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Bilinearization and new multi-soliton solutions of mKdV hierarchy with time-dependent coefficients

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Abstract: In this paper, Hirota's bilinear method is extended to a new modified Kortweg–de Vries (mKdV) hierarchy with time-dependent coefficients. To begin with, we give a bilinear form of the mKdV hierarchy. Based on the bilinear form, we then obtain one-soliton, two-soliton and three-soliton solutions of the mKdV hierarchy. Finally, a uniform formula for the explicit N-soliton solution of the mKdV hierarchy is summarized. It is graphically shown that the obtained soliton solutions with time-dependent functions possess time-varying velocities in the process of propagation.

Keywords: Hirota's bilinear method; bilinear form; soliton solution; mKdV hierarchy with time-dependent coefficients

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1 Introduction

In 1965, the concept of a soliton was first defined by Zabusky and Kruskal [1]. With the development of soliton theory, Zabusky and Kruskal's classical solitons have been generalized to a more general class of solitons with at least time-varying velocities, such as those observed in nonlinear optics and Bose-Einstein condensate. It was Serkin, Hasegawa and Belyaeva [2] who substantially extended the concept of classical solitons to nonautonomous solitons. From a mathematical point of view, solitons are a class of special solutions of nonlinear partial differential equations (PDEs). As the generalization of classical differential equations with integer order, fractional-order differential calculus and its applications have attracted much attention [3–7]. In 2010, Fujioka, Espinosa and Rodríguez [8] described soliton propagation of an extended nonlinear Schrödinger equation which incorporates fractional dispersion and a fractional nonlinearity. Recently, Yang *et al.* [9] modeled fractal waves on shallow water surfaces via a local fractional KdV equation. In the past several decades, finding soliton solutions of nonlinear PDEs has gradually developed into a significant direction and some effective methods have been proposed for constructing soliton solutions, such as the inverse scattering method [10], Bäcklund transformation [11], Darboux transformation [12], Hirota's bilinear method [13], Wronskian technique [14], tanh method [15], homogeneous balance method [16], Jacobi elliptic function expansion method [17], sub-equation method [18], exp-function method [19], transformed rational function method [20], and the first integral method [21, 22].

As a direct method, Hirota's bilinear method [13] has been widely used to construct multi-soliton solutions of many nonlinear PDEs [23–34]. However, there is very little research work in extending Hirota's bilinear method for a whole hierarchy of nonlinear PDEs (see. e.g., [35]). This is because it is difficult to find a bilinear form suitable for all the nonlinear PDEs in a given hierarchy. In this paper, we shall give a bilinear form of the following new mKdV hierarchy with time-dependent coefficients

$$v_t = \sum_{m=0}^{n} \alpha_{2m+1}(t) F^m v_x,\ n = 0, 1, 2, \cdots, \tag{1}$$

and then construct its multi-soliton solutions through Hirota's bilinear method. Here the recursion operator employed is

$$F = \partial^2 + 4v^2 + 4v_x\partial^{-1}v,$$

$$\partial = \frac{\partial}{\partial x},\ \partial^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} \mathrm{d}x - \int_{x}^{\infty} \mathrm{d}x\right).$$

When $\alpha_{2m+1}(t) = 1$ and $m = n$, Equation (1) becomes the known constant-coefficient mKdV hierarchy [30] $v_t = F^n v_x$, $n = 0, 1, 2, \cdots$. The first equation of the mKdV hierarchy (1) is the trivial linear variable-coefficient equation $v_t = \alpha_1(t)v_x$, and the second and third equations are the following non-trivial nonlinear equations with variable-coefficients:

$$v_t = \alpha_1(t)v_x + \alpha_3(t)v_{xxx} + 6\alpha_3(t)v^2 v_x, \tag{2}$$

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$$\begin{aligned} v_t &= \alpha_1(t)v_x + \alpha_3(t)v_{xxx} + 6\alpha_3(t)v^2v_x \\ &+\alpha_5(t)v_{xxxxx} + 10\alpha_5(t)v^2v_{xxx} \\ &+40\alpha_5(t)vv_xv_{xx} + 10\alpha_5(t)v_x^3 + 30\alpha_5(t)v^4v_x. \end{aligned} \tag{3}$$

Obviously, Equations (2) and (3) include the famous mKdV equation $v_t = v_{xxx} + 6v^2v_x$ as a special case.

In Section 2, a bilinear form of the mKdV hierarchy (1) is obtained through Hirota's bilinear method. In Section 3, starting from the obtained bilinear form, we first construct one-soliton, two-soliton, and three-soliton solutions of the mKdV hierarchy (1). Based on the obtained soliton solutions, we then summarize a uniform formula for the explicit N-soliton solution of the mKdV hierarchy (1). In addition, some spatial structures and propagations of the obtained one-, two- and three-soliton solutions are shown in figures. In Section 4, we conclude this paper.

