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Duality relation among the Hamiltonian structures of a parametric coupled Korteweg-de Vries system

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Abstract: We obtain the full Hamiltonian structure for a parametric coupled KdV system. The coupled system arises from four different real basic lagrangians. The associated Hamiltonian functionals and the corresponding Poisson structures follow from the geometry of a constrained phase space by using the Dirac approach for constrained systems. The overall algebraic structure for the system is given in terms of two pencils of Poisson structures with associated Hamiltonians depending on the parameter of the Poisson pencils. The algebraic construction we present admits the most general space of observables related to the coupled system. We then construct two master lagrangians for the coupled system whose field equations are the ϵ-parametric Gardner equations obtained from the coupled KdV system through a Gardner transformation. In the weak limit $\epsilon \to 0$ the lagrangians reduce to the ones of the coupled KdV system while, after a suitable redefinition of the fields, in the strong limit $\epsilon \to \infty$ we obtain the lagrangians of the coupled modified KdV system. The Hamiltonian structures of the coupled KdV system follow from the Hamiltonian structures of the master system by taking the two limits $\epsilon \to 0$ and $\epsilon \to \infty$.

Keywords: partial differential equations; integrable systems; Lagrangian and Hamiltonian approach

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1 Introduction

Coupled Korteweg-de Vries (KdV) systems describe several physical interactions of interest. Hirota and Satsuma [1] proposed a model that describes interactions of two long waves with different dispersion relations. Gear and Grimshaw [2, 3] considered a coupled KdV system to describe linearly-stable internal waves in a stratified fluid.

More recently Lou, Tong, Hu and Tang [4] proposed models which may be used in the description of atmospheric and oceanic phenomena. Coupled KdV systems were also analyzed in [5–7]. An important area of interest of high energy physics related to coupled systems is provided by the supersymmetric extensions of KdV equations [8–14] and more generally by operator and Clifford valued extensions of KdV equation [15, 16].

In this work we consider a parametric coupled KdV system. For some values of the parameter, $\lambda < 0$, the system corresponds to the complexification of KdV equation. For $\lambda = 0$ the system corresponds to one of the Hirota-Satsuma coupled KdV systems, while for $\lambda > 0$ the system is equivalent to two decoupled KdV equations. We analyze the Hamiltonian formulation and the associated Poisson bracket structure of the system. Although some properties of the complexification of KdV arise directly from the analogous ones on the solutions of the KdV equation there are new properties, in particular, the full Hamiltonian structure, which does not have an analogous on the original real equation. In fact, the complexification approach gives rise only to holomorphic observables on phase space. The full Hamiltonian structure of the complex system give rise to self-adjoint Hamiltonian functionals, whose Hamiltonian flow are the complex KdV equations, and it provides the full structure of observables on phase space, not only the holomorphic ones.

The goal of this paper is to analyze the different Poisson structures associated with the coupled KdV system in order to determine which of them are compatible, in the sense that a linear combination of them is also a Poisson structure. This goal is important from the point of view of the integrability of the system. Moreover, it may also be an important first step in order to understand the quantization of the system in the sense of quantization deformation of Poisson structures [17].

The approach we will follow in our analysis is to construct a family of lagrangians from which the coupled KdV system is obtained by taking independent variations of the associated functional action with respect to the fields

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defining the lagrangian function. It turns out that these lagrangians are singular ones. This implies that the Hamiltonian construction, via a Legendre transformation is formulated on a constrained phase space. In all the cases we will consider the constraints turn out to be primary constraints and of the second class. The unconstrained phase space is equipped with a Poisson bracket structure, however since there are second class constraints we must obtain the Poisson bracket structure on the constrained submanifold of the phase space. This Poisson bracket is provided by the Dirac brackets [18]. It satisfies all the properties of a Poisson bracket, in particular the Jacobi identity. In this way starting from a lagrangian for the system we can construct a Poisson bracket structure, together with a Hamiltonian functional. This approach was followed for the KdV equation in [19, 20]. It provides a geometrical picture on phase space of the Hamiltonian structure of the integrable system. The other way to proceed is to find a Hamiltonian operator together with a Hamiltonian functional. Afterwards we may construct a Poisson bracket structure provided the Hamiltonian operator satisfies a differential restriction [21] ensuring that the Jacobi identity is satisfied. In this approach the set of allowed observables is only a subset of the space of observables of the more general formulation in terms of the constrained phase space approach.

In Section 2 we introduce the parametric coupled KdV system we wish to analyze. In Section 3 we obtain two lagrangian densities. The functional Gateaux derivatives of the corresponding actions give rise to the parametrically coupled KdV system. By a Legendre transformation we obtain the corresponding hamiltonians and associated Poisson structures. In Section 4 we introduce a Miura transformation which allows the construction of two additional lagrangian densities with the corresponding Hamiltonian and Poisson structures. In Section 5 we study the compatibility of the Poisson structures obtained in Sections 3 and 4. We obtain a pencil of Poisson bracket structures each of them associated to a Hamiltonian functional. In particular this implies compatibility between some of the Poisson structures. In Section 6 we construct a duality relation among the Hamiltonian structures. We construct two master lagrangians for the parametric coupled system whose field equations are the Gardner equations obtained from the coupled KdV system though a ϵ-parametric Gardner transformation. In the weak limit $\epsilon \to 0$ the lagrangians reduce to the ones of the coupled KdV system while, after a suitable redefinition of the fields, in the strong limit $\epsilon \to \infty$ we obtain the lagrangians of the coupled modified KdV system. The Hamiltonian structures of the coupled KdV system follow from the Hamiltonian structures of the master system by taking the two limits $\epsilon \to 0$ and $\epsilon \to \infty$.

2 The parametric coupled KdV system

We consider a coupled Korteweg-de Vries (KdV) system,formulated in terms of two real differentiable functions $u(x, t)$ and $v(x, t)$, given by the following partial differential equations:

$$u_t + uu_x + u_{xxx} + \lambda vv_x = 0 \tag{1}$$

$$v_t + u_x v + v_x u + v_{xxx} = 0, \tag{2}$$

where λ is a real parameter.

Here and in the sequel u and v belong to the real Schwartz space defined by

$$C^\infty_\downarrow = \left\{ w \in C^\infty(\mathbb{R}) / \lim_{x\to\pm\infty} x^p \frac{\partial^q}{\partial x^q} w = 0; p, q \geq 0 \right\}.$$

By a redefinition of v given by $v \to \frac{v}{\sqrt{|\lambda|}}$ we may reduce the values of $\lambda > 0$ to be $+1$ and $\lambda < 0$ to be -1. The systems for $\lambda = +1$, $\lambda = -1$ and $\lambda = 0$ are not equivalent. The $\lambda = -1$ case corresponds to the complexification of KdV equation.

