\tilde{\omega}'(\xi), \quad \xi \geq \xi_2. \quad (49)$$

Substituting ξ with s in (49), multiplying both sides by $G(\xi, s)$ and then integrating it with respect to s from ξ_2 to ξ , we get that

$$\int_{\xi_2}^{\xi} G(\xi, s) \left\{ \varpi - \frac{\varrho}{4} \right\} ds \leq - \int_{\xi_2}^{\xi} G(\xi, s) \tilde{\omega}'(s) ds \quad (50)$$

and thus,

$$\begin{aligned} \int_{\xi_2}^{\xi} G(\xi, s) \left\{ \varpi - \frac{\varrho}{4} \right\} ds &\leq -G(\xi, \xi) \tilde{\omega}(\xi) \\ &\quad + G(\xi, \xi_2) \tilde{\omega}(\xi_2) \\ &\quad + \int_{\xi_2}^{\xi} G_s(\xi, s) \tilde{\omega}(s) ds \\ &\leq G(\xi, \xi_2) \tilde{\omega}(\xi_2). \end{aligned}$$

Then,

$$\int_{\xi_2}^{\xi} G(\xi, s) \left\{ \varpi - \frac{\varrho}{4} \right\} ds \leq G(\xi, \xi_0) \tilde{\omega}(\xi_2) \quad (51)$$

and

$$\begin{aligned} \int_{\xi_0}^{\xi} G(\xi, s) \left\{ \varpi - \frac{\varrho}{4} \right\} ds &= \int_{\xi_0}^{\xi_2} G(\xi, s) \left\{ \varpi - \frac{\varrho}{4} \right\} ds \\ &\quad + \int_{\xi_2}^{\xi} G(\xi, s) \left\{ \varpi - \frac{\varrho}{4} \right\} ds \\ &\leq G(\xi, \xi_0) \tilde{\omega}(\xi_2) + G(\xi, \xi_0) \int_{\xi_0}^{\xi_2} \left| \varpi - \frac{\varrho}{4} \right| ds. \end{aligned}$$

So,

$$\begin{aligned} \limsup_{\xi \rightarrow \infty} \frac{1}{G(\xi, \xi_0)} \left\{ \int_{\xi_0}^{\xi} G(\xi, s) \left\{ \varpi - \frac{1}{4} \varrho \right\} ds \right\} \\ \leq \tilde{\omega}(\xi_2) + \int_{\xi_0}^{\xi_2} \left| \varpi - \frac{1}{4} \varrho \right| ds < \infty \end{aligned}$$

which contradicts (48). So the proof is complete. $\square$

Letting $\phi(\xi) = \xi$ in (45),

3 Applications of the results

Example 1. Consider the nonlinear fractional differential equation with damping term

$$D_t^{1/2} \left[t^{1/4} D_t^{1/2} D_t^{1/2} x(t) \right] + \frac{\Gamma(3/2)}{\sqrt{t}} D_t^{1/2} D_t^{1/2} x(t) + t^{-1} x(t) \left(1 + \sin^2(x(t)) \right) = 0, \quad t \geq 2. \quad (52)$$

This corresponds to (1) with $t_0 = 2$; $\alpha = \frac{1}{2}$; $a(t) = t^{1/4}$; $r(t) = 1$; $p(t) = \Gamma(3/2)/\sqrt{t}$; $q(t) = t^{-1}$ and $f(x) = x + x \sin^2 x$. So, $f(x)/x = x(1 + \sin^2 x)/x \geq 1 = k$; $\xi_0 = 2^{1/2}/\Gamma(3/2)$; $\tilde{a}(\xi) = \sqrt{\xi \Gamma(3/2)}$; $\tilde{p}(\xi) = \xi^{-1}$; $\tilde{q}(\xi) = (\xi \Gamma(3/2))^{-2}$. Furthermore, $A(\xi) = \exp\left((\Gamma(3/2))^{-1/2} \int_{\xi_0}^{\xi} s^{-3/2} ds\right) = \exp\left((\Gamma(3/2))^{-1/2} [2\xi_0^{-1/2} - 2\xi^{-1/2}]\right)$ which implies $1 < A(\xi) \leq \left(\exp\left(2[\Gamma(3/2)]^{-1/2}\right) \xi_0^{-1/2}\right)$. On the other hand,