2 Bilinearization

For the bilinear form of the mKdV hierarchy (1), we have the following theorem.

Theorem 1. *The mKdV hierarchy* (1) *possesses a bilinear form:*

$$D_x^2 f^* \cdot f = 0, \tag{4}$$

$$\left(D_t - \sum_{m=0}^{n} \alpha_{2m+1}(t) D_x^{2m+1}\right) f^* \cdot f = 0, \tag{5}$$

where f^* denotes the conjugate of $f = f(x, t)$, and D_x and D_t are Hirota's differential operators defined by

$$\begin{aligned} &D_x^m D_t^n f(x,t) \cdot g(x,t) \\ &\quad = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x,t) \cdot g(x',t')|_{x'=x,t'=t}. \end{aligned}$$

Proof. Firstly, we reduce the general term $\alpha_{2m+1}(t)F^m v_x$ in Equation (1). For the sake of convenience, we introduce

$$v_{t_{2m+1}} = \alpha_{2m+1}(t)F^m v_x, m = 0, 1, 2, \cdots, n. \tag{6}$$

For any integer $m \geq 1$, Equation (6) can be rewritten as

$$\begin{aligned} v_{t_{2m+1}} &= \frac{\alpha_{2m+1}(t)}{\alpha_{2m-1}(t)} F v_{t_{2m-1}} = \frac{\alpha_{2m+1}(t)}{\alpha_{2m-1}(t)} [v_{t_{2m-1}xx} \\ &+4v^2 v_{t_{2m-1}} + 4v_x \partial^{-1} v v_{t_{2m-1}}]. \end{aligned} \tag{7}$$

Integrating Equation (7) with respect to x once, we have

$$\partial^{-1} v_{t_{2m+1}} = \frac{\alpha_{2m+1}(t)}{\alpha_{2m-1}(t)} \left[v_{t_{2m-1}x} + 2v\partial^{-1}(v^2)_{t_{2m-1}}\right]. \tag{8}$$

Secondly, we take a logarithmic transformation

$$v = i(\ln \frac{f^*}{f})_x, f = f(x, t_1, t_3, \cdots), \tag{9}$$

where i is the imaginary unit. Then Equation (8) becomes

$$\begin{aligned} &\frac{1}{f^*f} D_{t_{2m+1}} f^* \cdot f \\ &\quad = \frac{\alpha_{2m+1}(t)}{\alpha_{2m-1}(t)} \left[\frac{1}{f^*f} D_x^2 D_{t_{2m-1}} f^* \cdot f \right. \\ &\quad -\frac{1}{f^{*2}f^2} (D_{t_{2m-1}} f^* \cdot f)(D_x^2 f^* \cdot f) \\ &\quad \left. -\frac{2}{f^*f} (D_x f^* \cdot f) \partial^{-1} \left(\frac{D_x^2 f^* \cdot f}{f^*f}\right)_{t_{2m-1}}\right]. \end{aligned} \tag{10}$$

If we set $D_x^2 f^* \cdot f = 0$, then Equation (10) is reduced to

$$D_{t_{2m+1}} f^* \cdot f = \frac{\alpha_{2m+1}(t)}{\alpha_{2m-1}(t)} D_x^2 D_{t_{2m-1}} f^* \cdot f. \tag{11}$$

From Equations (6), (7) and (11), we have

$$D_{t_{2m+1}} f^* \cdot f = \alpha_{2m+1}(t) D_x^{2m+1} f^* \cdot f, m = 0, 1, 2, \cdots, n. \tag{12}$$

It is convenient to introduce $t_1, t_2, \cdots, t_n$ for the iterative reduction of the general term $\alpha_{2m+1}(t)F^m v_x$ in Equation (6). For example, Equation (6) gives

$$v_{t_1} = \alpha_1(t)v_x, v_{t_3} = \alpha_3(t)Fv_x = \frac{\alpha_3(t)}{\alpha_1(t)} F v_{t_1}, \tag{13}$$

$$v_{t_5} = \alpha_5(t)F^2 v_x = \frac{\alpha_5(t)}{\alpha_3(t)} F v_{t_3}, \tag{14}$$

and so forth.