The case $\lambda = +1$ corresponds to two decoupled KdV equations.

The system (1),(2) for $\lambda = -1$ describes a two-layer liquid model studied in references [2–4, 22]. It is a very interesting evolution system. It is known to have solutions developing singularities on a finite time [23]. Also, a class of solitonic solutions was reported in [24] via the Hirota approach [25].

The system (1),(2) (introduced in [26]) for $\lambda = 0$ corresponds to the ninth Hirota-Satsuma [1] coupled KdV system given in [6] (for the particular value of $k = 0$) (see also [5]) and is also included in the interesting study which relates integrable hierarchies with polynomial Lie algebras [7].

A Bäcklund transformation, the permutability theorem, the Gardner transformation as well as the Gardner equations for the coupled KdV system (1), (2), were obtained in [27]. Also a class of multisolitonic solutions, a class of periodic solutions and a regular static solution, which can be interpreted as a non-trivial background of the theory, were found in [27].

3 Poisson structures

In this and in the following section we will show that there exists four basic Hamiltonians and four associated basic Poisson structures for the coupled KdV system we are considering. We will use the method of Dirac for constrained systems to deduce them. The Hamiltonian as defined in quantum physics must be a selfadjoint operator conjugate to the time, hence our four Hamiltonians will be four real functionals in terms of the real fields which define them.

We rewrite (1) and (2) in terms of the Casimir potentials w and y given by

$$\begin{aligned} u(x,t) &= w_x(x,t) \\ v(x,t) &= y_x(x,t) \end{aligned}$$

and we obtain

$$\begin{aligned} w_{xt} + P[w,y] &= 0, \quad P[w,y] = w_x w_{xx} + w_{xxxx} + \lambda y_x y_{xx} \\ yxt + Q[w,y] &= 0, \quad Q[w,y] = w_{xx} y_x + y_{xx} w_x + y_{xxxx}. \end{aligned}$$

We use the Helmholtz procedure to obtain a lagrangian density from the field equations. We notice that the matrix constructed from the Gateaux derivatives of P and λQ is self-adjoint. We then get

$$\mathcal{L}_1 = -\frac{1}{2} w_x w_t + \int_0^1 \left(wP[\mu w, \mu y] + y\lambda Q[\mu w, \mu y]\right) d\mu$$

which explicitly gives

$$\mathcal{L}_1 = -\frac{1}{2} w_x w_t - \frac{1}{6} {w_x}^3 + \frac{1}{2} {w_{xx}}^2 - \frac{\lambda}{2} w_x {y_x}^2 - \frac{\lambda}{2} y_x y_t + \frac{\lambda}{2} {y_{xx}}^2,$$

for $\lambda \neq 0$.

We define the functional action $L_1(w,y) = \int_0^T dt \int_{-\infty}^{+\infty} dx\, \mathcal{L}_1$.

By taking independent variations of L_1 with respect to w and to y we obtain the field equations

$$\frac{\delta L_1}{\delta w} = 0 \quad , \quad \frac{\delta L_1}{\delta y} = 0,$$

which are the same as equations (1), (2). Here $\frac{\delta}{\delta w}$ and $\frac{\delta}{\delta y}$ denote the Gateaux functional derivative defined by

$$\frac{\delta}{\delta w} F(w) = \lim_{\epsilon \to 0} \frac{F(w+\epsilon h) - F(w)}{\epsilon}.$$

We now introduce a second functional action $L_2(w,y) = \int_0^T dt \int_{-\infty}^{+\infty} dx\, \mathcal{L}_2$ where

$$\mathcal{L}_2 = -\frac{1}{2} w_x y_t - \frac{1}{2} w_t y_x + \int_0^1 \left(yP[\mu w, \mu y] + wQ[\mu w, \mu y]\right) d\mu$$

which explicitly gives

$$\mathcal{L}_2 = -\frac{1}{2} w_x y_t - \frac{1}{2} w_t y_x - \frac{1}{2} w_x^2 y_x - y_x w_{xxx} - \frac{\lambda}{6} y_x^3$$

for any λ.

By taking independent variations of L_2 with respect to w and y we obtain the same field equations.

We will now construct the Hamiltonian structure associated to each of the lagrangians $\mathcal{L}_i$; $i = 1, 2$. We start by considering the lagrangian $\mathcal{L}_1$. We introduce the conjugate momenta associated to w and y, we denote them p and q respectively, we have

$$p = \frac{\partial \mathcal{L}_1}{\partial w_t} = -\frac{1}{2} w_x \quad , \quad q = \frac{\partial \mathcal{L}_1}{\partial y_t} = -\frac{\lambda}{2} y_x.$$

We define

$$\phi_1 \equiv p + \frac{1}{2} w_x \quad , \quad \phi_2 = q + \frac{\lambda}{2} y_x.$$

Hence $\phi_1 = \phi_2 = 0$. ϕ_1 and ϕ_2 do not have any w_t nor any y_t dependence, they are constraints on the phase space. It turns out that these are the only constraints on the phase space.

The Hamiltonian may be obtain directly from $\mathcal{L}_1$ by performing a Legendre transformation,

$$\mathcal{H}_1 = p w_t + q y_t - \mathcal{L}_1.$$

We obtain

$$\mathcal{H}_1 = \frac{1}{6} w_x^3 - \frac{1}{2} w_{xx}^2 + \frac{\lambda}{2} w_x y_x^2 - \frac{\lambda}{2} y_{xx}^2$$

and the corresponding Hamiltonian is $H_1 = \int_{-\infty}^{+\infty} dx\, \mathcal{H}_1$.

We introduce a Poisson structure on the phase space defined by

$$\begin{aligned} \left\{w(x), p(\hat{x})\right\}_{PB} &= \delta(x - \hat{x}) \\ \left\{y(x), q(\hat{x})\right\}_{PB} &= \delta(x - \hat{x}) \end{aligned}$$

with all other brackets between these variables being zero.

We then have that ϕ_1, ϕ_2 are second class constraints. In fact,

$$\begin{aligned} \left\{\phi_1(x), \phi_1(\hat{x})\right\}_{PB} &= \delta_x(x - \hat{x}) \\ \left\{\phi_1(x), \phi_2(\hat{x})\right\}_{PB} &= 0 \\ \left\{\phi_2(x), \phi_2(\hat{x})\right\}_{PB} &= \delta_x(x - \hat{x}). \end{aligned}$$

The lagrangian system is degenerate (singular), such system can be hamiltonized only by the use of Dirac's theory of constraints [28, 29].