$$\begin{aligned} \tilde{\delta}_1(\xi, \xi_2) &= \int_{\xi_2}^{\xi} (1/A(s)) \tilde{a}(s) ds \\ &\geq [\Gamma(3/2)]^{-1/2} \exp\left(-2[\Gamma(3/2)]^{-1/2} \xi_0^{-1/2}\right) \int_{\xi_2}^{\xi} \frac{1}{\sqrt{s}} ds \\ &= 2[\Gamma(3/2)]^{-1/2} \exp\left(-2[\Gamma(3/2)]^{-1/2} \xi_0^{-1/2}\right) \\ &\quad \times \left(\sqrt{\xi} - \sqrt{\xi_2}\right) \end{aligned}$$

which implies $\lim_{\xi \rightarrow \infty} \tilde{\delta}_1(\xi, \xi_2) = \infty$, and so, (9) holds. Then, there exists a sufficiently large $T > \xi_2$ such that $\tilde{\delta}_1(\xi, \xi_2) > 1$ on $[T, \infty)$. In (10),

$$\int_{t_0}^{\infty} \frac{\alpha t^{\alpha-1}}{\Gamma(1+\alpha)r(t)} dt = \int_{\xi_0}^{\infty} \frac{1}{r(s)} ds = \int_{\xi_0}^{\infty} ds = \infty. \quad (53)$$

In (11),

$$\begin{aligned} &\int_{\xi_0}^{\infty} \frac{1}{\tilde{r}(\zeta)} \int_{\zeta}^{\infty} \frac{1}{A(\tau) \tilde{a}(\tau)} \int_{\tau}^{\infty} A(s) \tilde{q}(s) ds d\tau d\zeta \\ &\geq [\Gamma(3/2)]^{-5/2} \exp\left(-2[\Gamma(3/2)]^{-1/2} \xi_0^{-1/2}\right) \\ &\quad \times \int_{\xi_0}^{\infty} \int_{\zeta}^{\infty} \frac{1}{\sqrt{\tau}} \int_{\tau}^{\infty} s^{-2} ds d\tau d\zeta \\ &= \infty. \end{aligned}$$

(54)

$$\begin{aligned} &\int_{\xi_0}^{\infty} \left[kA(s) \tilde{q}(s) \tilde{\phi}(s) - \frac{\tilde{r}(s) [\tilde{\phi}'(s)]^2}{4\tilde{\delta}_1(s, \xi_2) \tilde{\phi}(s)} \right] ds \\ &= \int_{\xi_0}^{\infty} \left[A(s) s \left[s\Gamma\left(\frac{3}{2}\right) \right]^{-2} - \frac{1}{4\tilde{\delta}_1(s, \xi_2) s} \right] ds \\ &= \int_{\xi_0}^T \left[A(s) \left[\Gamma\left(\frac{3}{2}\right) \right]^{-2} - \frac{1}{4\tilde{\delta}_1(s, \xi_2)} \right] \frac{1}{s} ds \\ &\quad + \int_T^{\infty} \left[A(s) \left[\Gamma\left(\frac{3}{2}\right) \right]^{-2} - \frac{1}{4\tilde{\delta}_1(s, \xi_2)} \right] \frac{1}{s} ds \quad (55) \\ &\geq \int_{u_0}^T \left[A(s) \left[\Gamma\left(\frac{3}{2}\right) \right]^{-2} - \frac{1}{4\tilde{\delta}_1(s, \xi_2)} \right] \frac{1}{s} ds \\ &\quad + \int_T^{\infty} \left(\left[\Gamma\left(\frac{3}{2}\right) \right]^{-2} - \frac{1}{4} \right) \frac{1}{s} ds \\ &= \infty. \end{aligned}$$

So, (52) is oscillatory by Theorem 6.

Example 2. Consider the nonlinear fractional differential equation with damping term

$$D_t^{2/3} \left[t^{2/9} D_t^{2/3} D_t^{2/3} x(t) \right] + t^{-2/3} D_t^{2/3} D_t^{2/3} x(t) + x(t) + x^5(t) = 0, \quad t \geq 2. \quad (56)$$