In fact, $t_1, t_2, \cdots, t_n$ can be seen as the same as t. It is then easy to see from Equations (11)–(14) that the right side of Equation (1) is expressed by

$$\sum_{m=0}^{n} \alpha_{2m+1}(t)F^m v_x = \frac{1}{f^*f} \sum_{m=0}^{n} \alpha_{2m+1}(t) D_x^{2m+1} f^* \cdot f \tag{15}$$

under the case of $D_x^2 f^* \cdot f = 0$. At the same time, for the left side of Equation (1) we have

$$v_t = \frac{1}{f^*f} D_t f^* \cdot f. \tag{16}$$

Finally, with the help of Equations (14) and (15) we obtain a bilinear form which consists of Equations (4) and (5) for the mKdV hierarchy (1). Thus, the proof is complete. □

3 Multi-soliton solutions

In this section, we use the bilinear form in Equations (4) and (5) to construct multi-soliton solutions of the mKdV hierarchy (1). In order to obtain the one-soliton solution, we suppose that

$$f = 1 + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \varepsilon^3 f^{(3)} + \cdots, \tag{17}$$

$$f^{\star} = 1 + \varepsilon f^{(1)^{\star}} + \varepsilon^2 f^{(2)^{\star}} + \varepsilon^3 f^{(3)^{\star}} + \cdots. \tag{18}$$

Substituting Equations (17) and (18) into Equations (4) and (5), and then collecting the coefficients of the same order of ε, we obtain a system of PDEs:

$$\partial_x^2 (f^{(1)} + f^{(1)^{\star}}) = 0, \tag{19}$$

$$\partial_x^2 (f^{(2)} + f^{(2)^{\star}}) = -D_x^2 f^{(1)} \cdot f^{(1)^{\star}}, \tag{20}$$

$$\partial_x^2 (f^{(3)} + f^{(3)^{\star}}) = -D_x^2 (f^{(1)} \cdot f^{(2)^{\star}} + f^{(1)^{\star}} \cdot f^{(2)}), \tag{21}$$

$$\left[\partial_t - \sum_{m=0}^{n} \alpha_{2m+1}(t)\partial_x^{2m+1}\right](f^{(1)} + f^{(1)^{\star}}) = 0, \tag{22}$$

$$\begin{aligned}&\left[\partial_t - \sum_{m=0}^{n} \alpha_{2m+1}(t)\partial_x^{2m+1}\right](f^{(2)} + f^{(2)^{\star}})\\ &= -\left(D_t - \sum_{m=0}^{n} \alpha_{2m+1}(t) D_x^{2m+1}\right) f^{(1)} \cdot f^{(1)^{\star}},\end{aligned} \tag{23}$$

$$\begin{aligned}&\left[\partial_t - \sum_{m=0}^{n} \alpha_{2m+1}(t)\partial_x^{2m+1}\right](f^{(3)} + f^{(3)^{\star}})\\ &= -\left(D_t - \sum_{m=0}^{n} \alpha_{2m+1}(t) D_x^{2m+1}\right)\\ &\quad \times (f^{(1)} \cdot f^{(2)^{\star}} + f^{(1)^{\star}} \cdot f^{(2)}),\end{aligned} \tag{24}$$

$$\vdots$$

and so forth. If we let

$$f^{(1)} = e^{\xi_1 + \frac{\pi}{2} i}, \xi_1 = k_1 x + \sum_{m=0}^{n} w_{1,2m+1} \int \alpha_{2m+1}(t) \mathrm{d}t \tag{25}$$

be a solution of Equations (19) and (22), we obtain

$$w_{1,2m+1} = k_1^{2m+1}. \tag{26}$$

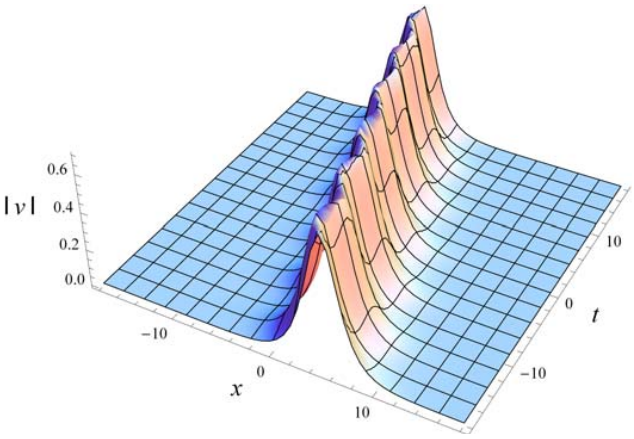


Figure 1: Spatial structure of one-soliton solution (29) with $k_1 = 0.7$, $\alpha_{2m+1}(t) = 1 + 0.5m \sin t$, $n = 20$.