Since we have a constrained phase space we must introduce the Dirac brackets corresponding to a Lie bracket structure on the constrained submanifold of phase space. For the second and third order degenerate lagrangians considered in this paper, one first needs to look for some kind of order reduction and then use the Dirac's theory of constraints to hamiltonize the system [19, 30, 31]. Equivalently, one can formulated the Dirac theory of constraints for higher order lagrangians using the above Poisson bracket structure and the relations

$$\left\{\partial_x^n w(x), \partial_{\hat{x}}^m p(\hat{x})\right\} = \partial_x^n \partial_{\hat{x}}^m \left\{w(x), p(\hat{x})\right\}.$$

The Dirac brackets between two functionals F and G on phase space is defined as

$$\begin{aligned} \{F, G\}_{DB} &= \{F, G\}_{PB} \\ &- \left\langle \left\langle \left\{F, \phi_i(x')\right\}_{PB} \mathbb{C}_{ij}(x', x'') \left\{\phi_j(x''), G\right\}_{PB} \right\rangle_{x'} \right\rangle_{x''}, \end{aligned} \quad (3)$$

where $<>_{x'}$ denotes integration on x' from $-\infty$ to $+\infty$. The indices $i, j = 1, 2$ and the $\mathbb{C}_{ij}(x', x'')$ are the components of the inverse of the matrix whose components are $\{\phi_i(x'), \phi_j(x''\}_{PB}$.

This matrix becomes

$$\begin{bmatrix} \partial_{x'}\delta(x'-x'') & 0 \\ 0 & \lambda\partial_{x'}\delta(x'-x'') \end{bmatrix}$$

and its inverse, satisfying

$$\begin{aligned} &\left\langle \begin{bmatrix} \partial_x\delta(x-x'') & 0 \\ 0 & \lambda\partial_x\delta(x-x'') \end{bmatrix} \begin{bmatrix} \mathbb{C}_{11}(x'',\hat{x}) & \mathbb{C}_{12}(x'',\hat{x}) \\ \mathbb{C}_{21}(x'',\hat{x}) & \mathbb{C}_{22}(x'',\hat{x}) \end{bmatrix} \right\rangle_{x''} = \\ &= \begin{bmatrix} \delta(x-\hat{x}) & 0 \\ 0 & \delta(x-\hat{x}) \end{bmatrix} \end{aligned}$$

is given by

$$\left[\mathbb{C}_{ij}(x', x'')\right] = \begin{bmatrix} \int^{x'} \delta(s-x'')ds & 0 \\ 0 & \frac{1}{\lambda}\int^{x'} \delta(s-x'')ds \end{bmatrix}.$$

It turns out, after some calculations, that

$$\begin{aligned} \left\{u(x), u(\hat{x})\right\}_{DB} &= -\partial_x\delta(x-\hat{x}), \\ \left\{v(x), v(\hat{x})\right\}_{DB} &= -\frac{1}{\lambda}\partial_x\delta(x-\hat{x}) \\ \left\{u(x), v(\hat{x})\right\}_{DB} &= 0. \end{aligned}$$

We notice that this Poisson bracket is not well defined for $\lambda = 0$. We have already assumed $\lambda \neq 0$.

From them we obtain the Hamilton equations, which are of course the same as (1), (2):

$$\begin{aligned} u_t &= \{u, H_1\}_{DB} = -uu_x - u_{xxx} - \lambda v v_x \\ v_t &= \{v, H_1\}_{DB} = -u_x v - v_x u - v_{xxx}. \end{aligned} \quad (4)$$

We notice that adding any function of the constraints to H_1 does not change the result, since the Dirac bracket of the constraints with any other local function of the phase space variables is zero.

Moreover, we may obtain directly the Dirac bracket of any two functionals $F(u, v)$ and $G(u, v)$ from (3) using the above bracket relations for u and v. We notice that the observables F and G in (3) may be functionals of w, y, p and q, not only of u and v. In this sense the phase space approach for singular lagrangians provides the most general space of observables. The same comment will be valid for the phase space construction using the action L_2 and L_1^M, L_2^M in the following sections.

The lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$ do not differ by a gauge term, despite that both of these lead to same equations of motion. Thus $\mathcal{L}_2$ represents an alternative lagrangian representation of the system. First, there can arise ambiguities in the association of symmetries with constants of the motion. Secondly, the same classical system, via alternative lagrangian description, can give rise to entirely different quantum mechanical systems. These points have been developed in the context of classical mechanics [32, 33]. The Noether theorem assures that associated to a differentiable symmetry of the lagrangian one has a conserved quantity. A symmetry of the action is also a symmetry of the field equations. However there may be symmetries of the field equations which are not symmetries of the action. In [27] we obtained an infinite sequence of polynomial local conserved quantities for equations (1) and (2). Most of them are not symmetries of the action L_1 nor of L_2.

We now consider the action L_2 and its associated Hamiltonian structure. In this case we denote the conjugate momenta to w and y by $\hat{p}$ and $\hat{q}$ respectively. We have

$$\hat{p} = -\frac{1}{2}y_x \quad , \quad \hat{q} = -\frac{1}{2}w_x.$$

The constraints become in this case

$$\widehat{\phi_1} = \hat{p} + \frac{1}{2}y_x = 0 \quad , \quad \widehat{\phi_2} = \hat{q} + \frac{1}{2}w_x = 0.$$

The corresponding Poisson brackets between ϕ_i and ϕ_j, $i, j = 1, 2$, are given by

$$\left\{\widehat{\phi_1}(x),\widehat{\phi_1}(x')\right\}_{PB}=0 \quad , \quad \left\{\widehat{\phi_2}(x),\widehat{\phi_2}(x')\right\}_{PB}=0,$$
$$\left\{\widehat{\phi_1}(x),\widehat{\phi_2}(x')\right\}_{PB}=\partial_x\delta(x-x').$$

The corresponding construction of the Dirac brackets yields

$$\{u(x),u(\hat{x})\}_{DB}=0 \quad , \quad \{v(x),v(\hat{x})\}_{DB}=0,$$
$$\{u(x),v(\hat{x})\}_{DB}=-\partial_x\delta(x-\hat{x}).$$

The Hamiltonian $H_2=\int_{-\infty}^{+\infty}dx\,\mathcal{H}_2$ is given by the Hamiltonian density

$$\mathcal{H}_2=\frac{1}{2}w_x^2y_x+y_xw_{xxx}+\frac{\lambda}{6}y_x^3.$$

The Hamilton equations

$$u_t(x)=\{u(x),H_2\}_{DB} \quad , \quad v_t(x)=\{v(x),H_2\}_{DB}$$

now using the corresponding Dirac brackets yields the same fields equations (1),(2) for any λ. We have thus constructed two Hamiltonian functionals and associated Poisson bracket structures. These two Hamiltonian structures arise directly from the basic actions L_1 and L_2. We will now construct two additional Hamiltonian structures by considering the Miura transformation.