This corresponds to (1) with $t_0 = 2$; $\alpha = \frac{2}{3}$; $a(t) = t^{2/9}$; $r(t) = 1$; $p(t) = t^{-2/3}$; $q(t) = 1$ and $f(x) = x + x^5$. So, $f(x)/x = x(1 + x^4)/x \geq 1 = k$; $\xi_0 = 2^{2/3}/\Gamma(5/3)$; $\tilde{a}(\xi) = (\xi \Gamma(5/3))^{1/3}$; $\tilde{r}(\xi) = 1$; $\tilde{p}(\xi) = (\xi \Gamma(5/3))^{-1}$; $\tilde{q}(\xi) = 1$. Furthermore, $A(\xi) = \exp\left((\Gamma(5/3))^{-4/3} \int_{\xi_0}^{\xi} s^{-4/3} ds\right) = \exp\left((\Gamma(5/3))^{-4/3} [3\xi_0^{-1/3} - 3\xi^{-1/3}]\right)$ which implies $1 < A(\xi) \leq \left(\exp\left(3[\Gamma(5/3)]^{-4/3}\right) \xi_0^{-1/3}\right)$. On the other hand,

$$\begin{aligned} \tilde{\delta}_1(\xi, \xi_2) &= \int_{\xi_2}^{\xi} (1/A(s)) \tilde{a}(s) ds \\ &\geq [\Gamma(5/3)]^{-1/3} \exp\left(-3[\Gamma(5/3)]^{-4/3} \xi_0^{-1/3}\right) \int_{\xi_2}^{\xi} s^{-1/3} ds \\ &= (3/2) [\Gamma(5/3)]^{-1/3} \exp\left(-3[\Gamma(5/3)]^{-4/3} \xi_0^{-1/3}\right) \\ &\quad \times \left(\xi^{2/3} - \xi_2^{2/3}\right) \end{aligned}$$

which implies $\lim_{\xi \rightarrow \infty} \tilde{\delta}_1(\xi, \xi_2) = \infty$ and so (9) holds. Then, there exists a sufficiently large $T > \xi_2$ such that $\tilde{\delta}_1(\xi, \xi_2) > 1$

on $[T, \infty)$. In (10),

$$\int_{t_0}^{\infty} \frac{\alpha t^{\alpha-1}}{\Gamma(1+\alpha)r(t)} dt = \int_{\xi_0}^{\infty} \frac{1}{r(s)} ds = \int_{\xi_0}^{\infty} ds = \infty. \quad (57)$$

In (11),

$$\begin{aligned} & \int_{\xi_0}^{\infty} \frac{1}{\tilde{r}(\zeta)} \int_{\zeta}^{\infty} \frac{1}{A(\tau)\tilde{a}(\tau)} \int_{\tau}^{\infty} A(s)\tilde{q}(s) ds d\tau d\zeta \\ & \geq [\Gamma(5/3)]^{-1/3} \exp\left(-3[\Gamma(5/3)]^{-4/3} \xi_0^{-1/3}\right) \\ & \times \int_{\xi_0}^{\infty} \int_{\zeta}^{\infty} \tau^{-1/3} \int_{\tau}^{\infty} ds d\tau d\zeta. \quad (58) \\ & = \infty \end{aligned}$$

Letting $\phi(\xi) = 1$ and $\lambda = 2$ in (44), for any sufficiently large l , we have

$$\begin{aligned} & \limsup_{\xi \rightarrow \infty} \int_l^{\xi} \left[(s-l)^{\lambda} \varpi \right. \\ & \quad \left. - \frac{\theta}{4} (s-l)^{\lambda-2} \left(\lambda + \frac{\tilde{\phi}'(s)}{\tilde{\phi}(s)} (s-l) \right)^2 \right] ds \\ & \geq \limsup_{\xi \rightarrow \infty} \int_l^{\xi} \left[(s-l)^2 - \frac{1}{4} (2)^2 \right] ds \\ & = \limsup_{\xi \rightarrow \infty} \int_l^{\xi} [(s-l)^2 - 1] ds = \infty \\ & \limsup_{\xi \rightarrow \infty} \int_l^{\xi} \left[(\xi-s)^{\lambda} \varpi \right. \\ & \quad \left. - \frac{\theta}{4} (\xi-s)^{\lambda-2} \left(\lambda - \frac{\tilde{\phi}'(s)}{\tilde{\phi}(s)} (\xi-s) \right)^2 \right] ds \\ & \geq \limsup_{\xi \rightarrow \infty} \int_l^{\xi} [(\xi-s)^2 - 1] ds = \infty. \end{aligned}$$

So (44) holds, and then we deduce that (56) is oscillatory by Corollary 2.

4 Conclusion

In this paper, we are concerned with the oscillation of solutions to nonlinear fractional differential equations with a damping term. Based on the variable transformation used in ξ , the fractional differential equations are converted into another differential equation of integer order. Then,

some new oscillation criteria for the equations are established by using inequalities, the integration average technique and the Riccati transformation. Consequently, it can be seen that this approach can also be applied to the oscillation of other fractional differential equations involving the modified Riemann-Liouville derivative.

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