Substituting Equations (25) and (26) into Equations (20) and (23), we can verify that if

$$f^{(2)} = f^{(2)^{\star}} = 0, \tag{27}$$

then Equations (21), (24) and all the other equations omitted in the above system of PDEs all hold. In this case, we write

$$f_1 = 1 + e^{\xi_1 + \frac{\pi}{2} i}, \tag{28}$$

and hence obtain the following one-soliton solution of the mKdV hierarchy (1):

$$v = i \left(\ln \frac{f_1^{\star}}{f_1}\right)_x = k_1 \mathrm{sech}\xi_1. \tag{29}$$

In Figure 1, the spatial structure of the one-soliton solution (29) is shown. Figure 2 displays the propagation of the one-soliton solution (29) along the negative x-axis. It can be seen from the velocity image in Figure 3 that the velocity of solution (29) periodically changes over time in the process of propagation.

For the two-soliton solution, we suppose that

$$f^{(1)} = e^{\xi_1 + \frac{\pi}{2} i} + e^{\xi_2 + \frac{\pi}{2} i}, \tag{30}$$

with

$$\xi_2 = k_2 x + \sum_{m=0}^{n} w_{2,2m+1} \int \alpha_{2m+1}(t) \mathrm{d}t.$$

Substituting Equation (30) into Equations (19) and (22) yields

$$w_{2,2m+1} = k_2^{2m+1}. \tag{31}$$

In view of Equation (23), we further suppose that

$$f^{(2)} = \mathrm{e}^{\xi_1+\xi_2+\pi i+\theta_{12}}, \tag{32}$$

where θ_{12} is a constant to be determined. Substituting Equations (30)–(32) into Equation (20), we obtain

$$\mathrm{e}^{\theta_{12}} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}. \tag{33}$$

It is easy to see that if we substitute Equations (30)–(32) into Equations (21) and (24), then we can verify that

$$f^{(3)} = f^{(3)*} = \cdots = 0 \tag{34}$$

satisfies Equations (21), (24) and all the other equations omitted in the above system of PDEs. In this case, we write

$$f_2 = 1 + \mathrm{e}^{\xi_1+\frac{\pi}{2}i} + \mathrm{e}^{\xi_2+\frac{\pi}{2}i} + \mathrm{e}^{\xi_1+\xi_2+\pi i+\theta_{12}} \tag{35}$$

and hence obtain the following two-soliton solution of the mKdV hierarchy (1):

$$v = i\left[\ln\left(\frac{1+\mathrm{e}^{\xi_1-\frac{\pi}{2}i}+\mathrm{e}^{\xi_2-\frac{\pi}{2}i}+\mathrm{e}^{\xi_1+\xi_2-\pi i+\theta_{12}}}{1+\mathrm{e}^{\xi_1+\frac{\pi}{2}i}+\mathrm{e}^{\xi_2+\frac{\pi}{2}i}+\mathrm{e}^{\xi_1+\xi_2+\pi i+\theta_{12}}}\right)\right]_x. \tag{36}$$

In Figure 4, the spatial structure of the two-soliton solution (36) is shown. Figure 5 displays the variable velocity propagation of the two-soliton solution (36) along the negative x-axis.

Similarly, we continue to construct the three-soliton solution. For this purpose, we suppose that

$$f^{(1)} = \mathrm{e}^{\xi_1+\frac{\pi}{2}i} + \mathrm{e}^{\xi_2+\frac{\pi}{2}i} + \mathrm{e}^{\xi_3+\frac{\pi}{2}i}, \tag{37}$$

with

$$\xi_3 = k_3 x + \sum_{m=0}^{n} w_{3,2m+1}\int \alpha_{2m+1}(t)\mathrm{d}t.$$

Substituting Equation (37) into Equations (19) and (22) yields

$$w_{3,2m+1} = k_3^{2m+1}. \tag{38}$$

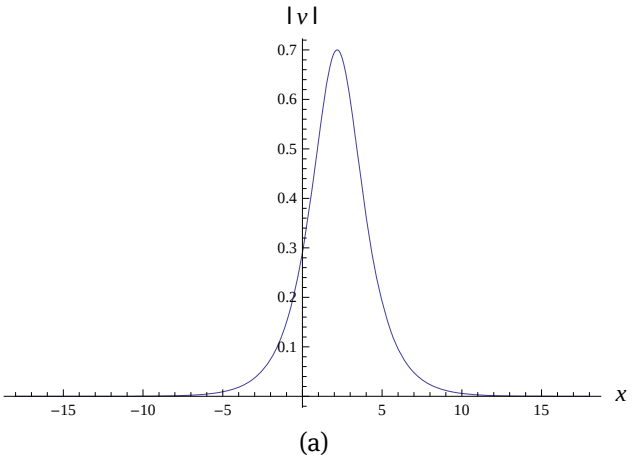
In view of Equations (20) and (23), we suppose that

$$f^{(2)} = \mathrm{e}^{\xi_1+\xi_2+\pi i+\theta_{12}} + \mathrm{e}^{\xi_1+\xi_3+\pi i+\theta_{13}} + \mathrm{e}^{\xi_2+\xi_3+\pi i+\theta_{23}}, \tag{39}$$

where θ_{13} and θ_{23} are unknown constants to be determined.