The Hamiltonians H_1 and H_2, H_1^M and H_2^M in the following section, were presented in [34].

4 The Miura transformation

We consider the Miura transformation

$$\begin{aligned} u&=\mu_x-\tfrac{1}{6}\mu^2-\tfrac{\lambda}{6}\nu^2\\ v&=\nu_x-\tfrac{1}{3}\mu\nu. \end{aligned} \tag{5}$$

The corresponding modified KdV system (MKdVS) is given by

$$\begin{aligned} &\mu_t+\mu_{xxx}-\tfrac{1}{6}\mu^2\mu_x-\tfrac{\lambda}{6}\nu^2\mu_x-\tfrac{\lambda}{3}\mu\nu\nu_x=0\\ &\nu_t+\nu_{xxx}-\tfrac{1}{6}\mu^2\nu_x-\tfrac{\lambda}{6}\nu^2\nu_x-\tfrac{1}{3}\mu\nu\mu_x=0. \end{aligned} \tag{6}$$

These equations may be obtained from two actions, which we will denote $L_1^M = \int_0^T dt\int_{-\infty}^{+\infty}dx\,\mathcal{L}_1^M$ and $L_2^M = \int_0^T dt\int_{-\infty}^{+\infty}dx\,\mathcal{L}_2^M$.

The lagrangian densities $\mathcal{L}_1^M$, formulated for $\lambda\neq 0$, and $\mathcal{L}_2^M$, formulated for any λ, expressed in terms of σ,ρ where $\mu=\sigma_x$, $\nu=\rho_x$ are given by

$$\begin{aligned} \mathcal{L}_1^M &= -\frac{1}{2}\sigma_t\sigma_x-\frac{\lambda}{2}\rho_t\rho_x-\frac{1}{2}\sigma_x\sigma_{xxx}-\frac{\lambda}{2}\rho_x\rho_{xxx}\\ &+ \frac{1}{72}{\sigma_x}^4+\frac{\lambda^2}{72}{\rho_x}^4+\frac{\lambda}{12}\rho_x^2\sigma_x^2 \end{aligned} \tag{7}$$

and

$$\begin{aligned} \mathcal{L}_2^M &= -\frac{1}{2}\sigma_t\rho_x-\frac{1}{2}\sigma_x\rho_t-\sigma_{xxx}\rho_x\\ &+ \frac{1}{18}{\sigma_x}^3\rho_x+\frac{\lambda}{18}{\rho_x}^3\sigma_x \end{aligned} \tag{8}$$

respectively. We will now construct the Hamiltonian structure associated to $\mathcal{L}_1^M$.

We denote by α and β the conjugate momenta associated to σ and ρ respectively. We have

$$\alpha=\frac{\delta\mathcal{L}_1^M}{\delta\sigma_t}=-\frac{1}{2}\sigma_x \quad , \quad \beta=\frac{\delta\mathcal{L}_1^M}{\delta\rho_t}=-\frac{\lambda}{2}\rho_x.$$

These are constraints on the phase space.

The Hamiltonian H_1^M corresponding to this lagrangian density $\mathcal{L}_1^M$ is given by

$$\mathcal{H}_1^M=v^2-u^2$$

$$H_1^M=\int_{-\infty}^{+\infty}\mathcal{H}_1^M\,dx,$$

where u and v are given in terms of μ and ν by the Miura transformation. $\mathcal{H}_1^M$ was obtained starting with the lagrangian $\mathcal{L}_1^M$ and performing the Legendre transformation. After some calculations it turns out that it can be rewritten in terms of the original variables u and v. However when evaluating the Dirac brackets one must use its expressions in terms of the Casimir potentials. $\mathcal{L}_1^M$ is of course formulated in terms of the Casimir potentials. At this point it may be interesting to determine if it is possible to obtain a formulation of the problem directly in terms of original variables [35].

The construction of the Dirac brackets follows in the usual way. We end up with the following Poisson structure on the constrained submanifold,

$$\begin{aligned} \{\mu(x),\mu(\hat{x})\}_{DB} &= -\partial_x\delta(x-\hat{x})\\ \{\nu(x),\nu(\hat{x})\}_{DB} &= -\frac{1}{\lambda}\partial_x\delta(x-\hat{x})\\ \{\mu(x),\nu(\hat{x})\}_{DB} &= 0. \end{aligned}$$

From these Poisson bracket structure we obtain for the original u and v fields

$$
\begin{aligned}
\{u(x), u(\hat{x})\}_{DB} &= \partial_{xxx}\delta(x-\hat{x}) + \frac{1}{3}u_x\delta(x-\hat{x}) \\
&+ \frac{2}{3}u\partial_x\delta(x-\hat{x}) \\
\{v(x), v(\hat{x})\}_{DB} &= \frac{1}{\lambda}\partial_{xxx}\delta(x-\hat{x}) + \frac{1}{3\lambda}u_x\delta(x-\hat{x}) \\
&+ \frac{2}{3\lambda}u\partial_x\delta(x-\hat{x}) \\
\{u(x), v(\hat{x})\}_{DB} &= \frac{1}{3}v_x\delta(x-\hat{x}) + \frac{2}{3}v\partial_x\delta(x-\hat{x})
\end{aligned}
$$

which defines the Poisson structure on the original fields inherited from the Poisson structure on the constrained submanifold on the phase space associated to the modified KdV system. This Poisson bracket is not well defined for $\lambda \neq 0$. We have already assumed $\lambda \neq 0$.

From the Dirac brackets of u and v we may obtain directly the Hamiltonian field equations

$$
\begin{aligned}
u_t = \{u, H_1^M\}_{DB} = -uu_x - u_{xxx} - \lambda v v_x \\
v_t = \{v, H_1^M\}_{DB} = -v_{xxx} - (uv)_x
\end{aligned}
\tag{9}
$$

which, as it should be, coincide with system (1), (2).

We have then obtained the Poisson structure associated to the Hamiltonian H_1^M. Notice that the Hamiltonian formulation includes all the effects of the constraints. The point is that by using the Dirac brackets the second class constraints commute with any other obervable. It is interesting to remark that the Miura-like transformation, more precisely the Gardner transformation allows to obtain an infinite sequence of local polynomial conserved quantities [27].

We now proceed to obtain a second Poisson structure starting from the Lagrangian $\mathcal{L}_2^M$.

The Hamiltonian obtained via a Legendre transformation is given by $H_2^M = \int_{-\infty}^{+\infty}(-uv)\,dx$ where u and v are functions of μ and ν according to the Miura transformation. We use as before $\mu = \sigma_x$, $\nu = \rho_x$.

We denote by $\hat{\alpha}$ and $\hat{\beta}$ the conjugate momenta associated to σ and ρ respectively.

The constraints on phase space become now

$$
\begin{aligned}
\hat{\alpha} &= -\frac{1}{2}\rho_x \\
\hat{\beta} &= -\frac{1}{2}\sigma_x.
\end{aligned}
$$

The Dirac brackets are

$$
\begin{aligned}
\{\mu(x), \mu(\hat{x})\}_{DB} &= 0 \\
\{\nu(x), \nu(\hat{x})\}_{DB} &= 0 \\
\{\mu(x), \nu(\hat{x})\}_{DB} &= -\partial_x\delta(x-\hat{x}).
\end{aligned}
$$

We then obtain, for any λ,

$$
\begin{aligned}
\{u(x), u(\hat{x})\}_{DB} &= \frac{\lambda}{3}v_x\delta(x-\hat{x}) + \frac{2\lambda}{3}v\partial_x\delta(x-\hat{x}) \\
\{v(x), v(\hat{x})\}_{DB} &= \frac{1}{3}v_x\delta(x-\hat{x}) + \frac{2}{3}v\partial_x\delta(x-\hat{x}) \\
\{u(x), v(\hat{x})\}_{DB} &= \partial_{xxx}\delta(x-\hat{x}) + \frac{1}{3}u_x\delta(x-\hat{x}) \\
&+ \frac{2}{3}u\partial_x\delta(x-\hat{x}).
\end{aligned}
$$

This is the Poisson bracket structure inherited from the second Poisson structure on the modified phase space. One may directly verify that the corresponding Hamilton equations exactly coincide with equations (1), (2). We have then constructed four basic lagrangians and associated Hamiltonian functionals together with four basic Poisson structures.

5 Two pencils of Poisson structures for the coupled system

In this section we show the existence of two pencils of Poisson structures for the system (1) and (2). The strategy will be to introduce two parametric lagrangian densities $\mathcal{L}_k$ and $\mathcal{L}_k^M$. For the value of the parameter $k = 1$, $\mathcal{L}_k$ reduces to the lagrangian density $\mathcal{L}_1$ while for $k = 0$ it reduces to the lagrangian density $\mathcal{L}_2$ in Section 3. Similarly for $k = 1$, $\mathcal{L}_k^M$ reduces to $\mathcal{L}_1^M$ and for $k = 0$ it reduces to $\mathcal{L}_2^M$ in Section 4. We then find the associated Poisson structures for the parametric Hamiltonians constructed via a Legendre transformation. Each Poisson structure is k dependent. By choosing suitable values of k we will show that the sum of the Poisson structures in Section 3 also determines a Poisson structure. In the same way the sum of the Poisson structures in Section 4 determines a Poisson structure. In this way we will show the existence ot two Poisson pencils.

We now introduce the parametric lagrangian density $\mathcal{L}_k$, where k is a real parameter, associated to the two basic actions L_1 and L_2.

We define the lagrangian density

$$\mathcal{L}_k = k\mathcal{L}_1 + (1-k)\mathcal{L}_2.$$

The field equations obtained from this lagrangian density are equivalent to (1) and (2) in the following cases: If $\lambda < 0$ for any k. If $\lambda = 0$, for $k \neq 1$. If $\lambda > 0$ for $k \neq \frac{1}{1+\sqrt{\lambda}}$ and $k \neq \frac{1}{1-\sqrt{\lambda}}$. From now on we will exclude these particular values of k. The corresponding Hamiltonian density constructed through the Legendre transformation is given by

$$\mathcal{H}_k = pw_t + qy_t - \mathcal{L}_k = k\mathcal{H}_1 + (1-k)\mathcal{H}_2$$

and the primary constraints by

$$\phi_1 \equiv \frac{k}{2}w_x + \frac{(1-k)}{2}y_x + p = 0 \tag{10}$$
$$\phi_2 \equiv \frac{\lambda k}{2}y_x + \frac{(1-k)}{2}w_x + q = 0. \tag{11}$$

These are the only constraints on phase space.

The Poisson brackets on the unconstrained phase space are

$$\begin{aligned}\left\{\phi_1(x), \phi_1(\hat{x})\right\}_{PB} &= k\partial_x\delta(x-\hat{x})\\ \left\{\phi_2(x), \phi_2(\hat{x})\right\}_{PB} &= \lambda k\partial_x\delta(x-\hat{x})\\ \left\{\phi_1(x), \phi_2(\hat{x})\right\}_{PB} &= (1-k)\,\partial_x\delta(x-\hat{x}).\end{aligned}$$

Hence they are second class constraints.

We will denote by $\{\}_{DB}^k$ the Dirac bracket corresponding to the parameter k.

The Dirac brackets are then given by

$$\begin{aligned}\left\{u(x), u(\hat{x})\right\}_{DB}^k &= \frac{\lambda k}{-\lambda k^2 + (1-k)^2}\partial_x\delta(x-\hat{x})\\ \left\{v(x), v(\hat{x})\right\}_{DB}^k &= \frac{k}{-\lambda k^2 + (1-k)^2}\partial_x\delta(x-\hat{x})\\ \left\{u(x), v(\hat{x})\right\}_{DB}^k &= \frac{1-k}{-\lambda k^2 + (1-k)^2}\left(-\partial_x\delta\left(x-\hat{x}\right)\right),\end{aligned}$$

where the denominator is different from zero for the values of k we are considering. They define the Poisson structure for the Hamiltonian $H_k = \int_{-\infty}^{+\infty} \mathcal{H}_k dx$.

The associated Hamilton equations coincide with the coupled equations (1), (2). It is interesting to notice that the above Poisson structure is a linear combination of the Dirac brackets associated to Hamiltonians H_1 and H_2. In the present notation H_2 corresponds to $k = 0$.