Substituting Equations (37)–(39) into Equations (20) and (23), we obtain

$$\mathrm{e}^{\theta_{13}} = \frac{(k_1-k_3)^2}{(k_1+k_3)^2}, \mathrm{e}^{\theta_{23}} = \frac{(k_2-k_3)^2}{(k_2+k_3)^2}. \tag{40}$$



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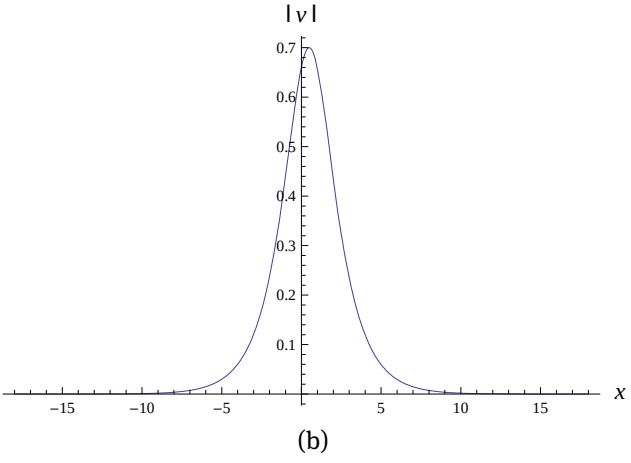

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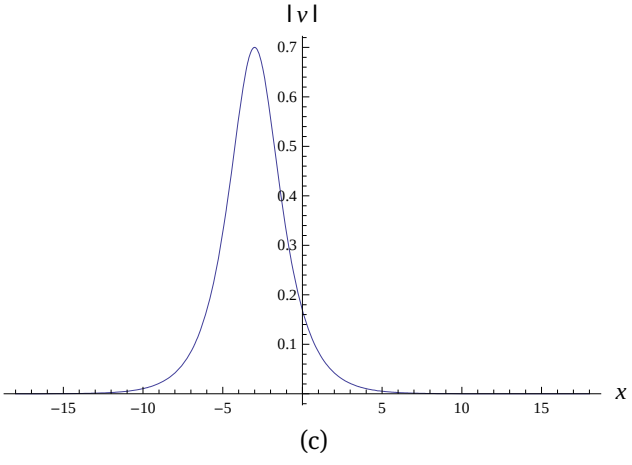


(c)

Figure 2: Propagation of one-soliton solution (29) with $k_1 = 0.7$, $\alpha_{2m+1}(t) = 1 + 0.5m\sin t$, $n = 20$ at different times (a) $t = -5$, (b) $t = 0$ and (c) $t = 5$.

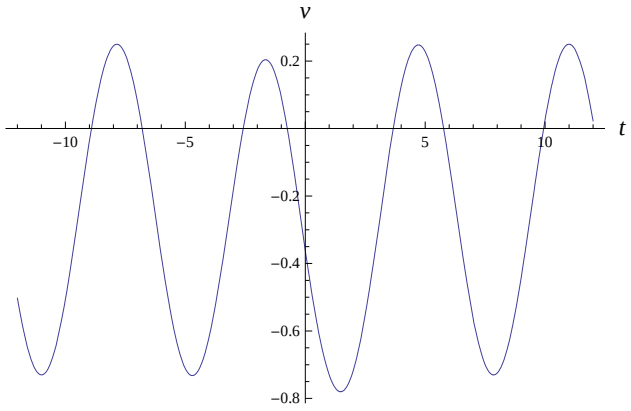
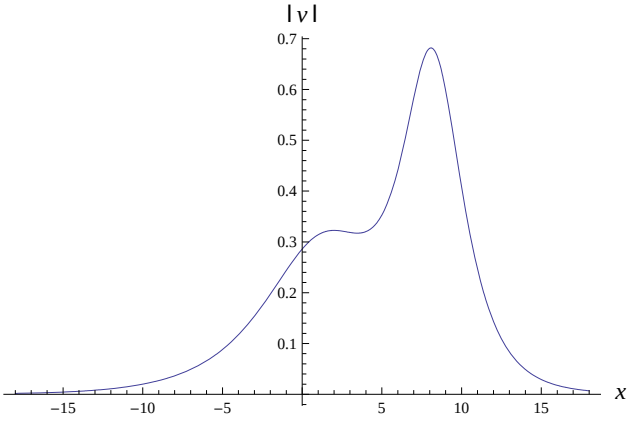


Figure 3: Velocity image of one-soliton solution (29) with $k_1 = 0.7$, $\alpha_{2m+1}(t) = 1 + 0.5m \sin t$, $n = 20$.



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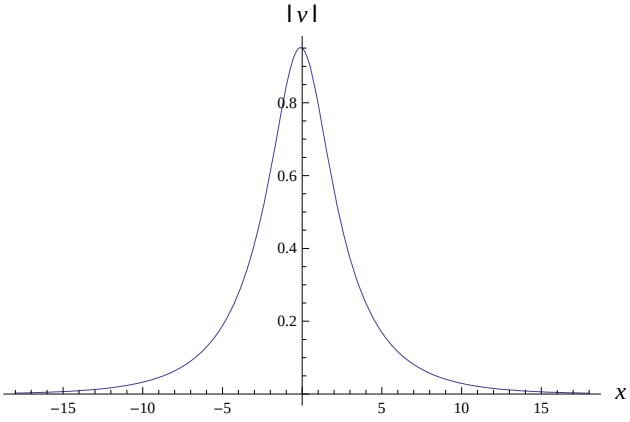

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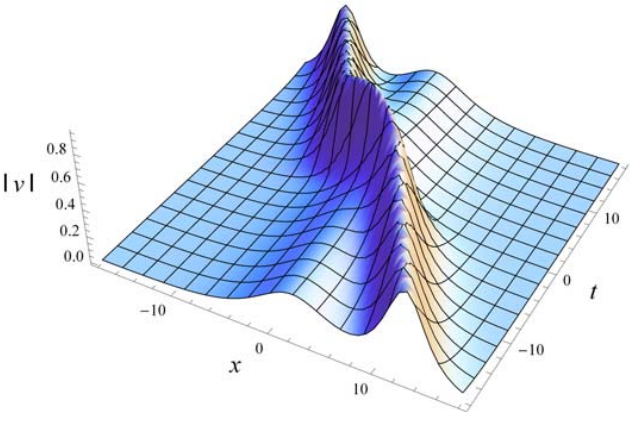
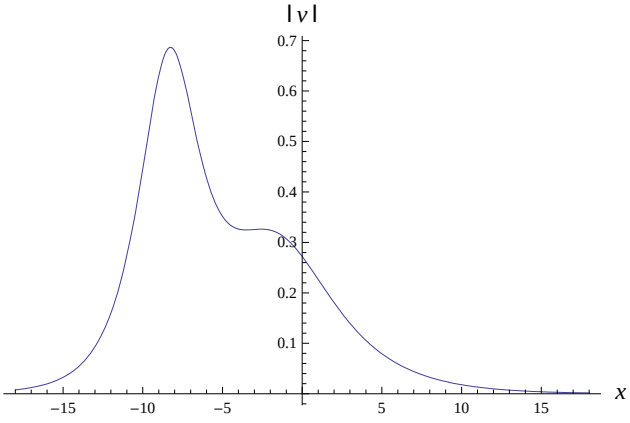


Figure 4: Spatial structure of two-soliton solution (36) with $k_1 = 0.3$, $k_2 = 0.6$, $\alpha_{2m+1}(t) = 1 + 2m\mathrm{sech}t$, $n = 120$.



(c)

Figure 5: Propagation of two-soliton solution (36) with $k_1 = 0.3$, $k_2 = 0.6$, $\alpha_{2m+1}(t) = 1 + 2m\mathrm{sech}t$, $n = 120$ at different times (a) $t = -10$, (b) $t = 0$ and (c) $t = 10$.

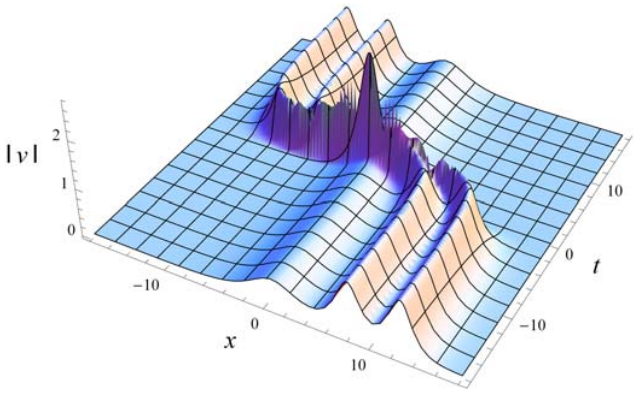


Figure 6: Spatial structure of three-soliton solution (43) with $k_1 = 0.5$, $k_2 = 1.05$, $k_3 = 1$, $\alpha_{2m+1}(t) = (1 + mt^2)^{-1}$, $n = 10$.