We then have

$$\begin{aligned}\{F, G\}_{DB}^k &= \frac{-\lambda k}{-\lambda k^2 + (1-k)^2}\{F, G\}_{DB}^1\\ &+ \frac{1-k}{-\lambda k^2 + (1-k)^2}\{F, G\}_{DB}^0\end{aligned}$$

where F, G are any functionals of u and v and $\{F, G\}_{DB}^1$, $\{F, G\}_{DB}^0$ are the Poisson brackets defined in Section 3. In particular for any λ different from one and zero, and $k = \frac{1}{1-\lambda}$, we obtain

$$\{F, G\}_{DB}^k = \{F, G\}_{DB}^1 + \{F, G\}_{DB}^0 .$$

Consequently, since $\{F, G\}_{DB}^k$ is a Poisson bracket, then the two basic Poisson brackets for every $\lambda \neq 0, 1$ are then compatible. In fact, if the sum determines a Poisson bracket the any linear combination of them also determines a Poisson bracket.

We also notice that for any k and $\lambda = -1$, using the above Poisson bracket structure, one gets

$$\left\{u(x) + iv(x), u(\hat{x}) - iv(\hat{x})\right\}_{DB}^k = 0, \tag{12}$$
$$\left\{u(x) + iv(x), u(\hat{x}) + iv(\hat{x})\right\}_{DB}^k = \tag{13}$$
$$-\frac{2}{k^2 + (1-k)^2}\partial_x\delta(x-\hat{x}).$$

We emphasize that only (13) arises from the complexification of the corresponding Poisson structure for real KdV. The relation (12) follows in our approach from first principles. It is not imposed by hand. The existence of a local real Hamiltonian H_k for each k is a non-trivial feature of the system (1), (2) and is not an algebraic consequence of the complexification of the real KdV equation.

We may now consider the case $\lambda = 0$. The Poisson bracket for any $k \neq 1$ becomes

$$\{F, G\}_{DB}^k = \frac{k}{2(1-k)^2}\{F, G\}_{DB}^{\frac{1}{2}} + \frac{1-2k}{(1-k)^2}\{F, G\}_{DB}^0 \tag{14}$$

in particular for $k = \frac{2}{5}$ the two coefficients are equal, hence the Poisson brackets for $k = \frac{1}{2}$ and $k = 0$ are compatible.

We have thus constructed a pencil of Poisson structures with an associated local real Hamiltonian $H_k = \int_{-\infty}^{+\infty} \mathcal{H}_k$.

We now introduce, as we have already done with $\mathcal{L}_1$ and $\mathcal{L}_2$, a parametric lagrangian density $\mathcal{L}_k^M = k\mathcal{L}_1^M + (1-k)\mathcal{L}_2^M$. The associated hamiltonian density is given by $\mathcal{H}_k^M = k\mathcal{H}_1^M + (1-k)\mathcal{H}_2^M$ in terms of the other two basic hamiltonian densities. The constraints on phase space are given by

$$\begin{aligned}\phi_1 &\equiv \alpha + \frac{k}{2}\sigma_x + \frac{(1-k)}{2}\rho_x = 0\\ \phi_2 &\equiv \beta\frac{\lambda k}{2}\rho_x + \frac{(1-k)}{2}\sigma_x = 0\end{aligned}$$

these constraints are the only ones on the phase space. The Poisson brackets on the unconstrained phase space are

$$\begin{aligned}\left\{\phi_1(x), \phi_1(\hat{x})\right\}_{PB} &= k\partial_x\delta(x-\hat{x})\\ \left\{\phi_2(x), \phi_2(\hat{x})\right\}_{PB} &= \lambda k\partial_x\delta(x-\hat{x})\\ \left\{\phi_1(x), \phi_2(\hat{x})\right\}_{PB} &= (1-k)\,\partial_x\delta(x-\hat{x}).\end{aligned}$$

Hence they are second class constraints.

The Dirac brackets are then given by

$$
\begin{aligned}
\left\{u(x), u(\hat{x})\right\}_{DB}^{k} &= -\frac{\lambda k}{-\lambda k^2 + (1-k)^2} \\
&\left(\partial_{xxx}\delta(x-\hat{x}) + \frac{1}{3}u_x\delta(x-\hat{x}) + \frac{2}{3}u\partial_x\delta(x-\hat{x})\right) \\
+ &\frac{\lambda(1-k)}{-\lambda k^2 + (1-k)^2}\left(\frac{1}{3}v_x\delta(x-\hat{x}) + \frac{2}{3}v\partial_x\delta(x-\hat{x})\right) \\
\left\{u(x), v(\hat{x})\right\}_{DB}^{k} &= \frac{(1-k)}{-\lambda k^2 + (1-k)^2} \\
&\left(\partial_{xxx}\delta(x-\hat{x}) + \frac{1}{3}u_x\delta(x-\hat{x}) + \frac{2}{3}u\partial_x\delta(x-\hat{x})\right) \\
- &\frac{\lambda k}{-\lambda k^2 + (1-k)^2}\left(\frac{1}{3}v_x\delta(x-\hat{x}) + \frac{2}{3}v\partial_x\delta(x-\hat{x})\right) \\
\left\{v(x), v(\hat{x})\right\}_{DB}^{k} &= -\frac{k}{-\lambda k^2 + (1-k)^2} \\
&\left(\partial_{xxx}\delta\left(x-\hat{x}\right) + \frac{1}{3}u_x\delta(x-\hat{x}) + \frac{2}{3}u\partial_x\delta(x-\hat{x})\right) \\
+ &\frac{(1-k)}{-\lambda k^2 + (1-k)^2}\left(\frac{1}{3}v_x\delta(x-\hat{x}) + \frac{2}{3}v\partial_x\delta(x-\hat{x})\right).
\end{aligned} \tag{15}
$$

It follows from the construction that the Hamilton equations in terms of the corresponding Poisson structure,

$$
u_t = \left\{u(x), H_k^M)\right\}_{DB}^{k} \quad , \quad v_t = \left\{v(x), H_k^M)\right\}_{DB}^{k}
$$

are equivalent to the coupled KdV system (1),(2).

As in the previous case the pencil of Poisson structures can be rewritten in terms of the basic Poisson structures which corresponds to $k = 1$ and $k = 0$ in (14):

$$
\begin{aligned}
\{F, G\}_{DB}^{k} &= \frac{-\lambda k}{-\lambda k^2 + (1-k)^2}\{F, G\}_{DB}^{1} \\
&+ \frac{1-k}{-\lambda k^2 + (1-k)^2}\{F, G\}_{DB}^{0},
\end{aligned}
$$

where $\{F, G\}_{DB}^{1}$ and $\{F, G\}_{DB}^{0}$ are the Poisson structures defined in Section 4.

We notice that this decomposition is the same as in previous case, however the basic Poisson structure are different.