Then the substitution of Equations (37)–(40) into Equations (21) and (24) gives

$$f^{(3)} = \mathrm{e}^{\xi_1+\frac{\pi}{2}i+\xi_2+\frac{\pi}{2}i+\xi_3+\frac{\pi}{2}i+\theta_{12}+\theta_{13}+\theta_{23}}.$$

If we further take

$$f^{(i)} = 0, f^{(i)\star} = 0, i = 4, 5, \cdots, \tag{41}$$

then it can be verified that all the other equations omitted in the above system of PDEs hold. In this case, we write

$$\begin{aligned} f_3 = {} & 1 + \mathrm{e}^{\xi_1+\frac{\pi}{2}i} + \mathrm{e}^{\xi_2+\frac{\pi}{2}i} + \mathrm{e}^{\xi_3+\frac{\pi}{2}i} \\ & + \mathrm{e}^{\xi_1+\xi_2+\pi i+\theta_{12}} + \mathrm{e}^{\xi_1+\xi_3+\pi i+\theta_{13}} \\ & + \mathrm{e}^{\xi_2+\xi_3+\pi i+\theta_{23}} + \mathrm{e}^{\xi_1+\xi_2+\xi_3+\theta_{12}+\theta_{13}+\theta_{23}+\frac{3}{2}\pi i}, \end{aligned} \tag{42}$$

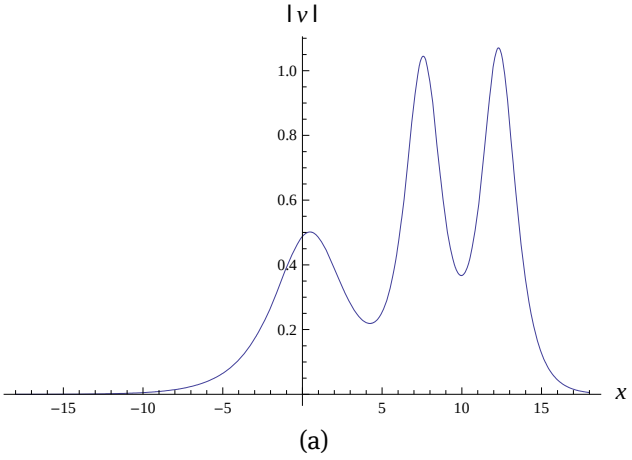
and hence obtain the following three-soliton solution of the the mKdV hierarchy (1):

$$v = i\left[\ln\left(\frac{\begin{array}{l} 1 + \mathrm{e}^{\xi_1-\frac{\pi}{2}i} + \mathrm{e}^{\xi_2-\frac{\pi}{2}i} \\ \quad + \mathrm{e}^{\xi_3-\frac{\pi}{2}i} + \mathrm{e}^{\xi_1+\xi_2-\pi i+\theta_{12}} \\ \quad + \mathrm{e}^{\xi_1+\xi_3-\pi i+\theta_{13}} + \mathrm{e}^{\xi_2+\xi_3-\pi i+\theta_{23}} \\ \quad + \mathrm{e}^{\xi_1+\xi_2+\xi_3+\theta_{12}+\theta_{13}+\theta_{23}-\frac{3}{2}\pi i} \end{array}}{\begin{array}{l} 1 + \mathrm{e}^{\xi_1+\frac{\pi}{2}i} + \mathrm{e}^{\xi_2+\frac{\pi}{2}i} \\ \quad + \mathrm{e}^{\xi_3+\frac{\pi}{2}i} + \mathrm{e}^{\xi_1+\xi_2+\pi i+\theta_{12}} \\ \quad + \mathrm{e}^{\xi_1+\xi_3+\pi i+\theta_{13}} + \mathrm{e}^{\xi_2+\xi_3+\pi i+\theta_{23}} \\ \quad + \mathrm{e}^{\xi_1+\xi_2+\xi_3+\theta_{12}+\theta_{13}+\theta_{23}+\frac{3}{2}\pi i} \end{array}}\right)\right]_x. \tag{43}$$

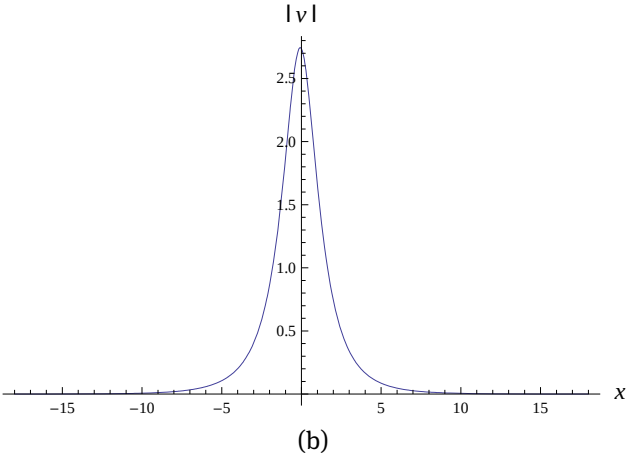
In Figure 6, the spatial structure of the three-soliton solution (43) is shown. Figure 7 displays the variable velocity propagation of the three-soliton solution (43) along the negative x-axis.

Generally, if we take

$$f_N = \sum_{\mu=0,1} \mathrm{e}^{\sum_{i=1}^{N}\mu_i(\xi_i+\frac{\pi}{2}i)+\sum_{1\le i<j}\mu_i\mu_j\theta_{ij}}, \tag{44}$$



(a)



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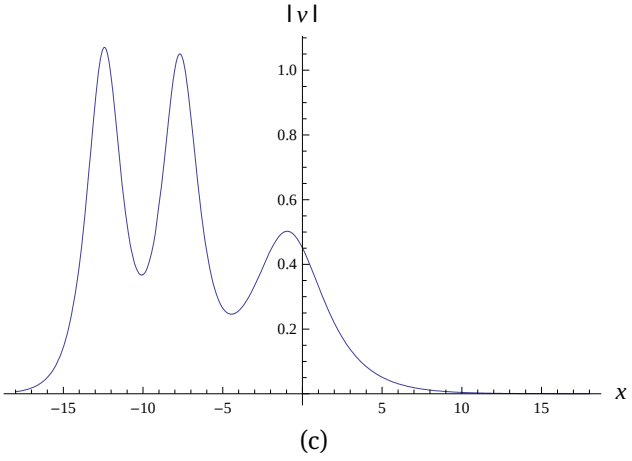

(c)

Figure 7: Propagation of three-soliton solution (43) with $k_1 = 0.5$, $k_2 = 1.05$, $k_3 = 1$, $\alpha_{2m+1}(t) = (1 + mt^2)^{-1}$, $n = 10$ at different times (a) $t = -15$, (b) $t = 0$ and (c) $t = 15$.

$$\xi_i = k_i x + \sum_{m=0}^{n} w_{i,2m+1} \int \alpha_{2m+1}(t)\mathrm{d}t,\ w_{i,2m+1} = k_i^{2m+1}, \quad (45)$$

$$\mathrm{e}^{\theta_{ij}} = \frac{(k_i - k_j)^2}{(k_i - k_j)^2} (i < j; i, j = 1, 2, \cdots, N), \quad (46)$$

then a uniform formula of the explicit N-soliton solution of the mKdV hierarchy (1) can be obtained as follows:

$$v = i \left\{ \ln \left[\frac{\sum\limits_{\mu=0,1} \mathrm{e}^{\sum\limits_{i=1}^{N} \mu_i(\xi_i - \frac{\pi}{2} i) + \sum\limits_{1 \le i < j} \mu_i \mu_j \theta_{ij}}}{\sum\limits_{\mu=0,1} \mathrm{e}^{\sum\limits_{i=1}^{N} \mu_i(\xi_i + \frac{\pi}{2} i) + \sum\limits_{1 \le i < j} \mu_i \mu_j \theta_{ij}}} \right] \right\}_x, \quad (47)$$

where the summation $\Sigma_{\mu=0,1}$ refers to all possible combinations of each $\mu_i = 0, 1$ for $i = 1, 2, \cdots, N$.

To the best of our knowledge, the one-soliton solution (29), two-soliton solution (36), three-soliton solution (43) and N-soliton solution (47) are new; they have not been reported in literature.

4 Conclusions

In summary, we have extended Hirota's bilinear method to the mKdV hierarchy (1) with time-dependent coefficients. In the procedure of extending Hirota's bilinear method to the mKdV hierarchy (1), one of the key steps is to reduce this mKdV hierarchy (1) to the bilinear form in Equations (4) and (5). The obtained one-soliton, two-soliton, three-soliton and N-soliton solutions (29), (36), (43) and (47) contain many time-dependent functions which provide enough freedom for us to describe spatial structures and propagations of these solutions. It is graphically shown that the one-soliton, two-soliton and three-soliton solutions (29), (36) and (43) possess time-varying velocities in the process of propagation. How to extend Hirota's bilinear method to some other hierarchies of nonlinear PDEs with time-dependent coefficients is worthy of study. This is our task in the future.

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