In particular for $k = \frac{1}{1-\lambda}$, $\lambda \neq 0, 1$, the $\{,\}_{DB}^{k}$ is the sum of the $\{,\}_{DB}^{1}$ and $\{,\}_{DB}^{0}$ basic Poisson structures. For $\lambda = 0$ and $k \neq 1$ the same relation (14) holds for the Poisson bracket we are now considering. These are then compatible Poisson structures.

We notice that by construction ϕ_1 and ϕ_2 as well as any functional of them, in all the cases we have considered, are Casimirs of the Poisson structure defined in terms of the Dirac brackets. In fact,

$$
\begin{aligned}
\left\{F, \phi_1)\right\}_{DB} &= 0 \\
\left\{F, \phi_2)\right\}_{DB} &= 0
\end{aligned}
$$

for any functional F on phase space. This is a general property of the Dirac bracket.

It is a non-trivial feature that for each real k, the parameter of the pencil of Poisson structures, there are Hamiltonians H_k and H_k^M which give rise to the coupled KdV system when the corresponding Poisson structure is used.

6 Duality among the Hamiltonian structures

In this section we show the existence of two ϵ-deformed Hamiltonians and their corresponding ϵ-deformed Poisson structures. In the limit $\epsilon \to 0$ they reduced to the Hamiltonians and Poisson structures obtained in Section 3. In the limit $\epsilon \to \infty$ they reduce to the Hamiltonians and Poisson structures obtained in Section 4. The ϵ-deformation for the coupled KdV system is analogous to the duality transformation in quantum field theory. One formulation can be analyzed in perturbation theory for small values of the coupling constant ϵ, the weak coupling limit, while the dual formulation can be analyzed perturbatively as an expansion in $\frac{1}{\epsilon}$, the strong coupling limit.

The Hamiltonian, constraints and Poisson structures we will present in this section are obtained following the same approach we have developed in previous sections. We will only provide the results.

The associated Gardner transformation and Gardner equations for the system (1), (2) are given by [27]

$$
u = r + \varepsilon r_x - \frac{1}{6}\varepsilon^2\left(r^2 + \lambda s^2\right) \tag{16}
$$

$$
v = s + \varepsilon s_x - \frac{1}{3}\varepsilon^2 rs \tag{17}
$$

and

$$
r_t + r_{xxx} + rr_x + \lambda ss_x - \frac{1}{6}\varepsilon^2\left[\left(r^2 + \lambda s^2\right)r_x + 2\lambda rss_x\right] = 0 \tag{18}
$$

$$
s_t + s_{xxx} + rs_x + sr_x - \frac{1}{6}\varepsilon^2\left[\left(r^2 + \lambda s^2\right)s_x + 2rsr_x\right] = 0. \tag{19}
$$

Any solution of the Gardner equations define through Gardner transformation a solution of the system (1), (2).

It follows as in the case of KdV equation and the corresponding Gardner transformation [36] that if we take the limit $\epsilon \to 0$ we obtain from (16), (17)

$$u = r \tag{20}$$

$$v = s \tag{21}$$

and from (18), (19) after replacing (20), (21) we get the original KdV coupled system (1), (2).

On the other hand, if we redefine

$$\mu \equiv \epsilon r \tag{22}$$

$$\nu \equiv \epsilon s \tag{23}$$

and rewrite (16), (17) we get

$$u = \frac{\mu}{\epsilon} + \mu_x - \frac{1}{6}\mu^2 - \frac{1}{6}\lambda\nu^2 \tag{24}$$

$$v = \frac{\nu}{\epsilon} + \nu_x - \frac{1}{3}\mu\nu. \tag{25}$$

Taking the limit $\epsilon \to \infty$ we have

$$\hat{u} = \mu_x - \frac{1}{6}\mu^2 - \frac{1}{6}\lambda\mu^2 \tag{26}$$

$$\hat{v} = \nu_x - \frac{1}{3}\mu\nu \tag{27}$$

which are exactly equations given by (5).

If we proceed in the same way with (18), (19) we obtain the modified KdV system equations (6). That means that if μ, ν satisfies the MKdV then $\hat{u}, \hat{v}$ satisfies equations (1), (2).

This result is analogous to the one known for the KdV equation and its Miura transformation.

We go now one step forward, beyond this relation between field equations, and prove that there exists a master lagrangian form which by taking variations with respect to r and s we obtain the ϵ-parametric Gardner equations (18), (19). Moreover, we will construct two different lagrangians with such property and the associated Hamiltonian structures. The corresponding ϵ-parametric Poisson structures are compatible. In the limit $\epsilon \to 0$ the ϵ-parametric Hamiltonian structures reduce to the compatible Hamiltonian structures obtained in Section 3 and with the $\epsilon \to \infty$ limit, after suitable redefinition of fields, we obtain the two compatible Hamiltonian structures obtained in Section 4. There is a duality relation between the Hamiltonian structure of the parametric coupled KdV system (1), (2) and the modified parametric system (MKdV) (6). In the weak limit $\epsilon \to 0$ we get one Hamiltonian structure and in the strong limit $\epsilon \to \infty$ we get the modified Hamiltonian structure.

By freezing the field v to zero we obtain from (1), (2) the KdV equation and the same occurs with the ϵ-parameter Hamiltonian structure of system (1), (2). It reduces to a ϵ-parametric Hamiltonian structure of KdV equation (which to our knowledge had not been constructed before).

In particular, the Poisson structure of the KdV system (1), (2) is a ϵ-deformation of the Poisson structure of MKdV system which is equivalent to the Virasoro algebra.

We now proceed to the construction of the master Gardner lagrangians.

We introduce the fields $w(x,t)$, $y(x,t)$ through

$$r = w_x \quad , \quad s = y_x .$$

The first master Gardner lagrangian density, which works for $\lambda \neq 0$, is given by

$$\begin{aligned}
\mathcal{L}_{G1} = \; & - \frac{1}{2}w_x w_t - \frac{1}{6}(w_x)^3 + \frac{1}{2}(w_{xx})^2 - \frac{\lambda}{2}w_x(y_x)^2 \\
& - \frac{\lambda}{2}y_x y_t + \frac{\lambda}{2}(y_{xx})^2 \\
& - \frac{1}{6}\epsilon^2\left[-\frac{1}{12}(w_x)^4 - \frac{\lambda}{2}(w_x)^2(y_x)^2\right] + \frac{\epsilon^2}{72}\lambda^2(y_x)^4
\end{aligned}$$

and the second master Gardner lagrangian density is given, for any λ, by

$$\begin{aligned}
\mathcal{L}_{G2} = \; & - \frac{1}{2}w_x y_t - \frac{1}{2}w_t y_x - \frac{1}{2}(w_x)^2 y_x - y_x w_{xxx} - \frac{\lambda}{6}(y_x)^3 \\
& + \frac{1}{18}\epsilon^2(w_x)^3 y_x + \frac{1}{18}\epsilon^2\lambda(y_x)^3 w_x .
\end{aligned}$$

If we take the limit $\epsilon \to 0$ of the above expressions, we obtain lagrangians densities $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively given in Section 3.

If we redefine

$$\sigma = \epsilon w \quad , \quad \rho = \epsilon y$$

$$\mathcal{L}^M_{G1} = \epsilon^2\mathcal{L}_{G1} \quad , \quad \mathcal{L}^M_{G2} = \epsilon^2\mathcal{L}_{G2}$$

and take the limit $\epsilon \to \infty$ we get

$$\begin{aligned}
\lim_{\epsilon\to\infty}\mathcal{L}^M_{G1}(\sigma,\rho) &= \mathcal{L}^M_1(\sigma,\rho) \\
\lim_{\epsilon\to\infty}\mathcal{L}^M_{G2}(\sigma,\rho) &= \mathcal{L}^M_2(\sigma,\rho).
\end{aligned}$$

The Hamiltonian structure associated to $\mathcal{L}^M_{G1}(\sigma,\rho)$ and $\mathcal{L}^M_{G2}(\sigma,\rho)$ arises in the standard way.

We introduce the conjugate momenta associated to w and y. We denote them p and q. We have

$$\begin{aligned}
p = \frac{\partial\mathcal{L}_{G1}}{\partial w_t} &= -\frac{1}{2}w_x \\
q = \frac{\partial\mathcal{L}_{G2}}{\partial y_t} &= -\frac{\lambda}{2}y_x
\end{aligned}$$

and define $\phi_1 \equiv p + \frac{1}{2}w_x$, $\phi_2 \equiv q + \frac{\lambda}{2}y_x$.

$\phi_1 = 0$ and $\phi_2 = 0$ are constraints on the phase space. They are second class constraints. Notice that this is the same structure of phase space as in Section 3 because the ϵ-parameter does not appear in the definition of the conjugate momenta. The Hamiltonian density may be obtained by performing a Legendre transformation

$$\mathcal{H}_{G1} = pw_t + qy_t - \mathcal{L}_{G1}.$$

We get

$$\begin{aligned}\mathcal{H}_{G1} &= \frac{1}{6}(w_x)^3 - \frac{1}{2}(w_{xx})^2 + \frac{\lambda}{2}w_x(y_x)^2 - \frac{\lambda}{2}(y_{xx})^2 \\ &+ \frac{1}{6}\epsilon^2\left[-\frac{1}{12}(w_x)^4 - \frac{\lambda}{2}(w_x)^2(y_x)^2 - \frac{1}{12}\lambda^2(y_x)^4\right].\end{aligned}$$

The Dirac brackets between the canonical variables w, p, y, q are the same as was obtained in section 3 since the ϵ-parameter does not appear in the constraints. However the corresponding brackets among u, v are different because the Gardner transformation involves the parameter ϵ. In the limit $\epsilon \to 0$ we obtain the same Dirac brackets for u and v as in Section 3. The same analysis can be performed for $\mathcal{L}_{G2}$. The conjugate momenta and contraints as well as the Dirac brackets among the canonical variables w, p, y, q are the same as in second part of Section 3. The Hamiltonian is given by

$$\begin{aligned}\mathcal{H}_{G2} &= \frac{1}{2}(w_x)^2y_x + y_xw_{xxx} + \frac{\lambda}{6}(y_x)^3 \\ &- \frac{1}{18}\epsilon^2(w_x)^3y_x - \frac{1}{18}\epsilon^2\lambda(y_x)^3w_x.\end{aligned}$$

In the limit $\epsilon \to 0$ it reduces to the Hamiltonian $\mathcal{H}_2$ in Section 3. In the same way by redefining $\epsilon^2\mathcal{H}_{G1}, \epsilon^2\mathcal{H}_{G2}$ and taking $\epsilon \to \infty$ we get the Hamiltonians $\mathcal{H}_1^M, \mathcal{H}_2^M$ respectively, presented in Section 4.

The Poisson structures in both limits are the ones explicitly given in in Sections 3 and 4.

The construction of the ϵ-parametric Gardner Hamiltonian and its Poisson structures unifies all the Hamiltonian structures of the coupled KdV system and make manifest the duality relation among the KdV and the modified KdV systems.

7 Conclusions

We obtained the full Hamiltonian structure for a coupled parametric KdV system. We started from four basic singular lagrangians. The associated Hamiltonian formulation on phase space is restricted by second class constraints. The Poisson structure on the constrained variety of phase space was obtained using the Dirac approach. The Dirac brackets on the constrained phase space yields the most general structure of observables. A subset of them are functionals of the original fields $u(x, t)$, $v(x, t)$ of the coupled KdV system. We then constructed two pencils of Poisson brackets each of them with an associated parametric Hamiltonian in terms of the same parameter of each pencil.

Each pencil of Poisson brackets is obtained from two compatible Poisson brackets of the same dimension. Consequently it is not possible to construct a hierarchy of higher dimensional Hamiltonians from them. However the two pencils of Poisson brackets are of different dimensions, hence one may construct a hierarchy of higher order Hamiltonians as in the KdV case.

Finally we constructed two master lagrangians for the parametric coupled KdV system whose field equations are the Gardner equations obtained from the coupled KdV through a ϵ-parametric Gardner transformation. The Gardner transformation is a non-linear transformation of the fields for each value of the parameter ϵ. It defines a duality transformation between the theories obtained in the two limits $\epsilon \to 0$ and $\epsilon \to \infty$.

In the weak limit $\epsilon \to 0$ the lagrangians reduce to the ones of the coupled KdV system while, after a suitable redefinition of the fields, in the strong limit $\epsilon \to \infty$ we obtain the lagrangians of the coupled modified KdV system. The Hamiltonian structures of the coupled KdV system follow from the Hamiltonian structures of the master system by taking the two limits $\epsilon \to 0$ and $\epsilon \to \infty$.

We have thus disentangled all the Hamiltonian structure (a very rich one) associated with the parametric coupled KdV system. This goal which we have fulfilled is the first step towards the quantization of the coupled KdV system in the sense of quantization deformation of the Poisson structures. The duality relation may be very important to relate the quantization of all the Poisson structures.

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